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Basis for the quotient space of matrices under equivalence

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For several of the results (chosen from [1]) in the section **Preliminaries** of the corresponding **Letter**, we give different proofs than those in [1]. We also give proofs for our main results (Proposition 1 and Theorem 4) in the **Letter**.

Theorem 1. For every $\bar{A} \in \mathcal{M}$, in $\langle \bar{A} \rangle$ there exists a unique irreducible element.

Corollary 1. For every given $A \in \mathcal{M}$, let A_0 be the only irreducible element of $\langle A \rangle$. Then

$$\langle A \rangle = \{A_0 \otimes I_s \mid s = 1, 2, \dots\}. \tag{1}$$

Theorem 2. Consider the algebraic system $(\mathcal{M}_\mu, \Vdash)$, where $\mu \in \mathbb{Q}_+$. The relation IE $\sim$ is a congruence with respect to $\Vdash$.

Proof. Arbitrarily chosen $\tilde{A}, A, \tilde{B}, B \in \mathcal{M}_\mu$, we assume $\tilde{A} \sim A$ and $\tilde{B} \sim B$. To prove this theorem, we need to verify $\tilde{A} \Vdash \tilde{B} \sim A \Vdash B$. We give an alternative proof compared to the one in [1].

By Theorem 1 and Corollary 1, there exist unique left irreducible matrices $A_0 \in \mathcal{M}_{\mu n \times n}$ and $B_0 \in \mathcal{M}_{\mu m \times m}$ such that

$$\begin{aligned} \tilde{A} &= A_0 \otimes I_s, \ A = A_0 \otimes I_t, \\ \tilde{B} &= B_0 \otimes I_p, \ B = B_0 \otimes I_q \end{aligned} \tag{2}$$

for some positive integers s, t, p, q. Set $T := \text{lcm}(ns, mp)$, $S := \text{lcm}(nt, mq)$, and $R := \text{lcm}(n, m)$. Then we have

$$\begin{aligned} \tilde{A} \Vdash \tilde{B} &= \left(A_0 \otimes I_s \otimes I_{T/ns}\right) + \left(B_0 \otimes I_p \otimes I_{T/mp}\right) \\ &= \left(A_0 \otimes I_{R/n} \otimes I_{T/R}\right) + \left(B_0 \otimes I_{R/m} \otimes I_{T/R}\right) \\ &= \left(A_0 \otimes I_{R/n} + B_0 \otimes I_{R/m}\right) \otimes I_{T/R}. \end{aligned} \tag{3}$$

Similarly, we have

$$A \Vdash B = \left(A_0 \otimes I_{R/n} + B_0 \otimes I_{R/m}\right) \otimes I_{S/R}. \tag{4}$$

(3) and (4) imply $\tilde{A} \Vdash \tilde{B} \sim A \Vdash B$.

Given $\mu \in \mathbb{Q}_+$, define the quotient space Σ_μ as

$$\Sigma_\mu := \mathcal{M}_\mu / \sim = \{\langle A \rangle \,|\, A \in \mathcal{M}_\mu\}. \tag{5}$$

Theorem 3. The quotient space $(\Sigma_\mu, \Vdash)$ is a vector space.

Proposition 1. For every positive rational number $\mu = p/q$, where $p, q \in \mathbb{Z}_+$ are co-prime, the vector space $(\Sigma_\mu, \Vdash)$ is of countably infinite dimension, and has a generator

$$\bigcup_{k \in \mathbb{Z}_+} \bigcup_{\substack{1 \leqslant i_{kp} \leqslant kp \\ 1 \leqslant j_{kq} \leqslant kq}} \left\{ \left\langle E^{kp \times kq}_{i_{kp} j_{kq}} \right\rangle \right\}, \tag{6}$$

where and hereinafter $E^{kp \times kq}_{i_{kp} j_{kq}}$ denotes the $kp \times kq$ matrix with the (i_{kp}, j_{kq})-th entry equal to 1 and all other entries 0.

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Proof. Arbitrarily chosen $\langle A\rangle$ in Σ_μ such that $A=(a_{ij})_{1\leqslant i\leqslant k_0p,1\leqslant j\leqslant k_0q}\in\mathcal{M}_{k_0p\times k_0q}$ is the unique irreducible matrix of $\langle A\rangle$, we have

$$\langle A\rangle = \underset{\substack{1\leqslant i\leqslant k_0p\\1\leqslant j\leqslant k_0q}}{\boxplus} a_{ij}\left\langle E_{ij}^{k_0p\times k_0q}\right\rangle$$

by Theorem 2. Hence the set (6) of countably infinite cardinality is a generator of the vector space $(\Sigma_\mu,\boxplus)$, and the dimension of $(\Sigma_\mu,\boxplus)$ is at most countably infinite.

We claim that the dimension of $(\Sigma_\mu,\boxplus)$ is countably infinite. Suppose the contrary: $(\Sigma_\mu,\boxplus)$ is of finite dimension, then there exists a positive integer l such that

$$\bigcup_{1\leqslant k\leqslant l}\ \bigcup_{\substack{1\leqslant i_{kp}\leqslant kp\\1\leqslant j_{kq}\leqslant kq}}\left\{\left\langle E_{i_{kp}j_{kq}}^{kp\times kq}\right\rangle\right\}$$

is a generator of $(\Sigma_\mu,\boxplus)$. Hence for all matrices A in $\mathcal{M}_{(l+1)p\times(l+1)q}$, $\langle A\rangle$ can be generated by

$$\bigcup_{\substack{k\leqslant l\\k|(l+1)}}\ \bigcup_{\substack{1\leqslant i_{kp}\leqslant kp\\1\leqslant j_{kq}\leqslant kq}}\left\{\left\langle E_{i_{kp}j_{kq}}^{kp\times kq}\right\rangle\right\},\tag{7}$$

where $k|(l+1)$ means k divides $l+1$.

It is not difficult to obtain that $\left\langle E_{12}^{(l+1)p\times(l+1)q}\right\rangle$ cannot be generated by (7), which is a contradiction, and hence completes the proof.

Note that for each positive integer l, the subset

$$\bigcup_{k=l}^{\infty}\ \bigcup_{\substack{1\leqslant i_{kp}\leqslant kp\\1\leqslant j_{kq}\leqslant kq}}\left\{\left\langle E_{i_{kp}j_{kq}}^{kp\times kq}\right\rangle\right\}$$

of (6) is a generator of the vector space $(\Sigma_\mu,\boxplus)$, since for all positive integers k_1,k_2 such that k_1 divides k_2,

$$\bigcup_{\substack{1\leqslant i_{k_1p}\leqslant k_1p\\1\leqslant j_{k_1q}\leqslant k_1q}}\left\{\left\langle E_{i_{k_1p}j_{k_1q}}^{k_1p\times k_1q}\right\rangle\right\}$$

is generated by

$$\bigcup_{\substack{1\leqslant i_{k_2p}\leqslant k_2p\\1\leqslant j_{k_2q}\leqslant k_2q}}\left\{\left\langle E_{i_{k_2p}j_{k_2q}}^{k_2p\times k_2q}\right\rangle\right\}.$$

That is, (6) is not a basis of $(\Sigma_\mu,\boxplus)$. Next we use (6) to construct a basis for the vector space $(\Sigma_\mu,\boxplus)$.

Theorem 4. For every positive rational number $\mu=p/q$, where $p,q\in\mathbb{Z}_+$ are co-prime, the following countably infinite set of equivalence classes

$$\mathsf{D}_\mu\cup\mathsf{N}_\mu\tag{8}$$

is a basis for the vector space $(\Sigma_\mu,\boxplus)$, where[1)]

$$\begin{aligned}\mathsf{D}_\mu&=\bigcup_{i=1}^{\infty}\ \bigcup_{\substack{1\leqslant j\leqslant i\\\gcd(i,j)=1}}\ \bigcup_{\substack{1\leqslant k\leqslant p\\1\leqslant l\leqslant q}}\left\{\left\langle E_{kl}^{p\times q}\otimes E_{jj}^{i\times i}\right\rangle\right\},\\ \mathsf{N}_\mu&=\bigcup_{i=2}^{\infty}\ \bigcup_{\substack{1\leqslant j_1,j_2\leqslant i\\j_1\neq j_2\\\gcd(i,j_1,j_2)=1}}\ \bigcup_{\substack{1\leqslant k\leqslant p\\1\leqslant l\leqslant q}}\left\{\left\langle E_{kl}^{p\times q}\otimes E_{j_1j_2}^{i\times i}\right\rangle\right\},\end{aligned}\tag{9}$$

where gcd() denote the greatest common divisor of numbers in ().

Proof. To prove this theorem, we need to show that i) (8) is a generator of $(\Sigma_\mu,\boxplus)$ and ii) every finite number of elements of (8) are linearly independent. Fix co-prime positive integers p and q.

i): We next verify that for all positive integers i,k,l,j_1,j_2 satisfying $1\leqslant k\leqslant p$, $1\leqslant l\leqslant q$ and $1\leqslant j_1,j_2\leqslant i$, $\left\langle E_{kl}^{p\times q}\otimes E_{j_1j_2}^{i\times i}\right\rangle$ is generated by (8). We divide the verification into two cases that $j_1=j_2=:j$ and $j_1\neq j_2$.

Case 1, $j_1=j_2$:

If $\gcd(i,j)=1$, then $\left\langle E_{kl}^{p\times q}\otimes E_{jj}^{i\times i}\right\rangle\in\mathsf{D}_\mu$, and $\left\langle E_{kl}^{p\times q}\otimes E_{jj}^{i\times i}\right\rangle$ is generated by (8). Next we assume that $\gcd(i,j)>1$. Write

$$\begin{aligned}f_1+f_2+\cdots+f_{m'}&=j,\\ g_1+g_2+\cdots+g_m&=j-1,\end{aligned}$$

1) Note that in (9), $E_{kl}^{p\times q}\otimes E_{jj}^{i\times i}$ is the unique irreducible element of $\left\langle E_{kl}^{p\times q}\otimes E_{jj}^{i\times i}\right\rangle$, and $E_{kl}^{p\times q}\otimes E_{j_1j_2}^{i\times i}$ is the unique irreducible element of $\left\langle E_{kl}^{p\times q}\otimes E_{j_1j_2}^{i\times i}\right\rangle$.

where

$$\begin{aligned}
f_1 &= \gcd(i, j), \\
f_2 &= \gcd(i, j - f_1), \\
&\vdots \\
f_{m'} &= \gcd\left(i, j - \sum_{\alpha=1}^{m'-1} f_\alpha\right), \\
g_1 &= \gcd(i, j - 1), \\
g_2 &= \gcd(i, j - 1 - g_1), \\
&\vdots \\
g_m &= \gcd\left(i, j - 1 - \sum_{\alpha=1}^{m-1} g_\alpha\right).
\end{aligned} \tag{10}$$

We have

$$E_{jj}^{i\times i} = \left(\biguplus_{n'=1}^{m'} E_{\bar{j}'_{n'}\bar{j}'_{n'}}^{\bar{i}'_{n'}\times\bar{i}'_{n'}}\right) \vdash \left(\biguplus_{n=1}^{m} E_{\bar{j}_n\bar{j}_n}^{\bar{i}_n\times\bar{i}_n}\right), \tag{11}$$

where $\bar{i}'_{n'} = i/f_{n'}$, $\bar{j}'_{n'} = (j - \sum_{\alpha=1}^{n'-1} f_\alpha)/f_{n'}$, $n' = 1, 2, \ldots, m'$; $\bar{i}_n = i/g_n$, $\bar{j}_n = (j - 1 - \sum_{\alpha=1}^{n-1} g_\alpha)/g_n$, $n = 1, 2, \ldots, m$.

Then $\bar{j}'_{n'} \leqslant \bar{i}'_{n'}$, $\gcd(\bar{i}'_{n'}, \bar{j}'_{n'}) = 1$, $\bar{j}_n \leqslant \bar{i}_n$, $\gcd(\bar{i}_n, \bar{j}_n) = 1$, $\left\langle E_{kl}^{p\times q} \otimes E_{\bar{j}'_{n'}\bar{j}'_{n'}}^{\bar{i}'_{n'}\times\bar{i}'_{n'}}\right\rangle \in \mathsf{D}_\mu$, $\left\langle E_{kl}^{p\times q} \otimes E_{\bar{j}_n\bar{j}_n}^{\bar{i}_n\times\bar{i}_n}\right\rangle \in \mathsf{D}_\mu$, $k = 1, 2, \ldots, p$, $l = 1, 2, \ldots, q$, $n' = 1, 2, \ldots, m'$, $n = 1, 2, \ldots, m$, and by Theorem 2,

$$\left\langle E_{kl}^{p\times q} \otimes E_{jj}^{i\times i}\right\rangle = \left(\biguplus_{n'=1}^{m'} \left\langle E_{kl}^{p\times q} \otimes E_{\bar{j}'_{n'}\bar{j}'_{n'}}^{\bar{i}'_{n'}\times\bar{i}'_{n'}}\right\rangle\right) \vdash \left(\biguplus_{n=1}^{m} \left\langle E_{kl}^{p\times q} \otimes E_{\bar{j}_n\bar{j}_n}^{\bar{i}_n\times\bar{i}_n}\right\rangle\right) \tag{12}$$

is generated by D_μ.

Case 2, $j_1 \neq j_2$:

Similar to the former case, if $\gcd(i, j_1, j_2) = 1$, then $\left\langle E_{kl}^{p\times q} \otimes E_{j_1j_2}^{i\times i}\right\rangle \in \mathsf{N}_\mu$, and $\left\langle E_{kl}^{p\times q} \otimes E_{j_1j_2}^{i\times i}\right\rangle$ is generated by (8). Next we consider the case $\gcd(i, j_1, j_2) > 1$, and assume that $j_1 < j_2$ without loss of generality. Write

$$f_1 + f_2 + \cdots + f_{m'} = j_1, \tag{13}$$

$$g_1 + g_2 + \cdots + g_m = j_1 - 1, \tag{14}$$

where

$$\begin{aligned}
f_1 &= \gcd(i, j_1, j_2), \\
f_2 &= \gcd(i, j_1 - f_1, j_2 - f_1), \\
&\vdots \\
f_{m'} &= \gcd\left(i, j_1 - \sum_{\alpha=1}^{m'-1} f_\alpha, j_2 - \sum_{\alpha=1}^{m'-1} f_\alpha\right), \\
g_1 &= \gcd(i, j_1 - 1, j_2 - 1), \\
g_2 &= \gcd(i, j_1 - 1 - g_1, j_2 - 1 - g_1), \\
&\vdots \\
g_m &= \gcd\left(i, j_1 - 1 - \sum_{\alpha=1}^{m-1} g_\alpha, j_2 - 1 - \sum_{\alpha=1}^{m-1} g_\alpha\right).
\end{aligned} \tag{15}$$

We have

$$E_{j_1j_2}^{i\times i} = \left(\biguplus_{n'=1}^{m'} E_{\bar{j}'_{1n'}\bar{j}'_{2n'}}^{\bar{i}'_{n'}\times\bar{i}'_{n'}}\right) \vdash \left(\biguplus_{n=1}^{m} E_{\bar{j}_{1n}\bar{j}_{1n}}^{\bar{i}_n\times\bar{i}_n}\right), \tag{16}$$

where $\bar{i}'_{n'} = i/f_{n'}$, $\bar{j}'_{1n'} = (j_1 - \sum_{\alpha=1}^{n'-1} f_\alpha)/f_{n'}$, $\bar{j}'_{2n'} = (j_2 - \sum_{\alpha=1}^{n'-1} f_\alpha)/f_{n'}$, $n' = 1, 2, \ldots, m'$; $\bar{i}_n = i/g_n$, $\bar{j}_{1n} = (j_1 - 1 - \sum_{\alpha=1}^{n-1} g_\alpha)/g_n$, $\bar{j}_{2n} = (j_2 - 1 - \sum_{\alpha=1}^{n-1} g_\alpha)/g_n$, $n = 1, 2, \ldots, m$.

Then $\bar{j}'_{1n'}, \bar{j}'_{2n'} \leqslant \bar{i}'_{n'}$, $\gcd(\bar{i}'_{n'}, \bar{j}'_{1n'}, \bar{j}'_{2n'}) = 1$, $\bar{j}_{1n}, \bar{j}_{2n} \leqslant \bar{i}_n$, $\gcd(\bar{i}_n, \bar{j}_{1n}, \bar{j}_{2n}) = 1$, $\left\langle E_{kl}^{p\times q} \otimes E_{\bar{j}'_{1n'}\bar{j}'_{2n'}}^{\bar{i}'_{n'}\times\bar{i}'_{n'}}\right\rangle \in \mathsf{N}_\mu$, $\left\langle E_{kl}^{p\times q} \otimes E_{\bar{j}_{1n}\bar{j}_{2n}}^{\bar{i}_n\times\bar{i}_n}\right\rangle \in \mathsf{N}_\mu$, $k = 1, 2, \ldots, p$, $l = 1, 2, \ldots, q$, $n' = 1, 2, \ldots, m'$, $n = 1, 2, \ldots, m$, and by Theorem 2,

$$\left\langle E_{kl}^{p\times q} \otimes E_{j_1j_2}^{i\times i}\right\rangle = \left(\biguplus_{n'=1}^{m'} \left\langle E_{kl}^{p\times q} \otimes E_{\bar{j}'_{1n'}\bar{j}'_{2n'}}^{\bar{i}'_{n'}\times\bar{i}'_{n'}}\right\rangle\right) \vdash \left(\biguplus_{n=1}^{m} \left\langle E_{kl}^{p\times q} \otimes E_{\bar{j}_{1n}\bar{j}_{2n}}^{\bar{i}_n\times\bar{i}_n}\right\rangle\right) \tag{17}$$

is generated by N_μ.

Based on the above, we have (6) is generated by (8), then by Proposition 1, (8) is a generator of $(\Sigma_\mu, \boxplus)$. Besides, by (12) and (17) we have every element of $(\Sigma_\mu, \boxplus)$ is represented uniquely as a linear combination of finitely many elements of (8).

ii): To prove that every finite number of elements of (8) are linearly independent, we need to prove that every finite number of elements of D_μ are linearly independent and every finite number of elements of N_μ are linearly independent, because in each $E_{jj}^{i\times i}$, only diagonal entries can be nonzero, and in each $E_{j_1j_2}^{i\times i}$ with $j_1 \neq j_2$, only nondiagonal entries can be nonzero.

To prove that every finite number of elements of D_μ are linearly independent, we need to prove that for all positive integers k, l, n satisfying that $1 \leqslant k \leqslant p$ and $1 \leqslant l \leqslant q$,

$$\bigcup_{i=1}^{n} \bigcup_{\substack{1\leqslant j\leqslant i \\ \gcd(i,j)=1}} \left\{ \left\langle E_{kl}^{p\times q} \otimes E_{jj}^{i\times i} \right\rangle \right\} \tag{18}$$

are linearly independent. Write $t := \mathrm{lcm}(1, 2, \ldots, n)$, then (18) equals

$$\bigcup_{i=1}^{n} \bigcup_{\substack{1\leqslant j\leqslant i \\ \gcd(i,j)=1}} \left\{ \left\langle E_{kl}^{p\times q} \otimes E_{jj}^{i\times i} \otimes I_{t/i} \right\rangle \right\}. \tag{19}$$

What is left is to prove the set

$$\{jt/i | 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant i, \gcd(i, j) = 1\} \tag{20}$$

has the same cardinality as (19), because for each $E_{jj}^{i\times i} \otimes I_{t/i}$, the $(jt/i, jt/i)$-th entry equals 1, and either $jt/i = t$ or the (s, s)-th entry equals 0 for each ineger $s > jt/i$.

Suppose there exist positive integers i_1, j_1, i_2, j_2 such that $1 \leqslant j_1 \leqslant i_1$, $1 \leqslant j_2 \leqslant i_2$, $\gcd(i_1, j_1) = \gcd(i_2, j_2) = 1$, and $j_1t/i_1 = j_2t/i_2$, then $i_1 = i_2$ and $j_1 = j_2$. Hence the set (20) has the same cardinality as the set (19).

To prove that every finite number of elements of N_μ are linearly independent, similar to the case for D_μ, we need to prove that for all positive integers k, l, n satisfying that $n \geqslant 2$, $1 \leqslant k \leqslant p$ and $1 \leqslant l \leqslant q$,

$$\bigcup_{i=2}^{n} \bigcup_{\substack{1\leqslant j_1,j_2\leqslant i \\ j_1\neq j_2 \\ \gcd(i,j_1,j_2)=1}} \left\{ \left\langle E_{kl}^{p\times q} \otimes E_{j_1j_2}^{i\times i} \right\rangle \right\} \tag{21}$$

are linearly independent. Write $\mathrm{lcm}(2, \ldots, n) =: t$, then (21) equals

$$\bigcup_{i=2}^{n} \bigcup_{\substack{1\leqslant j_1,j_2\leqslant i \\ j_1\neq j_2 \\ \gcd(i,j_1,j_2)=1}} \left\{ \left\langle E_{kl}^{p\times q} \otimes E_{j_1j_2}^{i\times i} \otimes I_{t/i} \right\rangle \right\}. \tag{22}$$

What is left is to prove the set

$$\{(j_1t/i, j_2t/i) | 1 \leqslant i \leqslant n, 1 \leqslant j_1, j_2 \leqslant i, j_1 \neq j_2, \gcd(i, j_1, j_2) = 1\} \tag{23}$$

has the same cardinality as (22), because for each $E_{j_1j_2}^{i\times i} \otimes I_{t/i}$, the $(j_1t/i, j_2t/i)$-th entry equals 1, and either $j_1t/i = t$, or $j_2t/i = t$, or the $(j_1t/i + s, j_2t/i + s)$-th entry equals 0 for each positive integer s.

Suppose there exist positive integers $i_1, j_1^1, j_2^1, i_2, j_1^2, j_2^2$ such that $1 \leqslant j_1^1, j_2^1 \leqslant i_1$, $1 \leqslant j_1^2, j_2^2 \leqslant i_2$, $j_1^1 \neq j_2^1$, $j_1^2 \neq j_2^2$, $\gcd(i_1, j_1^1, j_2^1) = \gcd(i_2, j_1^2, j_2^2) = 1$, $j_1^1t/i_1 = j_1^2t/i_2$, and $j_2^1t/i_1 = j_2^2t/i_2$, then $i_1 = i_2$, $j_1^1 = j_1^2$ and $j_2^1 = j_2^2$. This follows from the discussion below. i) Suppose $\gcd(i_1, j_1^1) = 1$ and $i_1 \neq i_2$, then $i_1 | i_2$, $i_2/i_1 =: t > 1$, and $\gcd(i_2, j_1^2, j_2^2) = \gcd(ti_1, tj_1^1, tj_2^1) = t > 1$, which is a contradiction. ii) Suppose $\gcd(i_1, j_1^1) > 1$, $i_1 \neq i_2$ and $\gcd(i_2, j_1^2) = 1$, then similarly $i_2 | i_1$, and $\gcd(i_1, j_1^1, j_2^1) = i_1/i_2 > 1$, which is also a contradiction. iii) Suppose $\gcd(i_1, j_1^1) =: t > 1$, $i_1 \neq i_2$ and $\gcd(i_2, j_1^2) =: s > 1$, then $\gcd(i_1/t, j_1^1/t) = \gcd(i_2/s, j_1^2/s) = 1$, $i_1/t = i_2/s$, $\gcd(t, s) | j_2^1$, $\gcd(t, s) | j_2^2$, $t \neq s$ (if $t = s$, then $i_1 = i_2$, a contradiction), $\gcd(i_1, j_1^1, j_2^1) \geqslant t/\gcd(t, s) > 1$ or $\gcd(i_2, j_1^2, j_2^2) \geqslant s/\gcd(t, s) > 1$, which is again a contradiction. Hence the set (23) has the same cardinality as the set (22), which completes the proof.

References

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