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• **Supplementary File** •

Dynamics and control of evolutionary congestion games

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Appendix A Illustrative example

We present an example to validate the theoretical results in this paper. Consider a congestion game with five players and two kinds of resources. The cost of each kind of resource is given by Table A1.

Table A1 The cost of resources

User number	1	2	3	4	5
resource 1	10	12	13	14	15
resource 2	5	9	14	20	28

In fact, the number of elements in Γ can be determined by the following algorithm.

Algorithm 1

Step 1: Calculate structure matrix L.

Step 2: Calculate fixed points and the limit cycles with length 2. Find a profile $\delta_{p^n}^{\beta_1} = \delta_{p^n}^{\Sigma_{j=1}^{n-1} p^j (i-1)+i} \in C$, $i = 1, \cdots, p$.

Step 3: Arbitrarily choose a fixed point $\delta_{p^n}^{\lambda}$ and set $\delta_{p^n}^{\beta_2} = L\delta_{p^n}^{\beta_1}$. Convert $\delta_{p^n}^{\beta_1}, \delta_{p^n}^{\beta_2}$, $\delta_{p^n}^{\lambda}$ into $(x_1^{\beta_1}, x_2^{\beta_1}, \cdots, x_n^{\beta_1})$, $(x_1^{\beta_2}, x_2^{\beta_2}, \cdots, x_n^{\beta_2})$, and $(x_1^{\lambda}, x_2^{\lambda}, \cdots, x_n^{\lambda})$, respectively.

Step 4: $r = \min\{|\{j \in N | x_j^{\beta_1} \neq x_j^{\lambda}\}|, |\{j \in N | x_j^{\beta_2} \neq x_j^{\lambda}\}|\}$.

Set $C_\beta = \{\delta_{p^n}^{\beta_1}, \delta_{p^n}^{\beta_2}\}$. If $r = |\{j \in N | x_j^{\beta_1} \neq x_j^{\lambda}\}|$, then add $\delta_{p^n}^{\beta_2}$ into Ω. Otherwise, add $\delta_{p^n}^{\beta_1}$ into Ω. Choose randomly one profile in each limit cycle that belongs to set $C \setminus C_\beta$ and add them into Ω. Then the set Ω can be obtained.

According to the parallel MBRA with minimum index priority, the dynamics of the congestion game can be converted into the following form.

$$x(t+1) = Lx(t),$$

where $L = \delta_{32}[32, 32, 32, 4, 32, 6, 7, 1, 32, 10, 11, 1, 13, 1, 1, 1, 32, 18, 19, 1, 21, 1, 1, 1, 25, 1, 1, 1, 1, 1, 1, 1]$. From the structure matrix L, it can be obtained that there are a cycle $C = \{\delta_{32}^1, \delta_{32}^{32}\}$ and a set of fixed points $\Lambda = \{\delta_{32}^4, \delta_{32}^6, \delta_{32}^7, \delta_{32}^{10}, \delta_{32}^{11}, \delta_{32}^{13}, \delta_{32}^{18}, \delta_{32}^{19}, \delta_{32}^{21}, \delta_{32}^{25}\}$.

By Algorithm 1, the number of players that act as controllers when certain strategy profiles occur is 2. Without loss of generality, we regard player 1 and 2 as intermittent controllers. The objective of this example is to design intermittent open-loop control and state feedback control, respectively, to make the game stabilize to NE (i.e., Λ).

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By taking intermittent control into consideration, we have

$$x(t+1) = \begin{cases} \tilde{L}u(t)x(t), & if\ x(t) \in \Omega, \\ Lx(t), & if\ x(t) \notin \Omega, \end{cases}$$

where $\tilde{L} = \delta_{32}[8,8,8,4,8,6,7,1,8,2,3,1,5,1,1,1,8,2,3,1,5,1,1,1,1,1,1,1,1,1,1,1,16,16,16,12,16,14,15,9,16,10,11,9,13,9,9,9,16,10,11,9,13,9,9,9,9,9,9,9,9,9,9,9,24,24,24,20,24,22,23,17,24,18,19,17,21,17,17,17,24,18,19,17,21,17,17,17,17,17,17,17,17,17,17,17,32,32,32,28,32,30,31,25,32,26,27,25,29,25,25,25,32,26,27,25,29,25,25,25,25,25,25,25,25,25,25,25]$ and $\Omega = \{\delta_{32}^{32}\}$.

First, we consider the open-loop control when the strategy profile reaches the set Ω, that is, $x(t) = \delta_{32}^{32}$. We can select $u = \delta_4^4$ such that $Col_4(\tilde{L}W_{[32,4]}x(t)) = \delta_{32}^{25} \in \Lambda$.

The number of element in set Ω is 1, so the state feedback control works only once. Then we design the state feedback control $u(t) = Kz(t)$ at time t_0. Assume that the state feedback gain matrix of intermittent control $K = \delta_4[v_1, v_2, \cdots, v_{32}]$. By solving $\delta_{32}^{\sigma_{32(v_i-1)+i}} \in \Lambda$, $i = 1, 2, \cdots, 32$, it can be obtained that $v_4 = v_6 = v_7 = 1$, $v_{10} = v_{11} = v_{13} = v_{18} = v_{19} = v_{21} = 2$, $v_8 = v_{12} = v_{14} = v_{15} = v_{16} = v_{20} = v_{22} = v_{23} = v_{24} = v_{25} = v_{26} = v_{27} = v_{28} = v_{29} = v_{30} = v_{31} = v_{32} = 4$, and the value of $v_1, v_2, v_3, v_5, v_9, v_{17}$ can be chosen randomly.

We choose one solution arbitrarily as $K = \delta_4[3,3,3,1,3,1,1,4,3,2,2,4,2,4,4,4,3,2,2,4,2,4,4,4,4,4,4,4,4,4,4,4]$. Then we have $x(t_0+1) = \tilde{L}K\Phi_5 x(t_0) = [24,24,24,4,24,6,7,25,24,10,11,25,13,25,25,25,24,10,11,25,13,25,25,25,25,25,25,25,25,25,25,25]x(t_0)$. It is obvious that all the profiles enter fixed points or will evolve to fixed points without control. Therefore, the evolutionary congestion game can be stabilized to NE under the effect of intermittent control.