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• **Supplementary File** •

Distributed algorithms for solving a class of convex feasibility problems

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Appendix A Proof of Theorem 1

To prove Theorem 1, the following lemmas are presented.

Lemma 1. [1,2] For a graph $\mathcal{G}(\mathcal{A})$, if $\mathcal{G}(\mathcal{A})$ strongly connected, then the Laplacian matrix L has one simple 0 eigenvalue and the other eigenvalues have positive real parts. Moreover, there exists a vector $w = [w_1 \cdots w_n]^T > 0$ such that $w^T L = 0$.

For ease of description, if $\mathcal{G}(\mathcal{A})$ strongly connected, we use $\lambda_1(L)$ to represent the 0 eigenvalue and $\lambda_i(L), i = 2, \cdots, n$ to represent other non-zero eigenvalues.

Lemma 2. [3] Given a symmetric matrix $P = (p_{ij})_{n\times n}$ and a vector $x = [x_1, \cdots, x_n]^T$, if $P\mathbf{1}_n = 0$, then $x^T Px = -\sum_{i=1}^{n}\sum_{j=i+1}^{n} p_{ij}(x_i - x_j)^2$.

Proof of Theorem 1.

Since the graph is strongly connected, by Lemma 1, there exists a vector $w = [w_1 \cdots w_n]^T > 0$ such that $w^T L = 0$. Consider a positive-definite Lyapunov function candidate $V(t) = \frac{1}{2}\sum_{i=1}^{n} w_i\|x_i(t) - x_0\|^2$, where $x_0 \in \mathbf{X}^*$. By the definition of g_i^+, we have $g_i^+(x_0) = \|x_0\|_X = 0$. Based on the property of subgradient, we have $\left\langle x_i(t) - x_0, \nabla g_i^+(x_i(t))\right\rangle \geqslant g_i^+(x_i(t))$. Taking the derivative of function $V(t)$ with respect to t yields

$$\begin{aligned}
\dot{V}(t) &= \sum_{i=1}^{n} w_i \langle x_i(t) - x_0, \dot{x}_i(t)\rangle \\
&= \sum_{i=1}^{n} w_i \Big\langle x_i(t) - x_0, \sum_{i\in N_i} a_{ij}(t)(x_j(t) - x_i(t)) - \tau[x_i(t) - P_{X_i}(x_i(t))] - \tau\nabla g_i^+(x_i(t))\Big\rangle \\
&= \sum_{i=1}^{n}\sum_{j\in N_i} w_i a_{ij}\langle x_i(t) - x_0, x_j(t) - x_i(t)\rangle - \tau\sum_{i=1}^{n} w_i \left\langle x_i(t) - x_0, x_i(t) - P_{X_i}(x_i(t))\right\rangle \\
&\quad - \tau\sum_{i=1}^{n} w_i \left\langle x_i(t) - x_0, \nabla g_i^+(x_i(t))\right\rangle .
\end{aligned} \tag{A1}$$

Denote $x(t) = \left[x_1^T(t), \cdots, x_n^T(t)\right]^T$, we have

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j\in N_i} w_i a_{ij}\langle x_i(t) - x_0, x_j(t) - x_i(t)\rangle \\
&= -\left(x(t) - (\mathbf{1}_n\otimes I_m)\,x_0\right)^T (WL\otimes I_m)\,x(t) + x_0^T\left(w^T L\otimes I_m\right)x(t) \\
&= x^T(t)\left(\frac{W(-L) + (-L)^T W}{2}\otimes I_m\right)x(t) \\
&= -\sum_{i=1}^{n}\sum_{j=i+1}^{n}\frac{w_i a_{ij} + w_j a_{ji}}{2}\|x_j(t) - x_i(t)\|^2 \\
&\leqslant 0
\end{aligned} \tag{A2}$$

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where $W = \mathrm{diag}(w)$ is a diagonal matrix formed by w and the last equation results from Lemma 2. Since $-\langle x_i(t) - x_0, x_i(t) - P_{X_i}(x_i(t))\rangle \leqslant -\frac{1}{2}\left\|x_i(t) - P_{X_i}(x_i(t))\right\|^2 \leqslant 0$, by (A1) and (A2), we have

$$\begin{aligned}\dot{V}(t) \leqslant & -\tau\sum_{i=1}^{n} w_i \left\|x_i(t) - P_{X_i}(x_i(t))\right\|^2 - \tau\sum_{i=1}^{n} w_i g_i^{+}(x_i(t)) \\ & - \sum_{i=1}^{n}\sum_{j=i+1}^{n} \frac{w_i a_{ij} + w_j a_{ji}}{2}\left\|x_j(t) - x_i(t)\right\|^2.\end{aligned} \tag{A3}$$

Note that $g_i^{+}(x_i(t)) \geqslant 0$. Thus, $\dot{V}(t) \leqslant 0$. Moreover, $V(t)$ is bounded by zero, therefore, $V(t)$ converges and $V(\infty)$ exists. By (A3), we have

$$\begin{cases}\int_0^{\infty}\sum_{i=1}^{n} w_i \left\|x_i(t) - P_{X_i}(x_i(t))\right\|^2 d_t \leqslant \frac{V(0) - V(\infty)}{\tau} < \infty \\ \int_0^{\infty}\sum_{i=1}^{n} w_i g_i^{+}(x_i(t)) d_t \leqslant \frac{V(0) - V(\infty)}{\tau} < \infty \\ \sum_{i=1}^{n}\sum_{j=i+1}^{n} \frac{w_i a_{ij} + w_j a_{ji}}{2}\int_0^{\infty}\|x_j(t) - x_i(t)\|^2 d_t \leqslant V(0) - V(\infty) < \infty.\end{cases} \tag{A4}$$

The first two inequalities in (A4) imply $\lim_{t\to\infty}\|x_i(t) - P_{X_i}(x_i(t))\| = 0$ and $\lim_{t\to\infty} g_i^{+}(x_i(t)) = 0$, respectively. Moreover, by the third inequality in (A4), we have $\lim_{t\to\infty}\|x_j(t) - x_i(t)\|^2 = 0$ for any $i, j \in \mathcal{V}$. Consequently, $\lim_{t\to\infty}\|x_i(t) - x^*\| = 0$ for some $x^* \in X^*$. This completes the proof.

Appendix B Proof of Theorem 2

Let $x(t) = \left[x_1^T(t), \cdots, x_n^T(t)\right]^T$ and $\phi(t) = \left[\phi_1^T(t), \cdots, \phi_n^T(t)\right]^T$, MAS (2) with (5) can be rewritten as

$$x(t+1) = \left((I - hL) \otimes I_m\right) x(t) + \phi(t). \tag{B1}$$

Define a variable $\hat{x}(t) = \sum_{i=1}^{n} \frac{w_i x_i(t)}{\sum_{i=1}^{n} w_i} = \left(\frac{w^T}{\mathbf{1}^T w} \otimes I_m\right) x(t)$, where $w = [w_1 \cdots w_n]^T$ is L's left eigenvector associated with 0 eigenvalue. Based on (B1), we have $\hat{x}(t+1) = \hat{x}(t) + \frac{(w^T \otimes I_m)}{\mathbf{1}^T w}\phi(t)$. Denote $e_i(t) = x_i(t) - \hat{x}(t)$ and $e(t) = \left[e_1^T(t), \cdots, e_n^T(t)\right]^T$. By (B1), we have

$$e(t+1) = ((I - hL) \otimes I_m)e(t) + \left(\left(I_n - \frac{\mathbf{1}_n w^T}{\mathbf{1}_n^T w}\right) \otimes I_m\right)\phi(t). \tag{B2}$$

To prove Theorem 2, the following lemmas are presented.

Lemma 3. ([4], Lemma 3.1) Let $\{\eta(t)\}$ be a scalar sequence, if $\lim\limits_{t\to\infty}\eta(t) = 0$ and $0 < \rho < 1$, then $\lim\limits_{t\to\infty}\sum_{\tau=0}^{t}\rho^{t-\tau}\eta(\tau) = 0$.

Lemma 4. Consider a linear system $x(t+1) = Ax(t) + \theta(t)$, where $A \in \mathbb{R}^{n\times n}$ and $x(t), \theta(t) \in \mathbb{R}^n$, if each eigenvalue $\lambda_i(A)$ of A satisfies $|\lambda_i(A)| < 1$ for any $i \in \{1, \cdots, n\}$ and $\lim\limits_{t\to\infty}\|\theta(t)\| = 0$, then we have $\lim\limits_{t\to\infty}\|x(t)\| = 0$.

Proof. From the theory of Schur's unitary triangularization, we know that for any $A \in \mathbb{R}^{n\times n}$, there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that

$$U^H A U = \begin{bmatrix}\lambda_1 & \lambda_{12} & \cdots & \lambda_{1n} \\ 0 & \lambda_2 & \lambda_{23} & \lambda_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda_n\end{bmatrix} \triangleq \Lambda$$

where λ_i is the eigenvalue of matrix A, $i = 1, \cdots, n$; U^H is the conjugate transpose matrix of U. Denote $y(t) = U^H x(t)$ and $r(t) = U^H\theta(t)$, we have

$$y(t+1) = \Lambda y(t) + r(t).$$

By the fact that $\lim\limits_{t\to\infty}\|\theta(t)\| = 0$, we have $\lim\limits_{t\to\infty}\|r(t)\| = 0$. Let $y(t) = [y_1(t), \cdots, y_n(t)]^T$ and $r(t) = [r_1(t), \cdots, r_n(t)]^T$, for any $i = 1, \cdots, n$, it is obvious that

$$y_i(t+1) = \lambda_i(A)y_i(t) + \left(\sum_{j=i+1}^{n}\lambda_{ij}(A)y_j(t) + r_i(t)\right).$$

It follows that

$$|y_i(t)| \leqslant |\lambda_i(A)|^t|y_i(0)| + \sum_{\tau=0}^{t}|\lambda_i(A)|^{t-\tau}\left(\sum_{j=i+1}^{n}|\lambda_{ij}(A)||y_j(\tau)| + |r_i(\tau)|\right).$$

(i) For $i = n$, we have

$$|y_n(t)| \leqslant |\lambda_n(A)|^t|y_n(0)| + \sum_{\tau=0}^{t}|\lambda_n(A)|^{t-\tau}|r_n(\tau)|.$$

By the fact that $|\lambda_n(A)|<1$ and using Lemma 3, we have $\lim\limits_{t\to\infty}|y_n(t)|=0$.
(ii) For $i=n-1$, we have

$$|y_{n-1}(t)|\leqslant|\lambda_{n-1}(A)|^t|y_{n-1}(0)|+\sum_{\tau=0}^{t}|\lambda_{n-1}(A)|^{t-\tau}\left(|\lambda_{(n-1)n}(A)||y_n(\tau)|+|r_{n-1}(\tau)|\right).$$

Note that $\lim\limits_{t\to\infty}(|\lambda_n(A)y_n(t)|+|r_{n-1}(t)|)=0$ and $|\lambda_{n-1}(A)|<1$, using Lemma 3 again, there is $\lim\limits_{t\to\infty}y_{n-1}(t)=0$.

In a similar fashion, we have $\lim\limits_{t\to\infty}\left(\sum\limits_{j=i+1}^{n}|\lambda_{ij}(A)||y_j(t)|+|r_i(t)|\right)=0$ and $\lim\limits_{t\to\infty}|y_i(t)|=0$ for $i=n-2,\cdots,1$. Thus, $\lim\limits_{t\to\infty}\|y(t)\|=0$. This and the fact $x(t)=Uy(t)$ imply $\lim\limits_{t\to\infty}\|x(t)\|=0$.

Proof of Theorem 2.

The proof consists of two parts. In part 1, we will prove that $y_i(t)$ asymptotically converges to set $\Omega_i=\{y|g_i^+(y)=0,y\in X_i\},\forall\, i\in\mathcal{V}$. In part 2, we will be committed to showing that consensus can be achieved asymptotically.

Part 1. Since the graph is strongly connected, by Lemma 1, there exists a vector $w=[w_1\cdots w_n]^T>0$ such that $w^TL=0$. Consider the positive-definite Lyapunov function candidate $V(t)=\sum\limits_{i=1}^{n}w_i\|x_i(t)-x_0\|^2$, where $x_0\in\mathbf{X}^*$. Taking the difference of function $V(t)$ yields

$$\begin{aligned}\Delta V(t)&=V(t+1)-V(t)\\&=\sum_{i=1}^{n}w_i\|y_i(t)+\phi_i(t)-x_0\|^2-\sum_{i=1}^{n}w_i\|x_i(t)-x_0\|^2\\&=\sum_{i=1}^{n}w_i\|y_i(t)-x_0\|^2+2\sum_{i=1}^{n}w_i\left\langle\phi_i(t),y_i(t)-x_0\right\rangle\\&\quad+\sum_{i=1}^{n}w_i\|\phi_i(t)\|^2-\sum_{i=1}^{n}w_i\|x_i(t)-x_0)\|^2.\end{aligned}\tag{B3}$$

Since $0<h<\frac{1}{\max\limits_{1\leqslant i\leqslant n}\left(\sum\limits_{j=1}^{n}a_{ij}\right)}$, we have $0<1-h\sum\limits_{j=1}^{n}l_{ij}<1$. By convexity of the norm square function, we have

$$\begin{aligned}\|y_i(t)-x_0\|^2&=\|(1-h\sum_{j\in N_i}l_{ij})(x_i(t)-x_0)+h\sum_{j\in N_i}l_{ij}(x_j(t)-x_0)\|^2\\&\leqslant(1-h\sum_{j\in N_i}l_{ij})\|x_i(t)-x_0\|^2+h\sum_{j\in N_i}l_{ij}\|x_j(t)-x_0\|^2.\end{aligned}$$

Then, we have

$$\begin{aligned}\sum_{i=1}^{n}w_i\|y_i(t)-x_0\|^2&\leqslant\sum_{i=1}^{n}w_i(1-h\sum_{j\in N_i}l_{ij})\|x_i(t)-x_0\|^2+h\sum_{i=1}^{n}w_i\sum_{j\in N_i}l_{ij}\|x_j(t)-x_0\|^2\\&=\sum_{i=1}^{n}w_i\|x_i(t)-x_0\|^2-h\sum_{i=1}^{n}w_i\left(\sum_{j=1}^{n}l_{ij}\right)\|x_i(t)-x_0\|^2\\&\quad+h\sum_{j=1}^{n}\left(\sum_{i=1}^{n}w_il_{ij}\right)\|x_j(t)-x_0\|^2\\&=\sum_{i=1}^{n}w_i\|x_i(t)-x_0\|^2\end{aligned}\tag{B4}$$

Submitting (B4) into (B3) yields

$$\Delta V(t)\leqslant2\sum_{i=1}^{n}w_i\left\langle\phi_i(t),y_i(t)-x_0\right\rangle+\sum_{i=1}^{n}w_i\|\phi_i(t)\|^2.\tag{B5}$$

Now we discuss the value of term $\|\phi_i(t)\|^2+2\left\langle\phi_i(t),y_i(t)-x_0\right\rangle$ in the following two cases.

Case 1. If $\varphi_i(t)=0$, then $\phi_i(t)=0$, which implies that $\|\phi_i(t)\|^2+2\left\langle\phi_i(t),y_i(t)-x_0\right\rangle=0$;

Case 2. If $\varphi_i(t)\neq0$, then $\frac{\xi_i(t)}{2\varphi_i(t)}>0$. By the properties of subgradient, we have

$$\begin{aligned}\left\langle\phi_i(t),y_i(t)-x_0\right\rangle&=-\left\langle\frac{\epsilon\xi_i(t)}{2\varphi_i(t)}(y_i(t)-P_{X_i}(y_i(t))+\nabla g_i^+(t),y_i(t)-x_0\right\rangle\\&\leqslant-\frac{\epsilon\xi_i(t)}{2\varphi_i(t)}\left(g_i^+(y_i(t))+\frac{1}{2}\|y_i(t)-P_{X_i}(y_i(t))\|^2\right)\\&=-\frac{\epsilon\xi_i^2(t)}{2\varphi_i(t)}\end{aligned}\tag{B6}$$

Moreover, we have

$$\begin{aligned}\|\phi_i(t)\|^2 &= \left(\frac{\epsilon\xi_i(t)}{2\varphi_i(t)}\right)^2 \|y_i(t) - P_{X_i}(y_i(t)) + \nabla g_i^+(t)\|^2 \\ &\leqslant \frac{1}{2}\left(\frac{\epsilon\xi_i(t)}{\varphi_i(t)}\right)^2 \left(\|y_i(t) - P_{X_i}(y_i(t))\|^2 + \|\nabla g_i^+(t)\|^2\right) \\ &= \frac{1}{2}\frac{\epsilon^2\xi_i^2(t)}{\varphi_i(t)}\end{aligned} \tag{B7}$$

By (B6) and (B7), one has $\|\phi_i(t)\|^2 + 2\langle \phi_i(t), y_i(t) - x_0\rangle \leqslant -\frac{1}{2}(2\epsilon - \epsilon^2)\frac{\xi_i(t)^2}{\varphi_i(t)}$ if $\varphi_i(t) \neq 0$.

Define

$$\Theta_i(t) = \begin{cases} \dfrac{\xi_i(t)^2}{\varphi_i(t)}, & if \quad \varphi_i(t) \neq 0, \\ 0, & if \quad \varphi_i(t) = 0. \end{cases} \tag{B8}$$

Note that $\varphi_i(t) \geqslant 0$, it knows $\Theta_i(t) \geqslant 0$. Combining Case 1 and Case 2, by (B5), one has

$$\Delta V(t) \leqslant -\frac{1}{2}(2\epsilon - \epsilon^2)\sum_{i=1}^{n} w_i\Theta_i(t). \tag{B9}$$

By (B8), we know $\Theta_i(t) \geqslant 0$ for any $i \in \mathcal{V}$. By the fact $0 < \epsilon < 2$ and $w_i > 0$, hence, $\Delta V(t) \leqslant 0$. Moreover, $V(t)$ is lower bounded by zero, thus, $V(t)$ converges and $V(\infty)$ exists. By the fact $\Delta V(t) \leqslant 0$, we have $V(t) \leqslant V(0)$ for any $t \geqslant 0$, which implies that $\|x_i(t)\|$ is bounded for any $i \in \mathcal{V}$. By (B9), we have

$$\begin{aligned}\sum_{t=0}^{\infty}\sum_{i=1}^{n} w_i\Theta_i(t) &\leqslant \frac{-2}{2\epsilon - \epsilon^2}\sum_{t=0}^{\infty}\Delta V(t) \\ &= \frac{2}{2\epsilon - \epsilon^2}(V(0) - V(\infty)) \\ &< \infty.\end{aligned}$$

Thus, $\sum_{t=0}^{\infty}\Theta_i(t) < \infty$, which implies $\lim_{t\to\infty}\Theta_i(t) = 0$. By (B8), we have $\lim_{t\to\infty}\varphi_i(t) = 0$ or $\lim_{t\to\infty}\frac{\xi_i(t)^2}{\varphi_i(t)} = 0$. Furthermore, by the boundedness of $\|x_i(t)\|$, $\forall\ i \in \mathcal{V}$, we know that $\|y_i(t)\|$ is bounded, which implies that $\|\varphi_i(t)\|$ is bounded, i.e., there exists a positive constant δ such that $\sup_{t\geqslant 0}\|\varphi_i(t)\| \leqslant \delta$.

On one hand, if $\lim_{t\to\infty}\varphi_i(t) = 0$, by the definition of $\varphi_i(t)$, we have $\lim_{t\to\infty}\|y_i(t) - P_{X_i}(y_i(t))\| = 0$ and $\lim_{t\to\infty} g_i^+(y_i(t)) = 0$;

On the other hand, if $\lim_{t\to\infty}\frac{\xi_i(t)^2}{\varphi_i(t)} = 0$, together with the fact that $0 \leqslant \frac{1}{\delta}\xi_i(t)^2 \leqslant \frac{\xi_i(t)^2}{\varphi_i(t)}$, we have $\lim_{t\to\infty}\xi_i(t) = 0$. Then, by the definition of $\xi_i(t)$, it follows that $\lim_{t\to\infty}\left\|y_i(t) - P_{X_i}(y_i(t))\right\| = 0$ and $\lim_{t\to\infty} g_i^+(y_i(t)) = 0$.

Thus, $\lim_{t\to\infty}\left\|y_i(t) - P_{X_i}(y_i(t))\right\| = 0$ and $\lim_{t\to\infty} g_i^+(y_i(t)) = 0$ hold for any $i \in \mathcal{V}$.

Part 2. Now we use $\frac{1}{\sqrt{w^Tw}}w$ to form a set of orthonormal basis on $\in \mathbb{C}^n$, denoted by $\frac{1}{\sqrt{w^Tw}}w, p_2, \cdots, p_n$. We define $P = (\frac{1}{\sqrt{w^Tw}}w, p_2, \cdots, p_n)$. It is obvious that P is a unitary matrix, so $I_n - hL$ can be written as

$$P^T(I_n - hL)P = I_n - hP^TLP = \begin{bmatrix} 1 & 0 \\ * & I_{n-1} - hL_1 \end{bmatrix}.$$

Denote a graph $\bar{\mathcal{G}} = (\mathcal{V}, \mathcal{E}, I_n - hL)$, since $\mathcal{G}(\mathcal{A})$ is strongly connected, we know that $\bar{\mathcal{G}}$ is strongly connected. Due to $0 < h < \frac{1}{\max\limits_{1\leqslant i\leqslant n}\left(\sum_{j=1}^{n} a_{ij}\right)}$, we know that $I_n - hL$ is a row stochastic matrix. By Corollary 3.5 in [2], we have $\max\limits_{2\leqslant i\leqslant n}|1 - h\lambda_i(L)| < 1$, which implies $|\lambda_i((I_{n-1} - hL_1)\otimes I_m))| < 1$ for any $i = 1, \cdots, n-1$. Denote $\tilde{e}(t) = P^Te(t)$ and partition $\tilde{e}(t)$ into two parts , i.e., $\tilde{e}(t) = [\tilde{e}_1^T(t), \tilde{e}_2^T(t)]^T$, then, by (B2), we have

$$\begin{cases} \tilde{e}_1(t+1) = \tilde{e}_1(t) + \left(\left(\dfrac{1}{\sqrt{w^Tw}}w^T - \dfrac{\frac{1}{\sqrt{w^Tw}}w^T\mathbf{1}_nw^T}{\mathbf{1}_n^Tw}\right)\otimes I_m\right)\phi(t) = \tilde{e}_1(t) \\ \tilde{e}_2(t+1) = ((I_{n-1} - hL_1)\otimes I_m)\tilde{e}_2(t) + \begin{bmatrix} (p_2^T - \frac{1}{n}p_2^T\mathbf{1}_n\mathbf{1}_n^T)\otimes I_m \\ \vdots \\ (p_n^T - \frac{1}{n}p_n^T\mathbf{1}_n\mathbf{1}_n^T)\otimes I_m. \end{bmatrix}\phi(t) \end{cases} \tag{B10}$$

For (B10), note that $\tilde{e}_1(0) = \frac{1}{\sqrt{w^Tw}}(w^T\otimes I_m)e(0) = \frac{1}{\sqrt{w^Tw}}(w^T\otimes I_m)\left(\left(I_n - \frac{\mathbf{1}_nw^T}{\mathbf{1}_n^Tw}\right)\otimes I_m\right)x(0) = 0$, then $\tilde{e}_1(t) = 0$ for any $t \geqslant 0$. Moreover, the result in Part 1 implies $\lim_{t\to\infty}\|\phi(t)\| = 0$, by Lemma 4 and $|\lambda_i((I_{n-1} - hL_1)\otimes I_m))| < 1$, we have $\lim_{t\to\infty}\|\tilde{e}_2(t)\| = 0$. Together with $\tilde{e}_1(t) = 0$, we have $\lim_{t\to\infty} e(t) = 0$. Then, MAS (2) with (5) reaches consensus asymptotically, i.e., $\lim_{t\to\infty}\|x_i(t) - x_j(t)\| = x^*$ for any $i, j \in \mathcal{V}$. Furthermore, from the expression of $y_i(t)$, it can be obtained that $\lim_{t\to\infty}\|x_i(t) - y_i(t)\| = 0$ for any $i, j \in \mathcal{V}$. Thus, $\lim_{t\to\infty}\|x_i(t) - x^*\| = 0$ for some $x^* \in \mathbf{X}^*$. This completes the proof.

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