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Mean-variance portfolio selection with discontinuous prices and random horizon in an incomplete market

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Dear editor,
The study of mean-variance portfolio selection problems began with a single-period setting by Markowitz [1]. Then, inspired by Li and Ng [2], Zhou [3] first introduced a linearly constrained linear quadratic (LQ) control approach to solve this type of problem with deterministic market parameters. Along this line, Lim [4] generalized the results in [3] to a case with random parameters in an incomplete market.

However, the mathematical models in the studies above are far from satisfactory. On the one hand, asset prices are usually not continuous. In this regard, Lim [5] assumed that Brownian motion and doubly Poisson processes drive asset prices. On the other hand, as Blanchet-Scalliet et al. [6] pointed out, investments usually exist for an uncertain amount time, but all studies above assume that the exit time is deterministic. To fill this gap, Yu [7] and Lv et al. [8] discussed mean-variance models with continuous asset prices and random horizon in complete and incomplete markets, respectively. Motivated by the above literature, we study a mean-variance problem in which asset prices have jumps and the exit time is random.

Formulation. Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space and $T > 0$. Suppose that $W(\cdot)$ is a d-dimensional Brownian motion and $\mathbb{F} = \{\mathcal{F}_t\}$ with $\mathcal{F}_T \subset \mathcal{A}$ is generated by $W(\cdot)$. Let $N(\cdot) = (N_1(\cdot), \dots, N_n(\cdot))'$, where $N_i(\cdot)$ is a doubly stochastic Poisson process with an $\mathbb{F}$-predictable non-negative intensity $\lambda_i(\cdot)$, and $\mathbb{D} = \{\mathcal{D}_t\}$ with $\mathcal{D}_T \subset \mathcal{A}$ is generated by $N(\cdot)$. Define the filtration as $\mathbb{G} = \{\mathcal{G}_t\}$, where $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t$. Define the set of $\mathbb{G}$-predictable square integrable processes as $\mathcal{P}^2(\mathbb{G}, \mathbb{R}^m)$, the set of $\mathbb{G}$-adapted continuous processes ϕ_t such that $\mathbb{E}^{\mathbf{P}}[\sup_{0\leqslant t\leqslant T} |\phi_t|^2] < +\infty$ as $\mathcal{S}^2(\mathbb{G}, \mathbb{R}^m)$, and the set of $\mathbb{G}$-adapted $\mathbf{P}$-essentially uniformly bounded processes as $\mathcal{L}^\infty(\mathbb{G}, \mathbb{R}^m)$.

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Consider one bond and m risky assets. The price of bond $P_0(t)$ satisfies

$$\begin{cases} \mathrm{d}P_0(t) = r(t)P_0(t)\mathrm{d}t, \quad 0 \leqslant t \leqslant T, \\ P_0(0) = p_0 > 0, \end{cases} \tag{1}$$

where $r(\cdot)$ is the risk-free interest rate. The price process of the i-th risky asset $P_i(t)$ satisfies

$$\begin{cases} \mathrm{d}P_i(t) = P_i(t)\mu_i(t)\mathrm{d}t + P_i(t)\sigma_i(t)\mathrm{d}W(t) \\ \qquad\qquad + P_i(t)\theta_i(t)\mathrm{d}N(t), \quad 0 \leqslant t \leqslant T, \\ P_i(0) = p_i > 0, \end{cases} \tag{2}$$

where $\mu_i(\cdot)$ and $\sigma_i(\cdot) = (\sigma_{i1}(\cdot), \dots, \sigma_{id}(\cdot))$ are the risky interest rate and volatility of the i-th asset, and the j-th component of $\theta_i(\cdot) = (\theta_{i1}(\cdot), \dots, \theta_{in}(\cdot))$ represents the relative size of the jump to $P_i(\cdot)$ given an arrival of $N_j(\cdot)$.

Assumption 1. (i) $r(\cdot)$, $\mu_i(\cdot)$, $\sigma_{ik}(\cdot)$, $\theta_{ij}(\cdot)$ and $\lambda_j(\cdot)$ are uniformly bounded processes;

(ii) A $\delta > 0$ exists such that for any $t \in [0, T]$, $\lambda_i(t) \geqslant \delta$, $\mathbf{P}$-almost surely;

(iii) A $\delta > 0$ exists such that $\Sigma(t) = \sigma(t)\sigma(t)' + \theta(t)D(t)\theta(t)' \geqslant \delta I$, $\forall t \in [0, T]$, $\mathbf{P}$-almost surely.

Thus, the compensated Poisson process $M_j(\cdot) = N_j(\cdot) - \int_0^{\cdot} \lambda_j(s)\mathrm{d}s$ is a $\mathbb{G}$-martingale, $j = 1, \ldots, n$. Let $M(\cdot) = (M_1(\cdot), \ldots, M_n(\cdot))'$. Then, we can rewrite (2) as

$$\begin{aligned}\mathrm{d}P_i(t) = P_i(t)[\mu_i(t) + \theta_i(t)\lambda(t)]\mathrm{d}t \\ + P_i(t)\sigma_i(t)\mathrm{d}W(t) + P_i(t)\theta_i(t)\mathrm{d}M(t).\end{aligned} \tag{3}$$

Now, assume an agent invests $\pi_i(t)$ of his/her wealth $x(t)$ in the i-th risky asset at time t, $i = 1, \ldots, m$. Let $\pi(\cdot) = (\pi_1(\cdot), \ldots, \pi_m(\cdot))'$. Then, the agent's wealth process $x(t)$ satisfies

$$\begin{cases} \mathrm{d}x(t) = [r(t)x(t) + \pi(t)'b(t)]\mathrm{d}t \\ \qquad + \pi(t)'\sigma(t)\mathrm{d}W(t) + \pi(t)'\theta(t)\mathrm{d}M(t), \\ x(0) = x_0, \end{cases} \tag{4}$$

where $b_i(\cdot) = \mu_i(\cdot) + \theta_i(\cdot)\lambda(\cdot) - r(\cdot)$ and $b(\cdot) = (b_1(\cdot), \ldots, b_m(\cdot))'$. We call $\pi(\cdot)$ an admissible portfolio if $\pi(\cdot) \in \mathcal{U} := \mathcal{P}^2(\mathbb{G}, \mathbb{R}^m)$.

Suppose that the exit time τ is a positive random variable measurable with respect to $\mathcal{A}$. Following [6], we introduce the conditional distribution of exit time $F(t) = \mathbf{P}(\tau \leqslant t|\mathcal{G}_t)$.

Assumption 2. $F(\cdot)$ is absolutely continuous with respect to Lebesgue's measure, with an $\mathbb{F}$-predictable bounded non-negative density $a(\cdot)$, i.e., $F(t) = \int_0^t a(s)\mathrm{d}s$, $t \in [0, T]$.

Assumption 3. An $\varepsilon > 0$ exists such that $F(T) \leqslant 1 - \varepsilon$, $\mathbf{P}$-almost surely.

Then, we can formulate the mean-variance problem under consideration as follows:

$$\min J_{\mathrm{MV}}(\pi(\cdot)) = \mathbb{E}^{\mathbf{P}}\bigg[\int_0^T a(s)(x(s) - z)^2\mathrm{d}s + (1 - F(T))(x(T) - z)^2\bigg],$$

$$\text{s.t.} \begin{cases} J_1(\pi(\cdot)) = \mathbb{E}^{\mathbf{P}}\bigg[\int_0^T a(s)x(s)\mathrm{d}s + (1 - F(T))x(T)\bigg] = z, \\ (x(\cdot), \pi(\cdot)) \text{ admissible.} \end{cases}$$

We denote this as Problem (MV). Moreover, Problem (MV) is feasible if there exists an admissible portfolio $\pi(\cdot) \in \mathcal{U}$ such that $J_1(\pi(\cdot)) = z$. An admissible portfolio $\pi^*(\cdot)$ is an optimal or efficient portfolio for Problem (MV) if it achieves the infimum of $J_{\mathrm{MV}}(\pi(\cdot))$ and $J_{\mathrm{MV}}(\pi^*(\cdot))$ is finite.

Feasibility. To solve Problem (MV), we should first ensure that it is feasible.

Proposition 1. Let $(\psi, \xi) \in \mathcal{S}^2(\mathbb{F}, \mathbb{R}) \times \mathcal{P}^2(\mathbb{F}, \mathbb{R}^d)$ be the unique solution to the backward stochastic differential equation (BSDE):

$$\begin{cases} -\mathrm{d}\psi(t) = [r(t)\psi(t) + a(t)]\mathrm{d}t - \xi(t)'\mathrm{d}W(t), \\ \psi(T) = 1 - F(T). \end{cases}$$

Then, for any $z \in \mathbb{R}$, Problem (MV) is feasible iff

$$\mathbb{E}^{\mathbf{P}}\left[\int_0^T |\psi(t)b(t)' + \xi(t)'\sigma(t)'|^2\mathrm{d}t\right] > 0. \tag{5}$$

We apply the Lagrange multiplier technique to solve Problem (MV). For any $\lambda \in \mathbb{R}$, we introduce $J(\pi(\cdot), \lambda) = J_{MV}(\pi(\cdot)) + 2\lambda(J_1(\pi(\cdot)) - z) = \mathbb{E}^{\mathbf{P}}[\int_0^T a(s)[x(s) + (\lambda - z)]^2\mathrm{d}s + (1 - F(T))[x(T) + (\lambda - z)]^2] - \lambda^2$, and consider the following unconstrained problem parameterized by the Lagrange multiplier $\lambda \in \mathbb{R}$:

$$\text{Problem (OP)} \begin{cases} \min J(\pi(\cdot), \lambda), \\ \text{s.t. } (x(\cdot), \pi(\cdot)) \text{ admissible.} \end{cases}$$

Wellposedness of related BSDEs. We introduce the following stochastic Riccati equation (SRE):

$$\begin{cases} \mathrm{d}p = \bigg[-2a - 2rp + \Big(b + \dfrac{\sigma\Lambda}{p}\Big)'\Sigma^{-1}\Big(b + \dfrac{\sigma\Lambda}{p}\Big)p\bigg]\mathrm{d}t + \Lambda'\mathrm{d}W, \\ p(T) = 2(1 - F(T)), \\ p(t) > 0, \quad t \in [0, T], \end{cases} \tag{6}$$

and the auxiliary BSDE:

$$\begin{cases} \mathrm{d}h = \bigg[-\dfrac{2a}{p} + \Big(r + \dfrac{2a}{p}\Big)h + \Big(b + \dfrac{\sigma\Lambda}{p}\Big)'\Sigma^{-1}\sigma\eta - \dfrac{\Lambda'\eta}{p}\bigg]\mathrm{d}t + \eta'\mathrm{d}W, \\ h(T) = 1. \end{cases} \tag{7}$$

Now we give the wellposedness of (6) and (7) (Appendix A provides the proof of Theorem 1).

Theorem 1. Let Assumptions 1–3 hold. Then, the SRE (6) admits a unique solution $(p, \Lambda) \in \mathcal{L}^\infty(\mathbb{F}, \mathbb{R}) \times \mathcal{P}^2(\mathbb{F}, \mathbb{R}^d)$ satisfying $k \leqslant p \leqslant K$ for some constants $K > k > 0$, $\mathbf{P}$-almost surely.

Theorem 2. Let Assumptions 1–3 hold. Then, the BSDE (7) admits a unique solution $(h, \eta) \in \mathcal{S}^2(\mathbb{F}, \mathbb{R}) \times \mathcal{P}^2(\mathbb{F}, \mathbb{R}^d)$. Moreover, we have $0 < h(t) \leqslant 1$ for any $t \in [0, T]$. Furthermore, if $r(t) > 0$ almost everywhere in $[0, T]$, then $0 < h(t) < 1$ for any $t \in [0, T)$.

Solution to Problem (OP). We consider the following feedback portfolio:

$$\pi^\lambda(t) = -\Sigma^{-1}(t)\bigg[\Big(b(t) + \frac{\sigma(t)\Lambda(t)}{p(t)}\Big)\Big(x^\lambda(t_-) + (\lambda - z)h(t)\Big) + (\lambda - z)\sigma(t)\eta(t)\bigg], \tag{8}$$

where $x^\lambda(\cdot)$ is the solution of (4) under $\pi^\lambda(\cdot)$.

Theorem 3. Let Assumptions 1–3 hold. Then, $\pi^\lambda(\cdot)$ given by (8) is admissible and an optimal control for Problem (OP). Moreover, $\inf_{\pi(\cdot)\in\mathcal{U}} J(\pi(\cdot),\lambda) = J(\pi^\lambda(\cdot),\lambda) = (\frac{1}{2}p(0)h^2(0) + \Delta-1)(\lambda-z)^2+(p(0)h(0)x_0-2z)(\lambda-z)+\frac{1}{2}p(0)x_0^2-z^2$, where the constant $\Delta = \mathbb{E}^{\mathbf{P}}\int_0^T\{\frac{1}{2}p(\eta'\eta - \eta'\sigma'\Sigma^{-1}\sigma\eta) + a(h-1)^2\}\mathrm{d}s \geqslant 0$.

Efficient portfolios and efficient frontier. After solving the unconstrained Problem (OP), we can give the efficient frontier of Problem (MV).

Theorem 4. Let Assumptions 1–3 and the condition (5) hold. Then, $\frac{1}{2}p(0)h^2(0) + \Delta - 1 < 0$. Moreover, the following gives an efficient portfolio corresponding to z:

$$\begin{aligned}\pi^*(t) &= \pi^{\lambda^*}(t)\\ &= -\Sigma^{-1}(t)\Big[\Big(b(t)+\frac{\sigma(t)\Lambda(t)}{p(t)}\Big)\Big(x^*(t_-)\\ &\quad+(\lambda^*-z)h(t)\Big)+(\lambda^*-z)\sigma(t)\eta(t)\Big],\end{aligned} \tag{9}$$

where $\lambda^* - z = \frac{2z-p(0)h(0)x_0}{p(0)h^2(0)+2\Delta-2}$ and $x^*(\cdot) = x^{\lambda^*}(\cdot)$ is the corresponding wealth process satisfying (4) under (9). Furthermore,

$$\begin{aligned}&\mathrm{Var}[x^*(T\wedge\tau)]\\ &= \frac{p(0)h^2(0)+2\Delta}{2-2\Delta-p(0)h^2(0)}\Big(z-\frac{p(0)h(0)}{p(0)h^2(0)+2\Delta}x_0\Big)^2\\ &\quad+\frac{p(0)\Delta}{p(0)h^2(0)+2\Delta}x_0^2.\end{aligned}$$

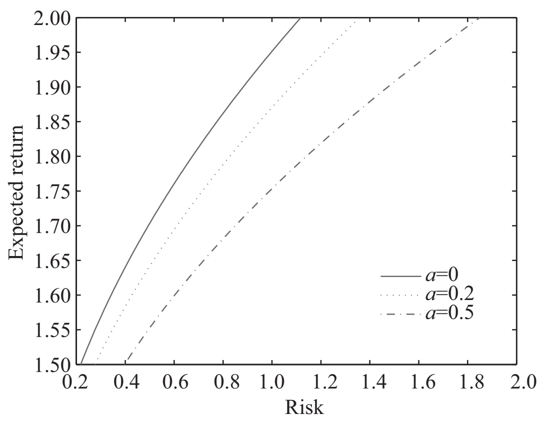

Figure 1 Efficient frontier.

Assume $r = 0.1$, $\mu = 0.3$, $\sigma = 0.5$, $\theta = 0.5$, and $T = 1$. Then, the efficient frontiers corresponding to the densities $a = 0, 0.2, 0.5$ are plotted in Figure 1, respectively.

Figure 1 shows that the larger the density of the random horizon is, the larger the minimal variance of terminal wealth is. This is reasonable because a larger density implies that more extra uncertainty owing to the random horizon is added to the model, resulting in the increment of the overall risk.

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Supporting information Appendix A. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

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