SCIENCE CHINA
Information Sciences

• **Supplementary File** •

Mean-variance portfolio selection with discontinuous prices and random horizon in an incomplete market

Haiyang WANG[1] & Zhen WU[2*]

[1] *School of Mathematics and Statistics, Shandong Normal University, Jinan* 250014*, China;*
[2] *School of Mathematics, Shandong University, Jinan* 250100*, China*

Appendix A Proof of Theorem 1

Theorem 1. Let Assumption 1–3 hold. Then the SRE (6) admits a unique solution $(p,\Lambda)\in\mathcal{L}^\infty(\mathbb{F},\mathbb{R})\times\mathcal{P}^2(\mathbb{F},\mathbb{R}^d)$ satisfying $k\leqslant p\leqslant K$ for some constants $K>k>0$, $\mathbf{P}$-a.s.

Proof. Consider the following simpler SRE:

$$\begin{cases} dp = \Big[(-2r+b'\Sigma^{-1}b)p+2b'\Sigma^{-1}\sigma\Lambda+\dfrac{\Lambda'\sigma'\Sigma^{-1}\sigma\Lambda}{p}\Big]dt \\ \qquad +\Lambda' dW, \\ p(T) = 2(1-F(T)), \\ p(t) > 0 \quad t\in[0,T]. \end{cases} \tag{A1}$$

It can be seen easily from Theorem 5.1 of [1] that (A1) admits a unique solution $(\bar p,\bar\Lambda)\in\mathcal{L}^\infty(\mathbb{F},\mathbb{R})\times\mathcal{P}^2(\mathbb{F},\mathbb{R}^d)$ satisfying $k\leqslant p\leqslant K$ for some constants $K>k>0$, $\mathbf{P}$-a.s. Thus $(\bar p,\bar\Lambda)$ is also a solution to the BSDE:

$$\begin{cases} dp = \Big[(-2r+b'\Sigma^{-1}b)p+2b'\Sigma^{-1}\sigma\Lambda+\dfrac{\Lambda'\sigma'\Sigma^{-1}\sigma\Lambda}{p\vee k}\Big]dt \\ \qquad +\Lambda' dW, \\ p(T) = 2(1-F(T)), \\ p(t) > 0. \end{cases} \tag{A2}$$

Furthermore, noting that the process

$$W^{\mathbf{Q}}(t)=W(t)+2\int_0^t \sigma(s)'\Sigma^{-1}(s)b(s)ds$$

is a standard Brownian motion under the probability $\mathbf{Q}$ defined by

$$\frac{d\mathbf{Q}}{d\mathbf{P}}=\exp\Big\{-2\int_0^T b(t)'\Sigma^{-1}(t)\sigma(t)dW(t)-\int_0^T |b(t)'\Sigma^{-1}(t)\sigma(t)|^2dt\Big\},$$

we can rewrite (A2) as follows:

$$\begin{cases} dp = \Big[(-2r+b'\Sigma^{-1}b)p+\dfrac{\Lambda'\sigma'\Sigma^{-1}\sigma\Lambda}{p\vee k}\Big]dt+\Lambda' dW^{\mathbf{Q}}, \\ p(T) = 2(1-F(T)), \\ p(t) > 0. \end{cases} \tag{A3}$$

On the other hand, let us consider the following quadratic BSDE truncated from the SRE (6):

$$\begin{cases} dp = \Big[-2f+(-2r+b'\Sigma^{-1}b)p+\dfrac{\Lambda'\sigma'\Sigma^{-1}\sigma\Lambda}{p\vee k}\Big]dt+\Lambda' dW^{\mathbf{Q}}, \\ p(T) = 2(1-F(T)). \end{cases} \tag{A4}$$

* Corresponding author (email: wuzhen@sdu.edu.cn)

Since r, μ, σ, θ and λ are all bounded, it follows from the existence and uniqueness results in [2] that the BSDE (A4) admits a unique bounded solution $(p^{(k)},\Lambda^{(k)})\in\mathcal{L}^\infty(\mathbb{F},\mathbb{R})\times\mathcal{P}^2(\mathbb{F},\mathbb{R}^d)$. Moreover, by Assumption 2 that $f(\cdot)\geqslant 0$ and comparison theorem also in [2], we have

$$p^{(k)}(t)\geqslant\bar{p}(t)\geqslant k,\quad \forall t\in[0,T].$$

So $(p,\Lambda):=(p^{(k)},\Lambda^{(k)})\in\mathcal{L}^\infty(\mathbb{F},\mathbb{R})\times\mathcal{P}^2(\mathbb{F},\mathbb{R}^d)$ is exactly a solution of SRE (6).

As for the uniqueness, suppose (p,Λ), $(\tilde{p},\tilde{\Lambda})$ are two different solutions to (6), satisfying $k<p<K$ and $\tilde{k}<\tilde{p}<\tilde{K}$ respectively for some $K>k>0$ and $\tilde{K}>\tilde{k}>0$. Denote $k^*=k\wedge\tilde{k}$. Then (p,Λ) and $(\tilde{p},\tilde{\Lambda})$ are both solutions to the following quadratic BSDE:

$$\begin{cases} dp=\Big[-2f+(-2r+b'\Sigma^{-1}b)p+\dfrac{\Lambda'\sigma'\Sigma^{-1}\sigma\Lambda}{p\vee k^*}\Big]dt+\Lambda'dW^{\mathbf{Q}},\\ p(T)=2(1-F(T)).\end{cases}\tag{A5}$$

This is a contradiction to the uniqueness of solution to (A5). Thus the proof is completed.

References

1 Lim A E B. Quadratic hedging and mean-variance portfolio selection with random parameters in an incomplete market. Math Oper Res, 2004, 29: 132–161

2 Kobylanski M. Backward stochastic differential equations and partial differential equations with quadratic growth. Ann Probab, 2000, 28: 558–602