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Dynamics and control of evolutionary congestion games

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Dear editor,

Congestion games (CGs) proposed in [1] are non-cooperative games in which a collection of players compete over a finite set of resources. In CGs, the cost incurred by each player depends only on the resource that the player chooses and the number of players choosing the same resource. Because of their potential across a range of fields, CGs have been applied to practical problems such as network routing and road pricing. Furthermore, CGs can be modeled as evolutionary congestion games (ECGs) if they are played repeatedly.

In contrast to previous work on CGs, this study applies a semi-tensor product (STP) of matrices [2] to study the dynamics of ECGs. Through STP, logic-based systems can be converted into formally equivalent discrete-time dynamic systems. STP has been applied in this way to Boolean networks [3, 4] and evolutionary games [5] with substantial success. Recently, STP has been applied to consider the optimization of facility-based systems via congestion game approach [6]. However, few studies have explored the control of ECGs through STP, thus motivating the present study.

A CG is a potential game that possesses at least one pure Nash equilibrium (NE). However, if all the players update their strategies simultaneously, the game may not converge to NEs. To deal with the EGC convergence problem, a stochastic update rule was designed by [7]. As an alternative to this approach, we apply control techniques in ECGs to influence the evolutionary process of the game and search for NEs. As real-world examples, the government often influences the evolutionary process of ECGs, such as in traffic networks, by intervening in the strategy choices of some players when special events occur. In this study, we assign some players the role of controllers at certain states; we call this intermittent control. This type of control is similar to the event-triggered control in [5, 8]. However, Refs. [5, 8] focused on evolutionary games and Boolean networks, respectively, whereas our model focuses on ECGs.

In this study, we consider the dynamics and control of ECGs. The main innovations are as follows: (i) Through STP, the matrix expression and rigorous dynamic equations of ECGs are established. (ii) Some properties of the fixed points and limit cycles are obtained and verified through STP. (iii) By designing an open-loop intermittent control and a state feedback intermittent control, two necessary and sufficient conditions to ensure stabilization of the ECGs to NEs are provided.

We investigate a particular congestion game $G = (N, P, (S_i)_{i\in N}, (\Xi_j)_{j\in P})$ that is stated as follows: $N = \{1, 2, \ldots, n\}$ and $P = \{1, 2, \ldots, p\}$ denote the finite sets of players and resources shared by players, respectively. S_i is the set of strategies of player i, where $s_i \in S_i$ and $s_i \subset P$ is the strategy of i. In contrast to the general congestion game, each player chooses exactly one resource, i.e., $|s_i| = 1$. Ξ_j describes the cost of resource $j \in P$, which monotonically increases as the number of players using the resource increases.

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Evolutionary dynamics. A finite congestion game, when repeated infinitely according to an update law, is called an evolutionary congestion game (ECG). In this section, we consider the dynamics of the following ECG:

$$x_i(t+1) = f_i(x_1(t), \ldots, x_n(t)), \ i = 1, \ldots, n, \quad (1)$$

where $x_i \in P$ denotes the strategy of player i.

To begin with, the strategy updating rule (SUR) used in this study is parallel myopic best response adjustment with minimum index priority.

$$x_i(t+1) = \begin{cases} x_i(t), & \text{if } x_i(t) \in O_i(x(t)), \\ \min\{O_i(x(t))\}, & \text{otherwise}, \end{cases} \quad (2)$$

where $O_i(x(t)) = \arg\min_{l\in P}\{c_i(l, x^{-i})\}$ is a set of best-response sets of strategies and $x^{-i} \in \prod_{j\neq i} S_j$. We call strategy j a best-response strategy of profile $x(t)$ if $n_j(t) = 0$ or $x_i = j$ implies $x_i \in O_i(x(t))$, for all $i \in N$, where $n_j(t)$ denotes the number of users of resource j at time t.

Then, by the above SUR, the dynamic equation (1) can be expressed as follows:

$$x_i(t+1) = L_i x(t), \quad i = 1, 2, \ldots, n, \quad (3)$$

where L_i is the structure matrix of f_i [2].

Setting $x(t) = \ltimes_{i=1}^n x_i(t)$, and multiplying the equations together, we have the following:

$$x(t+1) = Lx(t), \quad (4)$$

where $L = L_1 * L_2 * \cdots * L_n$ and $*$ denotes the Khari-Rao product [2].

For ECG (4) with SUR (2), we can prove the following results using the STP method. These results also arose in [7], but Ref. [7] does not contain the second item in Theorem 2 below. Put simply, the default length of limit cycles in this study is greater than one.

Theorem 1 ([7])**.** Consider ECG (4) with SUR (2). $\delta_{p^n}^q$ is a Nash equilibrium if and only if $(L)_{qq} = 1$.

Theorem 2 ([7])**.** ECG (4) with SUR (2) has the following properties.

(i) Fixed points are those profiles for which all the strategies are best-response.

(ii) Each profile in limit cycles has at least one strategy not chosen by players.

(iii) In each profile in limit cycles, there exists exactly one strategy which is not best-response.

(iv) The length of each limit cycle is equal to 2.

Intermittent control. Since congestion games evolve according to SUR (2), it is possible that the initial state reaches a limit cycle. Once the state is in a limit cycle, fixed points cannot appear again. Since fixed points are desirable under our approach, we opt to design an external intermittent control to intervene in the strategy choice of some players, in order to stabilize the game to NEs. We regard intermittent control as a kind of special control that operates only when special events occur. In this way, control time and cost are reduced, while the number of controllers is also reduced.

We denote the set of limit cycles with length 2 by $C = \{C_1, \ldots, C_\alpha\}$, and we denote the set of fixed points by Λ. Let $\Gamma = \{i_1, i_2, \ldots, i_r\}$ be the set of players affected by external intermittent control. The value of r can be determined using an algorithm given in Appendix A. We further denote $x^1 = \ltimes_{j=i_1}^{i_r} x_j$, $x^2 = \ltimes_{j=1, j\notin\Gamma}^n x_j$ and $u = \ltimes_{j=1}^r u_j$. It then follows from (3) that

$$\begin{aligned} x_i(t+1) &= L_i(\ltimes_{j=1}^r (I_{p^{j-1}} \otimes W_{[p, p^{i_j - j}]}))x^1(t)x^2(t) \\ &= M_i x^1(t)x^2(t). \end{aligned}$$

Defining $z(t) = x^1(t)x^2(t)$, we then use the STP method to obtain the following:

$$x^1(t+1) = \begin{cases} u(t), & \text{if } z(t) \in \Omega, \\ M^f z(t), & \text{if } z(t) \notin \Omega, \end{cases}$$

and

$$x^2(t+1) = M^r z(t),$$

where $M^f = M_{i_1} * \cdots * M_{i_r}$ and $M^r = *_{j=1, j\notin\Gamma}^n M_j$. Ω is the set of special states that will be determined later on. In the case of $z(t) \in \Omega$, the players in set Γ act as controllers with whose response the system intervenes in a designed way (see below). Otherwise, all players update their strategies according to the SUR. Since $u(t)$ operates only when $z(t) \in \Omega$, we call $u(t)$ an intermittent control.

Then it can be obtained that

$$z(t+1) = \begin{cases} \tilde{M}u(t)z(t), & \text{if } z(t) \in \Omega, \\ Mz(t), & \text{if } z(t) \notin \Omega, \end{cases} \quad (5)$$

where $\tilde{M} = I_{p^r} \otimes M^r$ and $M = M^f * M^r$.

We denote the control signal as follows:

$$\omega = \begin{cases} \delta_2^1, & \text{if } z(t) \in \Omega, \\ \delta_2^2, & \text{if } z(t) \notin \Omega. \end{cases}$$

Thus, the overall dynamics of the ECG become

$$z(t+1) = [\tilde{M}u(t) \ M]\omega z(t). \quad (6)$$

The aim of the following section is to design open-loop intermittent control and state feedback intermittent control such that the congestion game is stabilized to NEs, that is, Λ.

First, we need to choose a proper set Ω. Since the number of profiles is finite, no matter which profile is chosen as the initial one, it will enter a limit cycle within finite steps. Hence, it is enough to control the profiles in limit cycles of length 2. Furthermore, because of the characteristics of limit cycles, only one profile in each limit cycle of length 2 needs to be controlled. Therefore, we set $\Omega = \{\delta_{p^n}^{\alpha_1}, \delta_{p^n}^{\alpha_2}, \ldots, \delta_{p^n}^{\alpha_l}\}$.

Then we consider open-loop intermittent control for ECGs.

Theorem 3. ECG (6) is stabilized to Nash equilibria if and only if there exists a control u such that $\mathrm{Col}_i\{\tilde{M}W_{[p^n,p^r]}z(t)\} \in \Lambda$ for some i, $i \in \{1,\ldots,p^r\}$, where $z(t) \in \Omega$.

Next, we investigate the state feedback intermittent control:

$$u(t) = Kz(t),\ t = t_k, \tag{7}$$

where K is the intermittent control gain matrix to be designed, and t_k is the control time.

The state feedback control operates only when $z(t) \in \Omega$. As the number of elements in set Ω is equal to l, the corresponding sequence of control times is defined as $t_0, t_1, \ldots, t_{l-1}$. The next control time t_l will not be generated, so we set $t_l = \infty$.

Plugging (7) into (5), we have

$$z(t+1) = \begin{cases} \tilde{M}K\Phi_n z(t), & \text{if } z(t) \in \Omega, \\ Mz(t), & \text{if } z(t) \notin \Omega, \end{cases} \tag{8}$$

where $\Phi_n = \mathrm{diag}\{\delta_{p^n}^1, \delta_{p^n}^2, \ldots, \delta_{p^n}^{p^n}\} \in \mathcal{L}_{p^{2n}\times p^n}$ is the power-reducing matrix [2].

Denoting $K = \delta_{p^r}[v_1, v_2, \ldots, v_{p^n}]$, we calculate directly that

$$\begin{aligned} K\Phi_n &= (K \otimes I_{p^n})\Phi_n \\ &= [\delta_{p^r}^{v_1} \otimes I_{p^n}, \delta_{p^r}^{v_2} \otimes I_{p^n}, \ldots, \delta_{p^r}^{v_{p^n}} \otimes I_{p^n}]\Phi_n \\ &= \delta_{p^{n+r}}[(v_1-1)p^n+1, (v_2-1)p^n+2, \\ &\quad \ldots, (v_{p^n}-1)p^n+p^n]. \end{aligned}$$

We then split structure matrix $\tilde{M}$ into p^r blocks as

$$\begin{aligned} \tilde{M} &= [Blk_1(\tilde{M}), Blk_2(\tilde{M}), \ldots, Blk_{p^r}(\tilde{M})] \\ &= [\tilde{M}_1, \tilde{M}_2, \ldots, \tilde{M}_{p^r}], \end{aligned}$$

where $Blk_i(\tilde{M}) \in \mathcal{L}_{p^n\times p^n}$. Then for $z(t) \in \Omega$,

$$\begin{aligned} z(t+1) &= \tilde{M}K\Phi_n z(t) \\ &= [\mathrm{Col}_1(\tilde{M}_{v_1}), Col_2(\tilde{M}_{v_2}), \ldots, \\ &\quad \mathrm{Col}_{p^n}(\tilde{M}_{v_{p^n}})]z(t). \end{aligned} \tag{9}$$

Based on the above analysis, we obtain the following.

Theorem 4. ECG (6) is stabilized to Nash equilibria under intermittent control $u(t) = Kz(t)$ if and only if for $z(t) = \delta_{p^n}^{\alpha_j} \in \Omega$,

$$\mathrm{Col}_{\alpha_j}(\tilde{M}_{v_{\alpha_j}}) \in \Lambda. \tag{10}$$

Remark 1. The gain matrix K can be designed. Setting $\tilde{M} = \delta_{p^n}[\sigma_1, \sigma_2, \ldots, \sigma_{p^{n+r}}]$, then for $z(t) = \delta_{p^n}^{\alpha_j} \in \Omega$,

$$\begin{aligned} z(t+1) &= \tilde{M}u(t)z(t) = \tilde{M}Kz(t)z(t) = \tilde{M}\delta_{p^r}^{v_{\alpha_j}}\delta_{p^n}^{\alpha_j} \\ &= \delta_{p^n}^{\sigma_{(v_{\alpha_j}-1)p^n+\alpha_j}}. \end{aligned}$$

Solving $\delta_{p^n}^{\sigma_{(v_i-1)p^n+i}} \in \Lambda$, we obtain v_i, $i = 1, 2, \ldots, p^n$. If v_i is not unique, then K is easily obtained by choosing the smallest v_i.

Conclusion. In this study, the dynamics and control of ECGs were investigated through the STP method. A rigorous dynamic equation for ECGs was established, and some properties were obtained. Furthermore, to make the congestion game dynamics stabilize to NEs, a class of intermittent controls was introduced. By designing an intermittent open-loop control and a state feedback control, two necessary and sufficient conditions for game stabilization were achieved.

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Supporting information Appendix A. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

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