SCIENCE CHINA
Information Sciences

• RESEARCH PAPER •
From CAS & CAE Members

July 2020, Vol. 63 172002:1–172002:21
https://doi.org/10.1007/s11432-020-2916-8

Observer-based multi-objective parametric design for spacecraft with super flexible netted antennas

Guang-Ren DUAN* & Tian-Yi ZHAO

Center for Control Theory and Guidance Technology, Harbin Institute of Technology, Harbin 150001*, China*

Received 9 March 2020/Accepted 28 April 2020/Published online 10 June 2020

Abstract A multi-objective parametric design method that based on the robust observer is proposed for the attitude control of satellites with super flexible netted antennas. First, a parametric observer-based controller is obtained based on the eigen-structure assignment theory. The closed-loop poles are assigned to desired positions or regions, and full degrees of freedom of the design, which are characterized by a set of parameters, are preserved under the proposed control law. Second, the obtained parameters are comprehensively optimized to make the closed-loop system have lower eigenvalue sensitivity, a smaller control gain, and stronger tolerance to high-order unmodeled dynamics and external disturbances. Finally, comparative simulations are carried out based on practical engineering parameters of a satellite in order to verify the effect of the proposed method, and also to show their superiority over the traditional proportional-integral-derivative (PID) controller with filters and the traditional dynamic compensators.

Keywords parametric design of control systems, multi-objective designs, robust observers, super flexible spacecraft, parameter perturbations, high-order flexible modes

Citation Duan G-R, Zhao T-Y. Observer-based multi-objective parametric design for spacecraft with super flexible netted antennas. Sci China Inf Sci, 2020, 63(7): 172002, https://doi.org/10.1007/s11432-020-2916-8

1 Introduction

Flexible structures (such as solar panels and satellite antennas) are widely common in modern spacecraft. The flexible structure may cause elastic vibration of the attitude of the spacecraft, thereby they impair the accuracy of the attitude control and even destroy the stability of the closed-loop system. For example, in 1958 the satellite named "Explorer-1" caused the energy dissipation in the system owing to the flexible vibration of the four whip antennas that eventually led to the attitude roll [1]. Therefore, vibration suppression and attitude control of spacecraft with large flexible attachments is a critical problem and it has received lots of attention [2–21]. Sliding mode control (SMC) and backstepping control are commonly used control methods in order to solve this problem. As representative results, Refs. [2, 5] investigated the problem of attitude tracking control of flexible spacecraft, and developed a fault-tolerant control approach based on SMC and an adaptive backstepping SMC scheme, respectively. In [3], an approach was presented in order to reduce vibration of flexible spacecraft during attitude maneuver. In [4], a robust control algorithm was proposed for stabilization of a three-axis stabilized flexible spacecraft in the presence of parametric uncertainty, external disturbances and control input nonlinearity/dead-zone. For more up-to-date results about applications of SMC and backstepping methods in flexible satellite attitude control, readers can refer to [6–8]. Robust H_∞ control is another effective control method for flexible spacecraft control. In [9, 10], robust H_∞ controllers are designed for the multi-objective attitude control

* Corresponding author (email: g.r.duan@hit.edu.cn)

Table 1 Symbols

Symbol	Meaning
$\mathrm{diag}\,(s_1, s_2, \ldots, s_n)$	Diagonal matrix with $s_1, s_2, \ldots, s_n$ as diagonal elements
$\lambda_i\,(M)$	The i-th eigenvalue of a matrix M
$\mathrm{trace}\,(M)$	Sum of diagonal elements of a matrix M
$\mathrm{Blockdiag}\,(M_1, M_2, \ldots, M_n)$	Block diagonal matrix with $M_1, M_2, \ldots, M_n$ as diagonal elements
$\mathrm{vec}([\,\eta_1\ \eta_2\ \cdots\ \eta_n\,])$	$[\,\eta_1^{\mathrm{T}}\ \eta_2^{\mathrm{T}}\ \cdots\ \eta_n^{\mathrm{T}}\,]^{\mathrm{T}}$
$\mathrm{unvec}([\,\eta_1^{\mathrm{T}}\ \eta_2^{\mathrm{T}}\ \cdots\ \eta_n^{\mathrm{T}}\,]^{\mathrm{T}})$	$[\,\eta_1\ \eta_2\ \cdots\ \eta_n\,]$
$A \otimes B$	Kronecker product of A and B

problem of a flexible spacecraft in the presence of disturbances, parameter uncertainties and actuator saturation based on LMI approach. The problems of robust H_∞ controller design with input constraints are considered in [11,12]. In addition to SMC, backstepping, and robust H_∞ control, many other control methods are also applied to the vibration suppression and attitude control of flexible satellites, such as disturbance observer-based control [13–15], adaptive control [16–20], and finite-time control [16,18,21].

Many of the current control methods are proposed for a single control target (e.g., [2–8,11–21]) to provide flexible spacecraft control. However, in actual engineering, we need to consider the accuracy, rapidity, smoothness, the influences of parameter perturbations, and high-order unmodeled dynamics at the same time. Therefore, the control of flexible spacecraft is a typical multi-objective design problem. Although there are a few multi-objective design methods available, most of them have strict requirements on the form of indices and parameter constraints (such as the LMI method in [9,10]), which greatly limits the scope of application of the method. According to this consideration, Ref. [22] proposed a dynamic compensator-based robust parametric multi-objective design method. A multi-objective comprehensive optimization of the free parameter vectors is performed in [22], which makes the control system with better robustness, stronger disturbance suppression ability and a smaller control gain that is based on the complete parametric form of the dynamic compensator.

The parametric control system design is a powerful tool to solve multi-objective design problems [23–30]. Different from the traditional control methods, the parametric method first establishes a fully parametric representation of the control law, and then comprehensively optimizes the free parameters in the control law to achieve the multi-objective design requirements of the control system, such as minimum eigenvalue sensitivity design [26,30], disturbance attenuating or decoupling design [25,27], and gain scheduling design [28,29]. Therefore, when it is compared with the other conventional multi-objective design methods (such as LMI method), the parametric design method is more convenient, and it has a wider range of applications.

Inspired by the idea of parametric control system design in [22], in this paper, a new parametric multi-objective design method based on the robust observer for large flexible spacecraft attitude control systems is proposed. Different from the dynamic compensator method in [22], the proposed method allows us to design the observer and controller separately. This reduces the difficulty of the design and it provides the application of the method to practical systems. According to the control law that is used for system design in this paper, the closed-loop system has the following characteristics: (1) closed-loop poles which are assigned to the desired positions or regions; (2) lower eigenvalue sensitivities; (3) stronger tolerance to high-order unmodeled dynamics; (4) a smaller control gain. The simulation result proves that the proposed controller outperforms the traditional proportional-integral-derivative (PID) controller and dynamic compensators [22] in dynamic response, tolerance to high-order unmodeled dynamics, and the peak of control torques.

The remainder of this paper is organized as follows. In Section 2, the model of the system is given and the statement of the problem to be solved is presented. In Section 3, complete parameterized expressions of the observer-based controller are obtained. Then, the free parameters obtained in Section 3 are comprehensively optimized in Section 4 to realize the multi-objective design. Finally, comparative simulations are carried out in Section 5 that is based on practical engineering parameters, and after this there is a brief conclusion. Symbols that are used in this paper are shown in Table 1.

2 Problem formulation

2.1 The model

This subsection briefly reviews the dynamic model of a spacecraft with a large flexible antenna, that is obtained in [22]. According to [22], the attitude system can be separated into three subsystems, namely a pitch subsystem, a roll subsystem, and a yaw subsystem. Because the three subsystems are approximately independent of each other, it is reasonable to design the control system separately. The models of the three subsystems are exactly the same, except for the specific values of the parameters. Therefore, only the control system design of the pitch attitude subsystem is dealt in this paper.

The pitch channel attitude dynamics model of the flexible spacecraft to be investigated is described by

$$I_y\ddot{\theta} + b_y\ddot{q}_y = u, \tag{1}$$

where θ is the pitch angle, u is the control torque in the direction of the pitch axis generated by the thruster and the momentum wheel, q_y is the flexible modal component associated with the pitch axis, I_y is the moment of inertia in the pitch direction, and b_y is the mode coefficient.

Strictly speaking, the dynamics of the flexural modality should be described by a distributed parameter system. Here we rationally use the following second-order lumped parameter model for approximate description:

$$\ddot{q}_y + 2\xi\Lambda_y\dot{q}_y + \Lambda_y^2 q_y + b_y\ddot{\theta} = d, \tag{2}$$

where d is the dynamic term of higher order mode, ξ is the damping coefficient, and Λ_y is the modal frequency.

Let

$$x = \begin{bmatrix} \theta & q_y & \dot{\theta} & \dot{q}_y \end{bmatrix}^{\mathrm{T}},$$

and then, Eqs. (1) and (2) can be transformed into the following state-space model:

$$\dot{x} = Ax + Bu + Dd, \tag{3}$$

where

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -b_y a_1 & 0 & -b_y a_2 \\ 0 & I_y a_1 & 0 & I_y a_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ -\gamma \\ \gamma b_y \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \\ \gamma b_y \\ -\gamma I_y \end{bmatrix} \tag{4}$$

with γ, a_1 and a_2 being given by

$$\gamma = -\left(I_y - b_y\right)^{-1}, \quad a_1 = \gamma\Lambda_y^2, \quad a_2 = 2\gamma\xi\Lambda_y. \tag{5}$$

It is known from the physical background that θ and $\dot{\theta}$ can be directly measured, which leads to the following output equation:

$$y = Cx, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{6}$$

Suppose that the perturbations only exist in the modal frequency and damping coefficients [22]. It follows from (5) that only a_1 and a_2 have perturbations. Thus we can define

$$a_1 = a_{10} + \Delta a_1, \quad a_2 = a_{20} + \Delta a_2, \tag{7}$$

where a_{10} and a_{20} represent the nominal parameters, and Δa_1 and Δa_2 are the perturbations of a_1 and a_2, respectively. Then, the matrix A given in (4) can be rewritten in the form of

$$A = A_0 + A_1\Delta a_1 + A_2\Delta a_2, \tag{8}$$

where

$$A_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -b_y a_{10} & 0 & -b_y a_{20} \\ 0 & I_y a_{10} & 0 & I_y a_{20} \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -b_y & 0 & 0 \\ 0 & I_y & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b_y \\ 0 & 0 & 0 & I_y \end{bmatrix}. \tag{9}$$

2.2 Statement of the problem

In this paper, an observer-based state feedback control method is applied to the system (3)–(9). The specific control law to be designed takes the form of

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu - L\left(y - C\hat{x}\right), \\ u = K\hat{x}, \end{cases} \tag{10}$$

where $\hat{x} \in \mathbb{R}^4$ is the state variable of the observer, and $K \in \mathbb{R}^{1\times 4}$ and $L \in \mathbb{R}^{4\times 2}$ are the gain matrices to be determined. Letting

$$z = \begin{bmatrix} x^{\mathrm{T}} & \hat{x}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}},$$

we can obtain the closed-loop system as

$$\begin{cases} \dot{z} = A_z z + D_z d, \\ y = C_z z, \end{cases} \tag{11}$$

where

$$A_z = \begin{bmatrix} A & BK \\ -LC & A + BK + LC \end{bmatrix}, \quad C_z = \begin{bmatrix} C & 0 \end{bmatrix}, \quad D_z = \begin{bmatrix} D \\ 0 \end{bmatrix}. \tag{12}$$

With the above preparations, the problem to be solved in this paper can be stated as follows.

Problem MOD. For given system (3)–(9), find gain matrices K and L in the observer-based control law (10), such that

(1) The poles of the closed-loop system (11), that is, the eigenvalues of A_z, are assigned to the desired positions or regions, and are as insensitive as possible to parameter perturbations Δa_1 and Δa_2;

(2) The effect of high-order unmodeled modality d on the output y is as small as possible;

(3) The F-norm of the gain matrix K is, at the same time, as small as possible.

3 Parametric design for observer-based controller

Problem MOD can be solved in two steps. The first step is the parameterization of the control law, that is, to establish a complete parametric form of the proposed control law. The second step is parameter optimization, that is, to optimize the obtained parameters to meet the proposed multi-objective design requirements. This section treats the first step, that is, to find complete parametric forms of the gain matrices K and L in the control law (10), such that the closed-loop poles, that is, the eigenvalues of A_z, are assigned to the desired positions or regions.

As we know from the well-known separation principle for observer-based control system design, the poles of the closed-loop system (11) are composed of

$$A_c = A + BK \ \text{ and } \ A_o = A + LC. \tag{13}$$

This allows us to investigate the nondefective eigenstructure assignment in A_c and A_o, respectively. The eigenstructure assignment result in [23] performs an important role in solving this problem.

3.1 Nondefective eigenstructure assignment for $A + BK$

Let us first give a result in [22].

Lemma 1. The system (3)–(6) is controllable (or observable) if and only if

$$a_1 \neq 0, \quad b_y \neq 0. \tag{14}$$

With the help of Theorem 2 in [23] and Lemma 1, a parametric expression of the gain matrix K can be obtained immediately. Before giving the result, let us introduce the following notations:

$$\zeta_1^c(j) = -a_2 b_y^2 \alpha_j - a_1 b_y^2 - \alpha_j^2 + I_y a_2 \alpha_j + I_y a_1, \tag{15}$$

$$\zeta_2^c = \gamma\alpha_2(a_2 b_y^2 + 2\alpha_1 - I_y a_2), \tag{16}$$

$$\zeta_3^c = f_1\alpha_1 - f_2\alpha_2, \quad \zeta_4^c = f_1\alpha_2 + f_2\alpha_1, \tag{17}$$

$$\zeta_5^c(j,k) = f_j\alpha_1^3 + (-1)^{k+1}\cdot 3f_k\alpha_1^2\alpha_2 - 3f_j\alpha_1\alpha_2^2 + (-1)^k \cdot f_k\alpha_2^3, \tag{18}$$

$$\zeta_6^c(j,k) = -b_y f_j \gamma(\alpha_1^2 - \alpha_2^2) + (-1)^k \cdot 2b_y f_k \gamma\alpha_1\alpha_2, \tag{19}$$

$$\zeta_7^c = f_1(\alpha_1^2 - \alpha_2^2) - 2f_2\alpha_1\alpha_2, \quad \zeta_8^c = f_2(\alpha_1^2 - \alpha_2^2) + 2f_1\alpha_1\alpha_2, \tag{20}$$

$$\zeta_9^c(j) = -\alpha_j^2 + I_y a_2\alpha_j + I_y a_1, \quad \zeta_{10}^c = 2\alpha_1\alpha_2 - I_y a_2\alpha_2, \tag{21}$$

where $f_i, \alpha_i \in \mathbb{R},\ i = 1, 2, 3, 4$, are parameters to be optimized later. With the above preparations, the result can be stated as follows.

Theorem 1. Suppose that the system (3)–(6) satisfies condition (14). Let $s_{1,2} = \alpha_1 \pm \alpha_2\mathrm{i},\ \alpha_1 < 0,\ \alpha_2 \neq 0$, and $s_i = \alpha_i < 0,\ i = 3, 4$. Then, all the matrix K which makes $s_i,\ i = 1, 2, 3, 4$, be the eigenvalues of A_c is given by

$$\begin{cases} K = W_0 V_0^{-1}, \\ V_0 = [\, v_1\ \ v_2\ \ v_3\ \ v_4 \,], \\ W_0 = [\, w_1\ \ w_2\ \ w_3\ \ w_4 \,], \end{cases} \tag{22}$$

and the corresponding eigenvector matrix V is given by

$$V = \left[\, v_1 + v_2\mathrm{i}\ \ v_1 - v_2\mathrm{i}\ \ v_3\ \ v_4 \,\right], \tag{23}$$

where

$$v_1 = \begin{bmatrix} -f_1\gamma(\zeta_1^c(1) + \alpha_2^2) - f_2\zeta_2^c \\ \zeta_6^c(1,2) \\ -\gamma\zeta_3^c(\zeta_1^c(1) + \alpha_2^2) - \zeta_4^c\zeta_2^c \\ -b_y\gamma\zeta_5^c(1,2) \end{bmatrix}, \quad v_3 = \begin{bmatrix} -f_3\gamma\zeta_1^c(3) \\ -b_y f_3\gamma\alpha_3^2 \\ -f_3\gamma\alpha_3\zeta_1^c(3) \\ -b_y f_3\gamma\alpha_3^3 \end{bmatrix}, \tag{24}$$

$$v_2 = \begin{bmatrix} -f_2\gamma(\zeta_1^c(1) + \alpha_2^2) + f_1\zeta_2^c \\ \zeta_6^c(2,1) \\ -\gamma\zeta_4^c(\zeta_1^c(1) + \alpha_2^2) + \zeta_3^c\zeta_2^c \\ -b_y\gamma\zeta_5^c(2,1) \end{bmatrix}, \quad v_4 = \begin{bmatrix} -f_4\gamma\zeta_1^c(4) \\ -b_y f_4\gamma\alpha_4^2 \\ -f_4\gamma\alpha_4\zeta_1^c(4) \\ -b_y f_4\gamma\alpha_4^3 \end{bmatrix}, \tag{25}$$

and

$$w_1 = \zeta_7^c(\zeta_9^c(1) + \alpha_2^2) + \zeta_{10}^c\zeta_8^c, \quad w_3 = f_3\alpha_3^2\zeta_9^c(3), \tag{26}$$

$$w_2 = \zeta_8^c(\zeta_9^c(1) + \alpha_2^2) - \zeta_{10}^c\zeta_7^c, \quad w_4 = f_4\alpha_4^2\zeta_9^c(3). \tag{27}$$

In Theorem 1, $\alpha_i,\ f_i \in \mathbb{R},\ i = 1, 2, 3, 4$ are parameters satisfying the following constraint.

Constraint C1. $\Delta_c \neq 0$, where

$$\Delta_c = -a_1^2 b_y^2 f_3 f_4 \gamma^2 \prod_{j,k=1,\ldots,4}^{j<k} (s_j - s_k)(f_1^2 + f_2^2). \tag{28}$$

For a proof of Theorem 1, please refer to Appendix A.

3.2 Nondefective eigenstructure assignment for $A + LC$

Similarly, based on Theorem 2 in [23] and Lemma 1, a parametric expression of the gain matrix L can also be obtained, which needs the following notations:

$$\zeta_1^o\,(j) = -\tilde{\alpha}_j^2 + I_y a_2 \tilde{\alpha}_j + I_y a_1, \quad \zeta_2^o\,(j) = g_{j2}(2\tilde{\alpha}_1\tilde{\alpha}_2 - I_y a_2 \tilde{\alpha}_2), \tag{29}$$

$$\zeta_3^o\,(j,k) = b_y g_{j2}(a_1 + a_2\tilde{\alpha}_1) + (-1)^j\, a_2 b_y g_{k2} \tilde{\alpha}_2, \tag{30}$$

$$\zeta_4^o = \tilde{\alpha}_2(\tilde{\alpha}_1^2 - \tilde{\alpha}_2^2) + 2\tilde{\alpha}_1^2\tilde{\alpha}_2 - I_y a_1 \tilde{\alpha}_2 - 2I_y a_2 \tilde{\alpha}_1\tilde{\alpha}_2, \tag{31}$$

$$\zeta_5^o = -\tilde{\alpha}_1(\tilde{\alpha}_1^2 - \tilde{\alpha}_2^2) + 2\tilde{\alpha}_1\tilde{\alpha}_2^2 + I_y a_1 \tilde{\alpha}_1 + I_y a_2(\tilde{\alpha}_1^2 - \tilde{\alpha}_2^2), \tag{32}$$

$$\zeta_6^o\,(j) = g_{j2}(-\tilde{\alpha}_j^3 + I_y a_2 \tilde{\alpha}_j^2 + I_y a_1 \tilde{\alpha}_j) - g_{j1}, \tag{33}$$

where $g_{ij}, \tilde{\alpha}_i \in \mathbb{R},\ i = 1,2,3,4,\ j = 1,2,$ are parameters to be optimized later.

Theorem 2. Suppose that the system (3)–(6) satisfies condition (14). Let $\tilde{s}_{1,2} = \tilde{\alpha}_1 \pm \tilde{\alpha}_2\mathrm{i},\ \tilde{\alpha}_1 < 0,\ \tilde{\alpha}_2 \neq 0$, and $\tilde{s}_i = \tilde{\alpha}_i < 0,\ i = 3,4$. Then, all the matrix L which makes $\tilde{s}_i,\ i = 1,2,3,4$, be the eigenvalues of A_o is given by

$$\begin{cases} L = T_0^{-\mathrm{T}} Z_0^{\mathrm{T}}, \\ T_0 = [\, t_1\ t_2\ t_3\ t_4 \,], \\ Z_0 = [\, z_1\ z_2\ z_3\ z_4 \,], \end{cases} \tag{34}$$

and the corresponding left eigenvector matrix T is given by

$$T = \left[\, t_1 + t_2\mathrm{i}\ \ t_1 - t_2\mathrm{i}\ \ t_3\ \ t_4 \,\right], \tag{35}$$

where

$$t_1 = \begin{bmatrix} g_{11} \\ a_1 b_y (g_{12}\tilde{\alpha}_1 - g_{22}\tilde{\alpha}_2) \\ g_{12}(\zeta_1^o\,(1) + \tilde{\alpha}_2^2) + \zeta_2^o\,(2) \\ \zeta_3^o\,(1,2) \end{bmatrix}, \quad t_3 = \begin{bmatrix} g_{31} \\ a_1 b_y g_{32}\tilde{\alpha}_3 \\ g_{32}\zeta_1^o\,(3) \\ b_y g_{32}(a_1 + a_2\tilde{\alpha}_3) \end{bmatrix}, \tag{36}$$

$$t_2 = \begin{bmatrix} g_{21} \\ a_1 b_y (g_{12}\tilde{\alpha}_2 + g_{22}\tilde{\alpha}_1) \\ g_{22}(\zeta_1^o\,(1) + \tilde{\alpha}_2^2) - \zeta_2^o\,(1) \\ \zeta_3^o\,(2,1) \end{bmatrix}, \quad t_4 = \begin{bmatrix} g_{41} \\ a_1 b_y g_{42}\tilde{\alpha}_4 \\ g_{42}\zeta_1^o\,(4) \\ b_y g_{42}(a_1 + a_2\tilde{\alpha}_4) \end{bmatrix}, \tag{37}$$

and

$$z_1 = \begin{bmatrix} g_{11}\tilde{\alpha}_1 - g_{21}\tilde{\alpha}_2 \\ g_{22}\zeta_4^o - g_{11} + g_{12}\zeta_5^o \end{bmatrix}, \quad z_3 = \begin{bmatrix} g_{31}\tilde{\alpha}_3 \\ \zeta_6^o\,(3) \end{bmatrix}, \tag{38}$$

$$z_2 = \begin{bmatrix} g_{11}\tilde{\alpha}_2 + g_{21}\tilde{\alpha}_1 \\ g_{22}\zeta_5^o - g_{21} - g_{12}\zeta_4^o \end{bmatrix}, \quad z_4 = \begin{bmatrix} g_{41}\tilde{\alpha}_4 \\ \zeta_6^o\,(4) \end{bmatrix}. \tag{39}$$

In Theorem 2, $g_{jk} \in \mathbb{R},\ j = 1,2,3,4,\ k = 1,2$ are parameters satisfying the following constraint.

Constraint C2. $\Delta_o \neq 0$, where

$$\begin{aligned} \Delta_o \;=\; & -a_1^2 b_y^2 \big(g_{31}g_{42}\left(g_{12}^2 + g_{22}^2\right)(\delta_{12} + \delta_{41} + \delta_{24}) \\ & + g_{32}g_{41}\left(g_{12}^2 + g_{22}^2\right)(\delta_{21} + \delta_{13} + \delta_{32}) \\ & + g_{32}g_{42}\left(g_{11}g_{12} + g_{21}g_{22}\right)(\delta_{31} + \delta_{14} + \delta_{23} + \delta_{42}) \\ & + \mathrm{i} g_{32}g_{42}\left(g_{11}g_{22} - g_{12}g_{21}\right)(\delta_{31} + \delta_{14} + \delta_{32} + \delta_{24} + 2\delta_{43}) \big), \end{aligned} \tag{40}$$

with $\delta_{jk}, j,k = 1,2,3,4$, being defined by

$$\delta_{jk} = \tilde{s}_j \tilde{s}_k \left(\tilde{s}_k - \tilde{s}_j\right). \tag{41}$$

For a proof of Theorem 2, please refer to Appendix B.

Remark 1. It is noted that the degrees of freedom are not only composed of the parameters $f_i, g_{jk} \in \mathbb{R}, j = 1,2,3,4, k = 1,2$, but also the desired eigenvalues $s_i, \tilde{s}_i, i = 1,2,3,4$. The desired eigenvalues can be set partially (or entirely) undetermined, and sought together with the other parameters to achieve additional design requirements in applications.

4 Multi-objective design

At the beginning of Section 3, we mentioned two steps to solve Problem MOD. This section further investigates the second step, that is, to optimize the obtained parameters to meet the proposed multi-objective design requirements.

4.1 Closed-loop eigenvalue sensitivities

Without loss of generality, we assume that

$$\lambda_i (A_z) = \lambda_i (A_c), \quad \lambda_{4+i} (A_z) = \lambda_i (A_o), \quad i = 1,2,3,4. \tag{42}$$

Then, according to the separation principle for robust pole assignment (see Theorem 3.1 in [31]), we only need to consider the closed-loop eigenvalue sensitivities of A_c and A_o instead of those of A_z. The well-known Hellman-Feynman theorem (see Lemma 1 in [26]) shows that in order to obtain the eigenvalue sensitivities of A_c and A_o, explicit expressions of V^{-1} and T^{-1} are needed. Before giving these expressions, let us introduce two sets of notations. The first set of symbols is defined based on s_i and $f_i, i = 1,2,3,4$:

$$\varepsilon_1 (j) = f_3 f_4 \left(f_1 + (-1)^j f_2 i \right), \quad \varepsilon_2 (j) = f_j \left(f_1^2 + f_2^2 \right), \tag{43}$$

$$F_1^c (j,k,l) = a_1 b_y^2 \gamma^2 s_j s_k s_l \prod_{m,n=j,k,l}^{m<n} (s_m - s_n), \tag{44}$$

$$\begin{aligned} F_2^c (j,k,l) = \; & b_y \gamma^2 \big[-\gamma^{-1} a_1^2 (s_k + s_l + s_j) - \gamma^{-1} a_1 a_2 (s_j s_k + s_j s_l + s_k s_l) \\ & + \left(a_1 - \gamma^{-1} a_2^2 \right) s_j s_k s_l \big] \prod_{m,n=j,k,l}^{m<n} (s_m - s_n), \end{aligned} \tag{45}$$

$$F_3^c (j,k,l) = b_y^2 \gamma^2 (a_1 s_j s_k + a_1 s_j s_l + a_1 s_k s_l + a_2 s_j s_k s_l) \prod_{m,n=j,k,l}^{m<n} (s_m - s_n), \tag{46}$$

$$F_4^c (j,k,l) = b_y \gamma^2 \big[-\gamma^{-1} a_1^2 + a_1 (s_j s_k + s_j s_l + s_k s_l) + a_2 s_j s_k s_l \big] \prod_{m,n=j,k,l}^{m<n} (s_m - s_n), \tag{47}$$

while the second set of symbols is defined based on $\tilde{s}_i$ and $g_{ij}, i = 1,2,3,4, j = 1,2$:

$$F_1^o (j) = a_1^2 b_y^2 g_{32} g_{42} (g_{12} - (-1)^j g_{22} \mathrm{i}) \prod_{k,l=j,3,4}^{k<l} (\tilde{s}_k - \tilde{s}_l), \tag{48}$$

$$F_2^o (j) = a_1^2 b_y^2 g_{j2} (g_{12}^2 + g_{22}^2) \prod_{k,l=1,2,j}^{k<l} (\tilde{s}_k - \tilde{s}_l), \tag{49}$$

$$\begin{aligned} F_3^o (j,k) = \; & (-1)^k b_y \big[(a_1 (\tilde{s}_j + \tilde{s}_4) + a_2 \tilde{s}_j \tilde{s}_4) g_{k2} g_{31} g_{42} (\tilde{s}_4 - \tilde{s}_j) \\ & + (a_1 (\tilde{s}_3 + \tilde{s}_j) + a_2 \tilde{s}_j \tilde{s}_3) g_{k2} g_{32} g_{41} (\tilde{s}_j - \tilde{s}_3) \\ & + (a_1 (\tilde{s}_4 + \tilde{s}_3) + a_2 \tilde{s}_3 \tilde{s}_4) g_{k1} g_{32} g_{42} (\tilde{s}_3 - \tilde{s}_4) \big], \end{aligned} \tag{50}$$

$$\begin{aligned} F_4^o(j) = b_y\big[& \left(g_{12}^2+g_{22}^2\right) g_{j1}\left(\tilde{s}_1-\tilde{s}_2\right)\left(a_1\left(\tilde{s}_1+\tilde{s}_2\right)+\tilde{s}_1 \tilde{s}_2 a_2\right) \\ & +\left(g_{11} g_{12}+g_{21} g_{22}\right) a_1 g_{j2}\left(\tilde{s}_2^2-\tilde{s}_1^2\right) \\ & +\mathrm{i}\left(g_{12} g_{21}-g_{11} g_{22}\right) a_1 g_{j2}\left(\tilde{s}_1^2+\tilde{s}_2^2\right) \\ & +2 \mathrm{i}\left(g_{11} g_{22}-g_{12} g_{21}\right) a_1 g_{j2} \tilde{s}_j^2 \\ & +a_2 g_{j2}\left(g_{11} g_{12}+g_{21} g_{22}\right)\left(\delta_{1j}+\delta_{j2}\right) \\ & +\mathrm{i} a_2 g_{j2}\left(g_{11} g_{22}-g_{12} g_{21}\right)\left(\delta_{1j}+\delta_{2j}\right)\big], \end{aligned} \tag{51}$$

$$\begin{aligned} F_5^o(j, k) = (-1)^k a_1^2 b_y^2\big[& g_{k2} g_{31} g_{42}\left(\tilde{s}_4-\tilde{s}_j\right) \\ & +g_{k2} g_{32} g_{41}\left(\tilde{s}_j-\tilde{s}_3\right)+g_{k1} g_{32} g_{42}\left(\tilde{s}_3-\tilde{s}_4\right)\big], \end{aligned} \tag{52}$$

$$\begin{aligned} F_6^o(j) = a_1^2 b_y^2\big[& \left(g_{j1}\left(g_{12}^2+g_{22}^2\right)-g_{j2}\left(g_{11} g_{12}+g_{21} g_{22}\right)\right)\left(\tilde{s}_1-\tilde{s}_2\right) \\ & +\mathrm{i} g_{j2}\left(g_{11} g_{22}-g_{12} g_{21}\right)\left(2 \tilde{s}_j-\tilde{s}_1-\tilde{s}_2\right)\big], \end{aligned} \tag{53}$$

$$\begin{aligned} F_7^o(j, k) = (-1)^k a_1 b_y\big[& g_{k2} g_{32} g_{41}\left(\tilde{s}_j-\tilde{s}_3\right)\left(\tilde{s}_j \tilde{s}_3+I_y a_1\right) \\ & +g_{k2} g_{31} g_{42}\left(\tilde{s}_4-\tilde{s}_j\right)\left(\tilde{s}_j \tilde{s}_4+I_y a_1\right) \\ & +g_{k1} g_{32} g_{42}\left(\tilde{s}_3-\tilde{s}_4\right)\left(\tilde{s}_3 \tilde{s}_4+I_y a_1\right)\big], \end{aligned} \tag{54}$$

$$\begin{aligned} F_8^o(j) = a_1 b_y\big[& \left(g_{12}^2+g_{22}^2\right) g_{j1}\left(\tilde{s}_2-\tilde{s}_1\right)\left(\tilde{s}_1 \tilde{s}_2+I_y a_1\right) \\ & +\left(g_{11}\left(g_{12}+\mathrm{i} g_{22}\right)+g_{21}\left(g_{22}-\mathrm{i} g_{12}\right)\right) g_{j2} \delta_{j1} \\ & +\left(g_{11}\left(g_{12}-\mathrm{i} g_{22}\right)+g_{21}\left(g_{22}+\mathrm{i} g_{12}\right)\right) g_{j2} \delta_{2j} \\ & +I_y a_1 g_{j2}(\left(g_{11} g_{12}+g_{21} g_{22}\right)\left(\tilde{s}_1-\tilde{s}_2\right) \\ & +\mathrm{i}\left(g_{11} g_{22}-g_{12} g_{21}\right)\left(\tilde{s}_1+\tilde{s}_2-2 \tilde{s}_j\right))\big], \end{aligned} \tag{55}$$

where $\delta_{jk}, j, k=1,2,3,4$, are also given by (41).

With the above preparations, the explicit expressions of V^{-1} and T^{-1} can be obtained by the following lemma.

Lemma 2. Let V and T be matrices given by (23) and (35), respectively. Let Constraints C1 and C2 hold. Then,

$$V^{-1}=\frac{1}{\Delta_c}\left[\begin{array}{llll} v_1^{\mathrm{adj}} & v_2^{\mathrm{adj}} & v_3^{\mathrm{adj}} & v_4^{\mathrm{adj}} \end{array}\right]^{\mathrm{T}}, \quad T^{-1}=\frac{1}{\Delta_o}\left[\begin{array}{llll} t_1^{\mathrm{adj}} & t_2^{\mathrm{adj}} & t_3^{\mathrm{adj}} & t_4^{\mathrm{adj}} \end{array}\right]^{\mathrm{T}}, \tag{56}$$

where

$$v_1^{\mathrm{adj}}=\varepsilon_1(1)\left[\begin{array}{c} F_1^c(2,3,4) \\ F_2^c(2,3,4) \\ -F_3^c(2,3,4) \\ -F_4^c(2,3,4) \end{array}\right], \quad v_2^{\mathrm{adj}}=\varepsilon_1(2)\left[\begin{array}{c} -F_1^c(1,3,4) \\ -F_2^c(1,3,4) \\ F_3^c(1,3,4) \\ F_4^c(1,3,4) \end{array}\right], \tag{57}$$

$$v_3^{\mathrm{adj}}=\varepsilon_2(4)\left[\begin{array}{c} F_1^c(1,2,4) \\ F_2^c(1,2,4) \\ -F_3^c(1,2,4) \\ -F_4^c(1,2,4) \end{array}\right], \quad v_4^{\mathrm{adj}}=\varepsilon_2(3)\left[\begin{array}{c} -F_1^c(1,2,3) \\ -F_2^c(1,2,3) \\ F_3^c(1,2,3) \\ F_4^c(1,2,3) \end{array}\right], \tag{58}$$

$$t_1^{\mathrm{adj}}=\left[\begin{array}{c} F_1^o(2) \\ -\left(F_3^o(2,1)+\mathrm{i} F_3^o(2,2)\right) \\ -\left(F_5^o(2,1)+\mathrm{i} F_5^o(2,2)\right) \\ F_7^o(2,1)+\mathrm{i} F_7^o(2,2) \end{array}\right], \quad t_2^{\mathrm{adj}}=\left[\begin{array}{c} -F_1^o(1) \\ F_3^o(1,1)-\mathrm{i} F_3^o(1,2) \\ F_5^o(1,1)-\mathrm{i} F_5^o(1,2) \\ -F_7^o(1,1)-\mathrm{i} F_7^o(1,2) \end{array}\right], \tag{59}$$

$$t_3^{\mathrm{adj}}=\begin{bmatrix}F_2^o(4)\\F_4^o(4)\\F_6^o(4)\\F_8^o(4)\end{bmatrix},\quad t_4^{\mathrm{adj}}=\begin{bmatrix}-F_2^o(3)\\-F_4^o(3)\\-F_6^o(3)\\-F_8^o(3)\end{bmatrix}. \tag{60}$$

Proof. The results can be directly deduced from (23) and (35).

Based on Lemma 1 in [26] and Lemma 2, the following result can be obtained.

Theorem 3. Suppose that A_c and A_o are given by (13), where A depends on perturbations Δa_1 and Δa_2, as described in (8) and (9). Let the relation (42) hold, and Constraints C1 and C2 be met. If A_c and A_o do not have common eigenvalues, then the eigenvalue sensitivities of system (11) to variations Δa_1 and Δa_2 are given by

$$\frac{\partial\lambda_i(A_c)}{\partial\Delta a_j}=\frac{1}{\Delta_c}\left(v_i^{\mathrm{adj}}\right)^{\mathrm{T}}A_jv_i,\quad \frac{\partial\lambda_i(A_o)}{\partial\Delta a_j}=\frac{1}{\Delta_o}t_i^{\mathrm{T}}A_jt_i^{\mathrm{adj}},\quad i=1,2,3,4,\quad j=1,2, \tag{61}$$

where v_i and $t_i, i=1,2,3,4$, are given by (23) and (35), respectively, and v_i^{adj}, $t_i^{\mathrm{adj}}, i=1,2,3,4$, are given by (56).

See Appendix C for a proof.

It is seen from Theorem 3.1 in [31] that the control gain matrix K and the observer gain matrix L can be designed separately to realize pole assignment with lower sensitivities. Thus, according to Theorem 3, the indeces to be optimized related to the matrices K and L are given by

$$J_r^c(s_i,f_i,i=1,2,3,4)=\sum_{i=1}^{4}\sum_{j=1}^{2}\left(\frac{1}{\Delta_c}\left(v_i^{\mathrm{adj}}\right)^{\mathrm{T}}A_jv_i\right)^2, \tag{62}$$

and

$$J_r^o(\tilde{s}_i,g_{ij},i=1,2,3,4,j=1,2)=\sum_{i=1}^{4}\sum_{j=1}^{2}\left(\frac{1}{\Delta_o}t_i^{\mathrm{T}}A_jt_i^{\mathrm{adj}}\right)^2, \tag{63}$$

respectively.

4.2 Disturbance attenuation index

It can be seen that the response of $y(t)$ in the frequency domain is given by

$$y(s)=G_c(s)d(s),\quad G_c(s)=C_z(sI-A_z)^{-1}D_z, \tag{64}$$

where A_z, C_z and D_z are given by (12). In order to suppress the influence of high-order unmodeled dynamics $d(s)$ on the output $y(s)$, $\|G_c(s)\|_2$ needs to be minimized. The following theorem is obtained, aiming at giving an explicit expression of $\|G_c(s)\|_2$.

Theorem 4. Suppose that the system (3)–(6) satisfies condition (14) and A_z, C_z and D_z are given by (12). Let

(1) $s_{1,2}=\alpha_1\pm\alpha_2\mathrm{i}, \alpha_1<0, \alpha_2\neq 0$, and $s_i=\alpha_i<0$, $i=3,4$;

(2) $\tilde{s}_{1,2}=\tilde{\alpha}_1\pm\tilde{\alpha}_2\mathrm{i}, \tilde{\alpha}_1<0, \tilde{\alpha}_2\neq 0$, and $\tilde{s}_i=\tilde{\alpha}_i<0$, $i=3,4$;

(3) $s_i\neq\tilde{s}_j,\ i,j=1,2,3,4$.

Then, when K and L are taken as (22) and (34), respectively, and Constraints C1 and C2 hold, we have

$$\|G_c(s)\|_2=\left(\mathrm{trace}\left(\Theta_1P_1^*\Theta_1^{\mathrm{T}}\right)\right)^{\frac{1}{2}}=\left(\mathrm{trace}\left(\Theta_2P_2^*\Theta_2^{\mathrm{T}}\right)\right)^{\frac{1}{2}}, \tag{65}$$

where

$$\begin{cases}P_1^*=\mathrm{unvec}_{2n,2n}\left[-\Psi^{-1}\mathrm{vec}\left(\Theta_2^{\mathrm{T}}\Theta_2\right)\right],\\ P_2^*=\mathrm{unvec}_{2n,2n}\left[-\Psi^{-1}\mathrm{vec}\left(\Theta_1^{\mathrm{T}}\Theta_1\right)\right],\\ \Theta_1=\begin{bmatrix}CV & -CVQ_*\end{bmatrix},\\ \Theta_2=\begin{bmatrix}D^{\mathrm{T}}V^{-\mathrm{T}}-D^{\mathrm{T}}TQ_*^{\mathrm{T}} & -D^{\mathrm{T}}T\end{bmatrix},\\ Q_*=\mathrm{unvec}_{n,n}\left[\Phi^{-1}\mathrm{vec}\left(V^{-1}BKT^{-\mathrm{T}}\right)\right],\end{cases} \tag{66}$$

with V and T being given by (23) and (35), respectively, and

$$\begin{cases} \Psi = (I_{2n} \otimes \Lambda_z) + (\Lambda_z \otimes I_{2n}), \\ \Phi = (\Lambda_o \otimes I_n) - (I_n \otimes \Lambda_c), \end{cases} \tag{67}$$

with

$$\begin{cases} \Lambda_z = \text{Blockdiag}\,(\Lambda_c,\ \Lambda_o), \\ \Lambda_c = \text{diag}\,(s_i,\ i = 1,2,3,4), \\ \Lambda_o = \text{diag}\,(\tilde{s}_i,\ i = 1,2,3,4). \end{cases} \tag{68}$$

See Appendix D for a proof.

According to the above theorem, the corresponding optimization index can be taken as

$$J_d\,(s_i, f_i, \tilde{s}_i, g_{ij}, i = 1,2,3,4, j = 1,2) = \text{trace}\left(\Theta_1 P_1^* \Theta_1^{\mathrm{T}}\right). \tag{69}$$

4.3 Control gain magnitude

In order to reduce the control torque as much as possible, we wish to minimize

$$J_u\,(s_i, i = 1,2,3,4) = \|K\|_{\mathrm{F}}, \tag{70}$$

where K is the gain matrix given by (22). Toward this goal, we give an explicit expression of the gain matrix K.

Theorem 5. Suppose the system (3)–(6) satisfies condition (14). Let s_1 and s_2 be a pair of self conjugate complex numbers with nonzero imaginary part and negative real part, and s_3 and s_4 be two given negative real numbers. Then, all the matrix K which makes $s_i, i = 1,2,3,4$, be the eigenvalues of A_c is given by

$$K = \frac{1}{a_1^2 b_y \gamma} \begin{bmatrix} \Upsilon_4 a_1 b_y \gamma \\ -I_y a_1^3 + \left(a_1\gamma - a_2^2\right)\Upsilon_4 - a_1 a_2 \Upsilon_3 - a_1^2 \Upsilon_2 \\ -a_1 b_y \Upsilon_3 \gamma - a_2 b_y \Upsilon_4 \gamma \\ -a_1 \Upsilon_3 \gamma - a_1^2 I_y a_2 - a_2 \Upsilon_4 \gamma + a_1^2 \Upsilon_1 \end{bmatrix}^{\mathrm{T}},$$

where $\Upsilon_i, i = 1,2,3,4$, are defined by

$$\begin{cases} \Upsilon_1 = s_1 + s_2 + s_3 + s_4, \\ \Upsilon_2 = s_1 s_2 + s_1 s_3 + s_1 s_4 + s_2 s_3 + s_2 s_4 + s_3 s_4, \\ \Upsilon_3 = s_1 s_2 s_3 + s_1 s_2 s_4 + s_1 s_3 s_4 + s_2 s_3 s_4, \\ \Upsilon_4 = s_1 s_2 s_3 s_4. \end{cases}$$

Proof. The result can be directly deduced from (22) and (24)–(27).

4.4 The algorithm

With all the above preparations, a specific procedure for solving Problem MOD can now be stated as follows.

(1) According to Theorems 1 and 2, establish a complete parametric form of the proposed observer-based control law (10).

(2) Define an index function as

$$J\,(s_i, f_i, \tilde{s}_i, g_{ij}, i = 1,2,3,4, j = 1,2) = \alpha_r^o J_r^o + \alpha_r^c J_r^c + \alpha_d J_d + \alpha_u J_u, \tag{71}$$

where J_r^o, J_r^c, J_d and J_u are given by (63), (62), (69) and (70), respectively, and $\alpha_r^o,\ \alpha_r^c,\ \alpha_d,\ \alpha_u \geqslant 0$ are proper weighting factors.

Table 2 Nominal values of parameters

Parameter	Value	Unit
I_y	20667.25	$\mathrm{kg \cdot m^2}$
b	-108.88	$\sqrt{\mathrm{kg}}\cdot\mathrm{m}$
ξ	0.005	–
Λ_y	$2\pi \times 0.151$	–

(3) Determine the ranges of the closed-loop poles:

$$\begin{aligned}
&\mathcal{S}_{12} : \left\{ (s_1, s_2) \middle| \, s_{1,2} = \alpha_1 \pm \alpha_2 \mathrm{i},\ \alpha_i^{\min} \leqslant \alpha_i \leqslant \alpha_i^{\max}, i = 1, 2 \right\},\\
&\mathcal{S}_{34} : \left\{ (s_3, s_4) \middle| \, \alpha_i^{\min} \leqslant s_i \leqslant \alpha_i^{\max}, i = 3, 4 \right\},\\
&\tilde{\mathcal{S}}_{12} : \left\{ (\tilde{s}_1, \tilde{s}_2) \middle| \, \tilde{s}_{1,2} = \tilde{\alpha}_1 \pm \tilde{\alpha}_2 \mathrm{i},\ \tilde{\alpha}_i^{\min} \leqslant \tilde{\alpha}_i \leqslant \tilde{\alpha}_i^{\max}, i = 1, 2 \right\},\\
&\tilde{\mathcal{S}}_{34} : \left\{ (\tilde{s}_3, \tilde{s}_4) \middle| \, \tilde{\alpha}_i^{\min} \leqslant \tilde{s}_i \leqslant \tilde{\alpha}_i^{\max}, i = 3, 4 \right\}.
\end{aligned}$$

(4) Solve the following optimization problem:

$$\begin{aligned}
\min\ & J\left(s_i, \tilde{s}_i, g_{ij}, i = 1, 2, 3, 4,\ j = 1, 2\right)\\
\text{s.t. } & \text{Constraints C1 and C2},\\
& (s_1, s_2) \in \mathcal{S}_{12},\ (s_3, s_4) \in \mathcal{S}_{34},\\
& (\tilde{s}_1, \tilde{s}_2) \in \tilde{\mathcal{S}}_{12},\ (\tilde{s}_3, \tilde{s}_4) \in \tilde{\mathcal{S}}_{34},
\end{aligned} \tag{72}$$

and obtain a sub-optimal solution, represented by $s_i^*, f_i^*, \tilde{s}_i^*, g_{ij}^*, i = 1, 2, 3, 4,\ j = 1, 2$.

(5) Substitute the optimal solution $s_i^*, f_i^*, \tilde{s}_i^*, g_{ij}^*, i = 1, 2, 3, 4, j = 1, 2$, into (22) and (34) to obtain the gain matrices K^* and L^*. Then, by replacing K and L in (10) with K^* and L^*, respectively, we finally obtain the proposed observer-based state feedback control law.

Remark 2. The optimization problem (72) is a nonlinear optimization problem with constraints. It is usually impossible to obtain a theoretical global optimal solution. However, we can always use some numerical optimization algorithms (such as genetic algorithms and annealing algorithms) to obtain some local optimal solutions. The optimization toolbox in MATLAB can be readily used.

Remark 3. Although the forms of Constraints C1 and C2 are complex, they can almost always be satisfied in applications [23]. Therefore, in practical design applications, these constraints can be simply ignored.

Remark 4. Because the system (3)–(9) is single-input, the expression of K depends only on $s_i, i = 1, 2, 3, 4$, and is independent of $f_i, i = 1, 2, 3, 4$ (see [23]). In fact, we can also prove that the index (71) is independent of $f_i, i = 1, 2, 3, 4$. Therefore, these parameters can be simply set to 1.

5 Numerical simulations

In this section, we perform numerical simulations based on practical engineering parameters in Table 2 [22]. As comparisons, simulations of other two methods presented in [22] are also carried out to verify the effectiveness and superiority of the proposed method.

5.1 Controller design

According to practical requirements, the ranges of the desired closed-loop poles are set as

$$\begin{cases}
\alpha_1^{\min} = -0.1383,\ \alpha_1^{\max} = -0.1022,\\
\alpha_2^{\min} = 0.1607,\ \alpha_2^{\max} = 0.2175,\\
\alpha_3^{\min} = -3.0873,\ \alpha_3^{\max} = -2.2819,\\
\alpha_4^{\min} = -0.0837,\ \alpha_4^{\max} = -0.0619,
\end{cases}$$

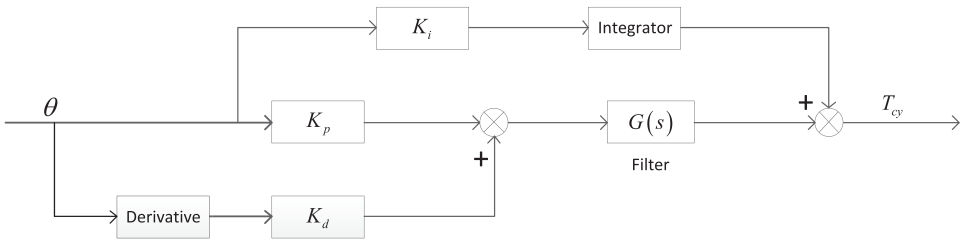

Figure 1 Structure of classic PID controller.

and

$$\begin{cases} \tilde{\alpha}_1^{\min} = -0.1915, \ \tilde{\alpha}_1^{\max} = -0.1416, \\ \tilde{\alpha}_2^{\min} = 0.2226, \ \tilde{\alpha}_2^{\max} = 0.3011, \\ \tilde{\alpha}_3^{\min} = -4.2748, \ \tilde{\alpha}_3^{\max} = -3.1596, \\ \tilde{\alpha}_4^{\min} = -0.1159, \ \tilde{\alpha}_4^{\max} = -0.0857. \end{cases}$$

In view of Remarks 3 and 4, we let $f_i = 1, i = 1, 2, 3, 4$, and ignore Constraints C1 and C2. Then, by setting the weighting factors as

$$\alpha_d = 10^2, \quad \alpha_u = 10^{-3}, \quad \alpha_r^o = \alpha_r^c = 10^{-8},$$

we obtain a set of solutions to the optimization problem (72) as follows:

$$\begin{aligned} &s_{1,2}^* = -0.1354 \pm 0.1625\mathrm{i}, \quad s_3^* = -2.7206, \quad s_4^* = -0.0754, \\ &\tilde{s}_{1,2}^* = -0.1637 \pm 0.2763\mathrm{i}, \quad \tilde{s}_3^* = -3.5989, \quad \tilde{s}_4^* = -0.1061, \\ &g_{11}^* = -2.0049, \quad g_{12}^* = -39.323, \quad g_{21}^* = 138.10, \quad g_{22}^* = -197.18, \\ &g_{31}^* = 142.26, \quad g_{32}^* = 27.254, \quad g_{41}^* = -188.61, \quad g_{42}^* = -130.46, \end{aligned}$$

which gives the index value $J^* = 18.1305$. The corresponding gain matrices are

$$K^* = \begin{bmatrix} -89.8389 & 90.3442 & -1767.3192 & -230.1825 \end{bmatrix}, \tag{73}$$

and

$$L^* = \begin{bmatrix} -0.060779 & -12.800622 \\ -1.186834 & -79.657154 \\ 0.019383 & -3.949398 \\ 3.507241 & -724.800195 \end{bmatrix}. \tag{74}$$

Substituting (74) and (73) into (10) gives the proposed observer-based state feedback control law.

Two control laws are presented in [22] for exactly the same system. The first one is a generalized PID controller commonly used in engineering, which takes the structure in Figure 1, with

$$G(s) = \frac{60s + 1}{1.5625s^2 + 3.5s + 1}, \tag{75}$$

$$K_p = K_d = 15, \quad K_i = 0.03. \tag{76}$$

The second one is a dynamic compensator:

$$\begin{cases} \dot{z} = K_{22}z + K_{21}y, \\ u = K_{12}z + K_{11}y, \end{cases} \tag{77}$$

with

$$K_{11} = \begin{bmatrix} -38.4632239 & -2248.07760 \end{bmatrix},$$

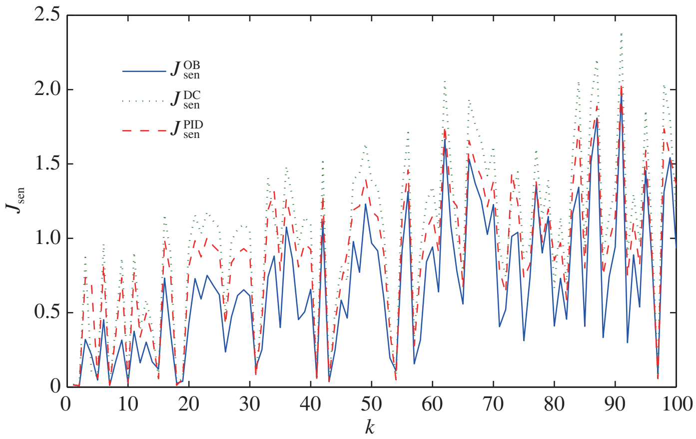

Figure 2 (Color online) The index values $J_{\rm sen}$ under the three control methods.

$$K_{12} = \begin{bmatrix} -0.14638871 & 5.90877238 \end{bmatrix},$$
$$K_{21} = \begin{bmatrix} 4.84008051 & 488.254137 \\ 1.53071924 & -126.193873 \end{bmatrix},$$
$$K_{22} = \begin{bmatrix} -0.01325631 & 2.15924884 \\ -0.18937241 & -2.28904292 \end{bmatrix}.$$

5.2 Simulation verification

5.2.1 *Verification of closed-loop pole sensitivities*

This subsection aims to verify the insensitivity of closed-loop poles to the parameter perturbations Δa_1 and Δa_2. In order to quantify the degree to which the closed-loop poles are affected by parameter perturbations, we introduce the following index:

$$J_{\rm sen} = \frac{1}{n}\sqrt{\sum_{i=1}^{n}\left(s_i^{\rm per} - s_i^{\rm nom}\right)^2},$$

where $s_i^{\rm nom}, i = 1, 2, \ldots, n$ are the nominal closed-loop poles, $s_i^{\rm per} i = 1, 2, \ldots, n$ represent the closed-loop poles when the parameters a_1 and a_2 are perturbed, and n is the number of closed-loop poles, which is equal to 6, 7 and 8, corresponding to the dynamic compensator, the PID cotroller, and the proposed method, respectively. It is obvious that the smaller the value of $J_{\rm sen}$ is, the less sensitive the closed-loop poles are to parameter perturbations.

In order to avoid the contingency of the experimental results as much as possible, we generate 100 sets of random parameter perturbations as follows:

$$\begin{cases} \Delta a_{1k} = {\rm GWN}\left(2\times 10^{-5}k, 0\right), \\ \Delta a_{2k} = {\rm GWN}\left(2\times 10^{-7}k, 0\right), \end{cases} \quad k = 1, 2, \ldots, 100,$$

where GWN $\left(\sigma^2, \mu\right)$ denotes a Gaussian white noise with variance σ^2 and mean μ. For each of these 100 cases, we calculate the index values corresponding to the three control methods, and connect the scattered points into three curves, as shown in Figure 2, where the superscript "OB" represents the result obtained by the proposed method, while the superscript "PID" and "DC" represent the results obtained by the PID controller and the dynamic compensator in [22], respectively.

It can be seen from Figure 2 that $J_{\rm sen}^{\rm OB}$ can almost always be smaller than $J_{\rm sen}^{\rm DC}$ and $J_{\rm sen}^{\rm PID}$, which fully reflects the superiority of the proposed method. It should be noted that although the dynamic compensator designed in [22] considers the eigenvalue sensitivities, it does not perform better than the

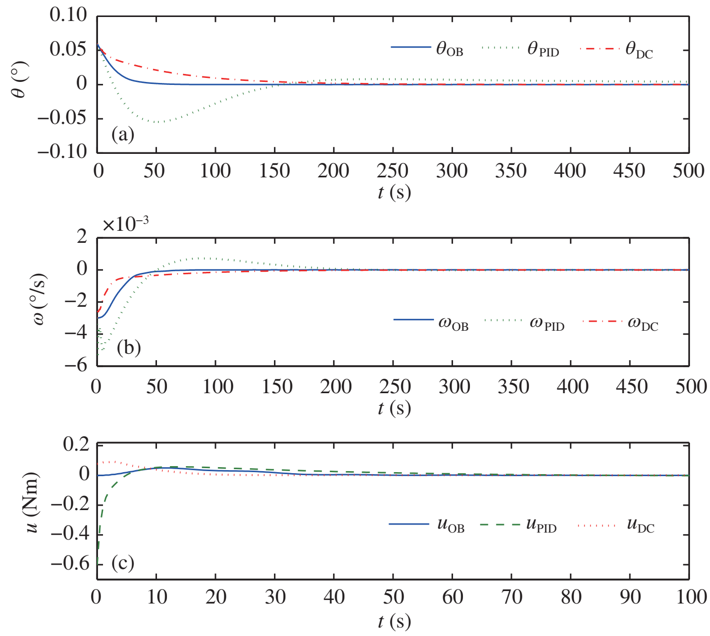

Figure 3 (Color online) (a) Pitch angle; (b) pitch angular velocity; (c) control torque.

PID method. The reason lies in that it aims to minimize an overall sensitivity index instead of the individual ones with respect to Δa_1 and Δa_2.

5.2.2 *Simulation results*

As in [22], the initial values of the attitude angle and its estimation are taken as $\theta_0 = \hat{\theta}_0 = 0.06°$, and those of the attitude angular velocity and its estimation are $\omega_0 = \hat{\omega}_0 = -0.003\,(°/\mathrm{s})$. The initial values of the remaining state variables are set to zero. The high-order unmodeled dynamics is set to $d = 0.1q_y^{(3)}$. The specific values of the parameter perturbations are chosen to be

$$\Delta a_1 = 2.83 \times 10^{-5}, \quad \Delta a_2 = -8.72 \times 10^{-7},$$

with the simulation step size being 0.5 s. The simulation results are shown in Figure 3. The subscript "OB" corresponds to the proposed method, while the subscripts "PID" and "DC" correspond to the PID controller and the dynamic compensator in [22], respectively.

It can be seen from Figures 3(a) and (b) that the proposed controller is superior to the traditional PID controller and the dynamical compensator in [22] in terms of transition time, overshoot, and convergence speed.

It can be seen from Figure 3(c) that among the three control methods, the one proposed in this paper has the smallest peak of control torque (about 0.05 nm), which satisfies the 0.1 nm limit specified in the engineering model task, while the peak of control torque of the PID controller has even exceeded 0.6 nm.

Remark 5. The proposed method has minimized the effect of the disturbances and parameter perturbations, and also the magnitude of the control gain. Thus it performs better than the PID control strategy. Our approach also performs better than the dynamical compensator design in [22] because, unlike the dynamical compensator design, we have considered the closed-loop eigenvalue sensitivities with respect to individual perturbed parameters Δa_1 and Δa_2 instead of the overall closed-loop matrix.

6 Conclusion

In this paper, a parametric design method is proposed for the attitude control of satellites with super flexible attachments. The superiority of the proposed method is significant owing to the following aspects.

(1) Different from many existing control methods that is proposed for a single objective (e.g., [2–8], [11–21]), this paper considers multi-objective design issues, and it aims to make the closed-loop system have simultaneously lower eigenvalue sensitivity, and also it aims a smaller control gain and a stronger tolerance for high-order unmodeled dynamics and disturbances.

(2) The design process of the observer-based control method in this paper is simpler and more convenient when it is compared with the multi-objective design method based on dynamic compensator in [22], because the gain matrices K and L can be designed separately.

(3) Because the parameter perturbations have specific forms, the proposed approach specifically minimizes an individual sensitivity index with respect to Δa_1 and Δa_2 instead of the overall one in [22], and it has been demonstrated to be more effective.

The proposed method shows a great potential because it is applicable and it has important advantages over the others. This situation is proved by the theoretical information and simulation results.

Acknowledgements This work was supported by Major Program of National Natural Science Foundation of China (Grant Nos. 61690210, 61690212), Self-Planned Task of State Key Laboratory of Robotics and System (HIT) (Grant No. SKLRS201716A), and National Natural Science Foundation of China (Grant No. 61333003). The authors are very grateful to the anonymous reviewers for their meaningful suggestions and comments.

References

1 Kida T, Yamaguchi I, Chida Y, et al. On-orbit robust control experiment of flexible spacecraft ETS-VI. J Guid Control Dyn, 1997, 20: 865–872
2 Xiao B, Hu Q, Zhang Y. Adaptive sliding mode fault tolerant attitude tracking control for flexible spacecraft under actuator saturation. IEEE Trans Control Syst Technol, 2012, 20: 1605–1612
3 Hu Q, Ma G. Variable structure control and active vibration suppression of flexible spacecraft during attitude maneuver. Aerospace Sci Tech, 2005, 9: 307–317
4 Hu Q, Ma G, Xie L. Robust and adaptive variable structure output feedback control of uncertain systems with input nonlinearity. Automatica, 2008, 44: 552–559
5 Jiang Y, Hu Q, Ma G. Adaptive backstepping fault-tolerant control for flexible spacecraft with unknown bounded disturbances and actuator failures. ISA Trans, 2010, 49: 57–69
6 Wu A G, Dong R Q, Zhang Y, et al. Adaptive sliding mode control laws for attitude stabilization of flexible spacecraft with inertia uncertainty. IEEE Access, 2019, 7: 7159–7175
7 Xu S, Cui N, Fan Y, et al. Flexible satellite attitude maneuver via adaptive sliding mode control and active vibration suppression. AIAA J, 2018, 56: 4205–4212
8 Huo J, Meng T, Song R, et al. Adaptive prediction backstepping attitude control for liquid-filled micro-satellite with flexible appendages. Acta Astronaut, 2018, 152: 557–566
9 Wu S, Chu W, Ma X, et al. Multi-objective integrated robust H_∞ control for attitude tracking of a flexible spacecraft. Acta Astronaut, 2018, 151: 80–87
10 Liu C, Sun Z, Shi K, et al. Robust dynamic output feedback control for attitude stabilization of spacecraft with nonlinear perturbations. Aerospace Sci Tech, 2017, 64: 102–121
11 Bai H, Huang C, Zeng J. Robust nonlinear H_∞ output-feedback control for flexible spacecraft attitude manoeuvring. Trans Inst Meas Control, 2019, 41: 2026–2038
12 Liu C, Shi K, Sun Z. Robust H_∞ controller design for attitude stabilization of flexible spacecraft with input constraints. Adv Space Res, 2019, 63: 1498–1522
13 Liu L, Cao D, Wei J, et al. Rigid-flexible coupling dynamic modeling and vibration control for a three-axis stabilized spacecraft. J Vib Acoust, 2017, 139: 041006
14 Yan R, Wu Z. Super-twisting disturbance observer-based finite- time attitude stabilization of flexible spacecraft subject to complex disturbances. J Vib Control, 2019, 25: 1008–1018
15 Yadegari H, Khouane B, Yukai Z, et al. Disturbance observer based anti-disturbance fault tolerant control for flexible satellites. Adv Aircraft Spacecr Sci, 2018, 5: 459–475
16 Zhong C, Wu L, Guo J, et al. Robust adaptive attitude manoeuvre control with finite-time convergence for a flexible spacecraft. Trans Institute Meas Control, 2018, 40: 425–435
17 Hu Q. Robust adaptive sliding mode attitude control and vibration damping of flexible spacecraft subject to unknown disturbance and uncertainty. Trans Inst Meas Control, 2012, 34: 436–447
18 Smaeilzadeh S M, Golestani M. A finite-time adaptive robust control for a spacecraft attitude control considering actuator fault and saturation with reduced steady-state error. Trans Inst Meas Control, 2019, 41: 1002–1009
19 Li L, Liu J. Neural-network-based adaptive fault-tolerant vibration control of single-link flexible manipulator. Trans Inst Meas Control, 2020, 42: 430–438
20 Fu Y, Liu Y, Huang D. Adaptive boundary control and vibration suppression of a flexible satellite system with input saturation. Trans Inst Meas Control, 2019, 41: 2666–2677

21 Zhong C, Guo Y, Yu Z, et al. Finite-time attitude control for flexible spacecraft with unknown bounded disturbance. Trans Inst Meas Control, 2016, 38: 240–249
22 Wu Y, Lin B, Zeng H. Parametric multi-objective design for spacecrafts with super flexible netted antennas (in Chinese). Control Theory Appl, 2019, 36: 766–773
23 Duan G-R. Solutions of the equation $AV + BW = VF$ and their application to eigenstructure assignment in linear systems. IEEE Trans Automat Contr, 1993, 38: 276–280
24 Duan G-R. Solution to matrix equation $AV + BW = EVF$ and eigenstructure assignment for descriptor systems. Automatica, 1992, 28: 639–642
25 Duan G-R, Irwin G W, Liu G P. Disturbance attenuation in linear systems via dynamical compensators: a parametric eigenstructure assignment approach. IEE Proc — Control Theor Appl, 2000, 147: 129–136
26 Duan G-R. Robust eigenstructure assignment via dynamical compensators. Automatica, 1993, 29: 469–474
27 Duan G-R, Liu G P, Thompson S. Disturbance decoupling in descriptor systems via output feedback-a parametric eigenstructure assignment approach. In: Proceedings of the 39th IEEE Conference on Decision and Control, Sydney, 2000. 3660–3665
28 Wu W J, Duan G-R. Gain scheduled control of linear systems with unsymmetrical saturation actuators. Int J Syst Sci, 2016, 47: 3711–3719
29 Wang Q, Zhou B, Duan G-R. Robust gain scheduled control of spacecraft rendezvous system subject to input saturation. Aerospace Sci Tech, 2015, 42: 442–450
30 Duan G-R. Simple algorithm for robust pole assignment in linear output feedback. IEE Proc D Control Theor Appl, 1992, 139: 465–469
31 Duan G-R, Thompson S, Liu G P. Separation principle for robust pole assignment-an advantage of full-order state observers. In: Proceedings of the 38th IEEE Conference on Decision and Control, Phoenix, 1999. 76–78

Appendix A Proof of Theorem 1

To prove this theorem, the following lemma which was presented in [22] is needed.

Lemma 3. Let the system (3)–(6) satisfy condition (14). Then the right coprime polynomial matrices $N(s)$ and $D(s)$ satisfying the following right coprime factorization (RCF):

$$(sI - A)^{-1} B = N(s) D^{-1}(s) \tag{A1}$$

are given by

$$N(s) = \begin{bmatrix} \gamma s^2 + a_2 s + a_1 \\ -b_y \gamma s^2 \\ \gamma s^3 + a_2 s^2 + a_1 s \\ -b_y \gamma s^3 \end{bmatrix}, \tag{A2}$$

$$D(s) = -s^4 + I_y a_2 s^3 + I_y a_1 s^2. \tag{A3}$$

According to the eigenstructure assignment result in [23], when the system (3)–(6) is controllable, that is, when the condition (14) holds, complete parametric forms of the gain matrix K and a corresponding nonsingular matrix V satisfying

$$(A + BK) V = V\Lambda_c, \tag{A4}$$

where Λ_c is shown in (68), can be given by

$$\begin{cases} K = WV^{-1}, \\ V = [\, \hat{v}_1 \;\; \hat{v}_2 \;\; \hat{v}_3 \;\; \hat{v}_4 \,], \\ W = [\, \hat{w}_1 \;\; \hat{w}_2 \;\; \hat{w}_3 \;\; \hat{w}_4 \,], \end{cases} \tag{A5}$$

with

$$\begin{cases} \hat{v}_1 = N(\alpha_1 + \alpha_2 \mathrm{i})(f_1 + f_2 \mathrm{i}), \\ \hat{v}_2 = N(\alpha_1 - \alpha_2 \mathrm{i})(f_1 - f_2 \mathrm{i}), \\ \hat{v}_3 = N(\alpha_3) f_3, \;\; \hat{v}_4 = N(\alpha_4) f_4, \end{cases}$$

and

$$\begin{cases} \hat{w}_1 = D(\alpha_1 + \alpha_2 \mathrm{i})(f_1 + f_2 \mathrm{i}), \\ \hat{w}_2 = D(\alpha_1 - \alpha_2 \mathrm{i})(f_1 - f_2 \mathrm{i}), \\ \hat{w}_3 = D(\alpha_3) f_3, \hat{w}_4 = D(\alpha_4) f_4, \end{cases}$$

where $N(s) \in \mathbb{R}^{4\times1}[s]$ and $D(s) \in \mathbb{R}[s]$ are a pair of polynomial matrices satisfying the RCF (A1), and $f_i, \alpha_i, i = 1, 2, 3, 4$, are parameters satisfying the following constraint:

$$\det(V) = \Delta_c \neq 0. \tag{A6}$$

It is known from Lemma 3 that such $N(s)$ and $D(s)$ can be given by (A2) and (A3), respectively. Then, through simple deductions, we can obtain the expression of Δ_c as shown in Constraint C1.

It is easy to see that $\hat{v}_1$ and $\hat{v}_2$ are complex conjugates to each other, so do $\hat{w}_1$ and $\hat{w}_2$. Therefore, assume that

$$\hat{v}_1 = \vartheta_{vR} + \vartheta_{vI}\mathrm{i}, \quad \hat{v}_2 = \vartheta_{vR} - \vartheta_{vI}\mathrm{i}, \tag{A7}$$

$$\hat{w}_1 = \vartheta_{wR} + \vartheta_{wI}\mathrm{i}, \quad \hat{w}_2 = \vartheta_{wR} - \vartheta_{wI}\mathrm{i}. \tag{A8}$$

In view of the first formula in (A5), we have

$$\hat{w}_i = K\hat{v}_i, \quad i = 1, 2, 3, 4. \tag{A9}$$

Substituting (A7) and (A8) into (A9), we obtain the following linear equation:

$$W_0 = KV_0,$$

where

$$W_0 = \begin{bmatrix} \vartheta_{wR} & \vartheta_{wI} & \hat{w}_3 & \hat{w}_4 \end{bmatrix} = \begin{bmatrix} w_1 & w_2 & w_3 & w_4 \end{bmatrix}, \tag{A10}$$

$$V_0 = \begin{bmatrix} \vartheta_{vR} & \vartheta_{vI} & \hat{v}_3 & \hat{v}_4 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix}, \tag{A11}$$

with $v_i, w_i, i = 1, 2, 3, 4$ being given by (24)–(27). Obviously, when Eq. (A6) holds, V_0 is also nonsingular. Thus the matrix K can be given by (22). Combining (A5), (A7) and (A11), gives the expression of V shown in (23). Then the proof is completed.

Appendix B Proof of Theorem 2

Similarly, to prove Theorem 2, the following result obtained in [22] is needed.

Lemma 4. Let the system (3)–(6) satisfy condition (14), and then the right coprime polynomial matrices $H(s)$ and $L(s)$ satisfying the following RCF:

$$\left(sI - A^{\mathrm{T}}\right)^{-1} C^{\mathrm{T}} = H(s) L^{-1}(s) \tag{B1}$$

are given by

$$H(s) = \begin{bmatrix} 1 & 0 \\ 0 & a_1 b_y s \\ 0 & -s^2 + I_y a_2 s + I_y a_1 \\ 0 & a_2 b_y s + a_1 b_y \end{bmatrix}, \tag{B2}$$

$$L(s) = \begin{bmatrix} s & 0 \\ -1 & -s^3 + I_y a_2 s^2 + I_y a_1 s \end{bmatrix}. \tag{B3}$$

Based on the eigenstructure assignment theory shown in [23], when $\left(A^{\mathrm{T}}, C^{\mathrm{T}}\right)$ is controllable, that is, when the condition (14) holds, complete parametric forms of the gain matrix L and a corresponding nonsingular matrix T satisfying

$$T^{\mathrm{T}}(A + LC) = \Lambda_o T^{\mathrm{T}}, \tag{B4}$$

where Λ_o is shown in (68), can be given by

$$\begin{cases} L = T^{-\mathrm{T}} Z^{\mathrm{T}}, \\ T = \begin{bmatrix} \hat{t}_1 & \hat{t}_2 & \hat{t}_3 & \hat{t}_4 \end{bmatrix}, \\ Z = \begin{bmatrix} \hat{z}_1 & \hat{z}_2 & \hat{z}_3 & \hat{z}_4 \end{bmatrix}, \end{cases} \tag{B5}$$

with

$$\begin{cases} \hat{t}_1 = H(\tilde{\alpha}_1 + \tilde{\alpha}_2 \mathrm{i})(g_1 + g_2 \mathrm{i}), \\ \hat{t}_2 = H(\tilde{\alpha}_1 - \tilde{\alpha}_2 \mathrm{i})(g_1 - g_2 \mathrm{i}), \\ \hat{t}_3 = H(\tilde{\alpha}_3) g_3, \ \hat{t}_4 = H(\tilde{\alpha}_4) g_4, \end{cases}$$

and

$$\begin{cases} \hat{z}_1 = L(\tilde{\alpha}_1 + \tilde{\alpha}_2 \mathrm{i})(g_1 + g_2 \mathrm{i}), \\ \hat{z}_2 = L(\tilde{\alpha}_1 - \tilde{\alpha}_2 \mathrm{i})(g_1 - g_2 \mathrm{i}), \\ \hat{z}_3 = L(\tilde{\alpha}_3) g_3, \hat{z}_4 = L(\tilde{\alpha}_4) g_4, \end{cases}$$

where $H(s) \in \mathbb{R}^{4\times 2}[s]$ and $L(s) \in \mathbb{R}^{2\times 2}[s]$ are a pair of polynomial matrices satisfying the RCF (B1), and $\tilde{\alpha}_i$, $g_i = \left[\begin{smallmatrix} g_{i1} \\ g_{i2} \end{smallmatrix}\right]$, $g_{i1}, g_{i2} \in \mathbb{R}$, $i = 1, 2, 3, 4$ are parameters satisfying the following constraint:

$$\det(T) = \Delta_o \neq 0. \tag{B6}$$

It is known from Lemma 4 that such $H(s)$ and $L(s)$ can be given by (B2) and (B3), respectively. Then, substituting (B2), (B3) and (B5) into (B6), gives the expression of Δ_o as shown in Constraint C2.

It is easy to see that $\hat{t}_1$ and $\hat{t}_2$ are complex conjugates to each other, so do $\hat{z}_1$ and $\hat{z}_2$. Therefore, assume that

$$\hat{t}_1 = \xi_{tR} + \xi_{tI}\mathrm{i}, \quad \hat{t}_2 = \xi_{tR} - \xi_{tI}\mathrm{i}, \tag{B7}$$

$$\hat{z}_1 = \xi_{zR} + \xi_{zI}\mathrm{i}, \quad \hat{z}_2 = \xi_{zR} - \xi_{zI}\mathrm{i}. \tag{B8}$$

Considering the first formula in (B5), we have

$$\hat{z}_i = L^{\mathrm{T}}\hat{t}_i, \quad i = 1,2,3,4. \tag{B9}$$

Substituting (B7) and (B8) into (B9), we obtain the following linear equation:

$$Z_0 = L^{\mathrm{T}}T_0,$$

where

$$Z_0 = \begin{bmatrix} \xi_{zR} & \xi_{zI} & \hat{z}_3 & \hat{z}_4 \end{bmatrix} = \begin{bmatrix} z_1 & z_2 & z_3 & z_4 \end{bmatrix}, \tag{B10}$$

$$T_0 = \begin{bmatrix} \xi_{tR} & \xi_{tI} & \hat{t}_3 & \hat{t}_4 \end{bmatrix} = \begin{bmatrix} t_1 & t_2 & t_3 & t_4 \end{bmatrix}, \tag{B11}$$

with $t_i, z_i, i = 1,2,3,4$ being given by (36)–(39). Obviously, when Eq. (B6) holds, T_0 is also nonsingular. Thus the matrix L can be given by (34). Combining (B5), (B7) and (B11), gives the expression of T shown in (35). Then the proof is completed.

Appendix C Proof of Theorem 3

Let

$$V^{-\mathrm{T}} = \begin{bmatrix} \tilde{v}_1 & \tilde{v}_2 & \tilde{v}_3 & \tilde{v}_4 \end{bmatrix}, \quad T^{-\mathrm{T}} = \begin{bmatrix} \tilde{t}_1 & \tilde{t}_2 & \tilde{t}_3 & \tilde{t}_4 \end{bmatrix}.$$

Then, according to (56), the following relations hold:

$$\tilde{v}_i = \frac{1}{\Delta_c}v_i^*, \quad \tilde{t}_i = \frac{1}{\Delta_o}t_i^*, \quad i = 1,2,3,4.$$

It can be seen that V and V^{-1} are, respectively, the right and the left eigenvector matrices of A_c. Thus, according to Lemma 1 in [26], we have

$$\begin{aligned} \frac{\partial\lambda_i\left(A_c\right)}{\partial\Delta a_j} &= \tilde{v}_i^{\mathrm{T}}\frac{\partial A_c}{\partial\Delta a_j}v_i = \tilde{v}_i^{\mathrm{T}}A_jv_i = \frac{1}{\Delta_c}\left(v_i^*\right)^{\mathrm{T}}A_jv_i, \\ i &= 1,2,3,4, \quad j = 1,2. \end{aligned}$$

Similarly, considering that T^{T} and $T^{-\mathrm{T}}$ are, respectively, the left and the right eigenvector matrices of A_o, it can be known from Lemma 1 in [26] that

$$\begin{aligned} \frac{\partial\lambda_i\left(A_o\right)}{\partial\Delta a_j} &= t_i^{\mathrm{T}}\frac{\partial A_o}{\partial\Delta a_j}\tilde{t}_i = t_i^{\mathrm{T}}A_j\tilde{t}_i = \frac{1}{\Delta_o}t_i^{\mathrm{T}}A_jt_i^*, \\ i &= 1,2,3,4, \quad j = 1,2, \end{aligned}$$

holds. Then, the proof is completed.

Appendix D Proof of Theorem 4

At first, let us discuss the eigenstructure of A_z as a preliminary. Let

$$T_z^{\mathrm{T}} = \begin{bmatrix} V^{-1} - Q_*T^{\mathrm{T}} & Q_*T^{\mathrm{T}} \\ -T^{\mathrm{T}} & T^{\mathrm{T}} \end{bmatrix}, \tag{D1}$$

and

$$V_z = \begin{bmatrix} V & -VQ_* \\ V & T^{-\mathrm{T}} - VQ_* \end{bmatrix}, \tag{D2}$$

where T and V are given by (23) and (35), respectively. It is known from Theorems 1 and 2 that, when K and L are taken as (22) and (34), respectively, the relations (A4) and (B4) hold. Thus, in view of (D1) and (D2), we can verify that

$$T_z^{\mathrm{T}}V_z = I, \tag{D3}$$

and

$$T_z^{\mathrm{T}}A_zV_z = \begin{bmatrix} \Lambda_c & -\Lambda_cQ_* + Q_*\Lambda_o + V^{-1}BKT^{-\mathrm{T}} \\ 0 & \Lambda_o \end{bmatrix}. \tag{D4}$$

According to the matrix equation theory, there exists a unique solution to the following linear matrix equation with respect to Q:

$$\Lambda_cQ - Q\Lambda_o = V^{-1}BKT^{-\mathrm{T}}. \tag{D5}$$

With the help of matrix vectorization operations, it can be easily verified that Q_* given by (66) is the unique solution of the matrix equation (D5). Thus, Eq. (D4) can be simplified as

$$T_z^{\mathrm{T}}A_zV_z = \Lambda_z, \tag{D6}$$

where Λ_z is given by (68).

Then, let us discuss the explicit expression of $\|G_c(s)\|_2$. Considering (D6), the function $\|G_c(s)\|_2$ can be transformed into

$$\|G_c(s)\|_2 = \left\|C_z V_z (sI - \Lambda_z)^{-1} T_z^{\mathrm{T}} D_z\right\|_2 .$$

According to Theorems 1 and 2, when K and L are taken as (22) and (34), respectively, both A_c and A_o are stable. Then, from (D6), we know that A_z is also stable. Therefore, it is known from Lemma 4.1 of the previous study[1)] that there exist unique symmetric positive definite solutions P_1 and P_2 to the following Lyapunov matrix equations:

$$\Lambda_z P_1 + P_1 \Lambda_z = -T_z^{\mathrm{T}} D_z D_z^{\mathrm{T}} T_z, \tag{D7}$$

and

$$\Lambda_z P_2 + P_2 \Lambda_z = -V_z^{\mathrm{T}} C_z^{\mathrm{T}} C_z V_z, \tag{D8}$$

and $\|G_c(s)\|_2$ can be given by

$$\left\|G_{dy_p}(s)\right\|_2 = \left(\mathrm{trace}\left(C_z V_z P_1 V_z^{\mathrm{T}} C_z^{\mathrm{T}}\right)\right)^{\frac{1}{2}} = \left(\mathrm{trace}\left(D_z^{\mathrm{T}} T_z P_2 T_z^{\mathrm{T}} D_z\right)\right)^{\frac{1}{2}} .$$

In view of (D1) and (D2), with the help of matrix vectorization operations, it can be verified that P_1^* and P_2^* which are given by (66) are the unique solutions of the matrix equations (D7) and (D8), respectively. Thus the result (65) can be obtained. Then the proof is completed.

1) Duan G-R, Liu G P, Thompson S. Disturbance attenuation in Luenberger function observer designs—a parametric approach. IFAC Proc Vol, 2000, 33: 41–46.

Profile of Guang-Ren DUAN

Prof. Guang-Ren DUAN received his B.S. degree in applied mathemetics, and both his M.S. and Ph.D. degrees in control systems sciences. He is a member of the Chinese Academy of Sciences, a fellow of the IEEE, IET, and CAA (Chinese Association of Automation), and the Founder and Director of the Center for Control Theory and Guidance Technology at HIT (Harbin Institute of Technology), China.

After receiving his Ph.D. degree from HIT in 1989, Prof. Duan took a post-doctoral position at HIT, where he became a professor of control systems theory when he finished the program in 1991. He visited the University of Hull, UK, and the University of Sheffield, UK, from December 1996 to October 1998, and worked as a member of academic staff at the Queen's University of Belfast, UK, from October 1998 to October 2002. He won the 4th Chinese National Youth Award of Science and Technology in 1994, the Fund for Cross-Century Talents of the MEPRC (Ministry of Education of China) in 1997, and the National Science Fund for Distinguished Young Scholars of NSFC (Natural Science Foundation of China) in 1999. One year later, he was selected by the Cheung Kong Scholars Program of the MEPRC. He was elected leader of a Cheung Kong Scholars Innovative Team sponsored by the MEPRC, in 2005, and leader of an Innovative Research Group sponsored by NSFC, in 2009.

He served at several international conferences and symposiums, as General Chair, Associate General Chair, Chair and Co-Chair of IPCs, and member of IPCs, etc., and also served as Associate Editor of a few academic journals. He has been invited as a keynote speaker at more than 20 international conferences, and has been a member of the Science and Technology Committee of the MEPRC, the Information Branch; Vice President of the Technical Committee on Control Theory, CAA; and Associate Director of the Foundation Committee for "Zhao-Zhi GUAN" Award.

His main research interests include parametric control systems design, robust control, quasi-linear control systems, descriptor systems, spacecraft control, and magnetic bearing control. He is the author and co-author of 5 books and more than 270 SCI journal publications. Particularly, he has published over 40 papers in *Automatica* and *IEEE Transactions*, and has published two books with CRC Press (Taylor & Francis), one with Springer, and one in Chinese with the Science Press. The last one has won two national book awards, namely, the 8th National Best Book Award of Science and Technology, and the 11th National Book Award of China. Furthermore, as a principal investigator, he has won two Chinese National Awards of Natural Sciences, one on "Parametric approaches for robust control systems design with applications", awarded in 2008, the other on "Parametric approaches for constrained control systems design with applications", awarded in 2015. Because of his achievements, he has been given the title of National Outstanding Scientific and Technical Worker.

He has supervised 75 master students and 68 Ph.D. students. Among these students, there is one who has been elected member of the Chinese Academy of Engineering, and one who has been selected by both the NSFC program of the National Science Fund for Distinguished Young Scholars and the Cheung Kong Scholars Program of the MEPRC. Besides these, there are also two winners of the Chinese National Best Ph.D. Thesis Award, and two winners of the Fund for New Century Talents by the MEPRC.

Parametric control systems design

Establish a set of formulas for controller parameterization, and propose a general theoretical framework for parametric control systems design, which provides an effective way of solving multi-objective control.

Stability is an essential requirement for all control systems design. If the whole set of stabilizing controllers of certain type can be characterized through parameterization, then a proper controller can be selected from this set by further constraining the design degrees of freedom to meet additional design objectives of the closed-loop system.

The problem of describing a set of controllers satisfying certain dominant requirement through parameterization is called controller parameterization. Probably the earliest trial on this aspect was the so-called Q-parameterization, and certain time-domain results also emerged in the 1980's. However, these results are mostly inexplicit, and are inconvenient to use. Prof. Duan established for linear systems complete parameterizations of stabilizing state and output feedback controllers in general explicit closed forms in the early 1990's [1–6], and later on extended the results to different types of systems and more complicated controllers [1,2,7–10]. A great feature of his formulas is that closed-loop eigenvalues are explicitly appeared as part of the design degrees of freedom, and general parametric expressions of the closed-loop eigenvectors are also provided. These formulas are called in citing papers as "neat", "elegant", "numerically reliable" and "has particular value". An overview paper in this aspect says that he has "performed extensive amount of research".

Prof. Duan also applies his formulas on controller parameterization to several control problems. The first one is robust control, for which he has developed parametric approaches for problems of robust stabilization, robust pole assignment, robust tracking and disturbance attenuation, etc., in various types of linear systems (e.g., [3,6]). It is recognized by researchers that the approach "may conveniently include closed-loop eigenvalues as optimizing parameters, and also possesses good numerical properties and a better optimality", and some of his work is considered to be "pioneering". The second is constrained control, for which he has investigated different types of systems with state and/or control constraints, particularly, systems with actuator saturation (e.g., [11]). "A parametric Lyapunov approach to deal with a series of control problems for systems subject to actuator saturation nonlinearity in a unified framework is proposed", as quoted from a citing paper. Work on this aspect is also called "excellent", "has numerical merits", and "greatly simplifies the expressions of our controllers and the subsequent analysis". Besides the above, he has also developed numerous results for parametric control of spacecraft attitude and orbit systems (e.g., [12]).

The work of Prof. Duan has been widely and frequently used. Among the publications which use his results, there are more than 70 ones which apply his basic results as technical lemmas. In some of these citations, his work is referred as "Duan approach", "Duan procedure", and "Duan method". Besides the many theoretical usages, many successful applications of his results to the designs, simulations, and experiments of certain practical systems have also appeared in certain publications, including structural vibration, quad-rotor rotorcraft, power generation, welding robots, and satellite attitude and orbit control. Particularly, his parametric approach has been successfully applied in the design of the attitude system of a practical satellite with a large netted antenna. In recent years, Prof. Duan also proposed parametric approaches for different types of quasi-linear systems and also applied them in spacecraft control.

Selected publications

- Duan G-R. Analysis and Design of Descriptor Linear Systems. New York: Springer, 2010
- Duan G-R. Generalized Sylvester Equations — Unified Parametric Solutions. Florida: CRC Press, 2015
- Duan G-R. Simple algorithm for robust pole assignment in linear output-feedback. IEE Proc D Control Theor Appl, 1992, 139: 465–469
- Duan G-R. Solution to matrix equation $AV + BW = EVF$ and eigenstructure assignment for descriptor systems. Automatica, 1992, 28: 639–642
- Duan G-R. Solutions of the equation $AV + BW = VF$ and their application to eigenstructure assignment in linear systems. IEEE Trans Automat Contr, 1993, 38: 276–280
- Duan G-R. Robust eigenstructure assignment via dynamic compensators. Automatica, 1993, 29: 469–474
- Duan G-R, Liu G P. Complete parametric approach for eigenstructure assignment in a class of second-order linear systems. Automatica, 2002, 38: 725–729
- Duan G-R. Parametric eigenstructure assignment in second-order descriptor linear systems. IEEE Trans Automat Contr, 2004, 49: 1789–1794
- Duan G-R. Eigenstructure assignment and response analysis in descriptor linear systems with state feedback control. Int J Control, 1998, 69: 663–694
- Duan G-R. Circulation algorithm for partial eigenstructure assignment via state feedback. Europ J Control, 2019, 50: 107–116
- Zhou B, Duan G-R, Lin Z L. A parametric Lyapunov equation approach to the design of low gain feedback. IEEE Trans Automat Contr, 2008, 53: 1548–1554
- Duan G-R, Yu H H, Tan F. Parametric control systems design with applications in missile control. Sci China Ser F-Inf Sci, 2009, 52: 2190–2200