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On pinning reachability of probabilistic Boolean control networks

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Dear editor,
Boolean networks (BNs) were introduced by Kauffman [1] to model complex and nonlinear biological systems. They have become a powerful tool for describing, analyzing, and simulating gene regulatory networks [2]. Probabilistic Boolean networks (PBNs) were proposed by Shmulevich et al. [3] as an extension of BNs. A PBN is a family of BNs, where at each discrete time point, the gene state transitions with respect to the rule of one BN. The rule for updating each gene is chosen randomly among several possible rules according to a fixed probability distribution. Semi-tensor product matrices have a wide range of applications in many subjects, for example, singular feedback shift registers [4], game theory [5], and PBNs (PBCNs) [6–8].

In analogy with the concept of reachability with a probability of one, a concept of stochastic reachability is studied. It was employed to describe the reachability probability ρ, $0<\rho<1$, from the initial state to any desired state within a finite time. Moreover, pinning control is one of the most useful available controllers. It does not control all nodes, so that it can save the costs in many real-world systems. Furthermore, it was employed in this study to control the system in order to attain a certain reachability level.

Preliminaries. Let $\Delta_k := \{\delta_k^i | i=1,\ldots,k\}$, where δ_k^i is the i-th column of the identity matrix I_k with degree k. An $m\times n$ matrix A is called a logical matrix, if the columns of A are elements of Δ_m. The set of all $m\times n$ logical matrices is denoted by $\mathcal{L}_{m\times n}$. A vector $r=(r_1\cdots r_n)^{\mathrm{T}}$ is called an n-dimensional probabilistic vector if $\sum_{i=1}^n r_i = 1$, $r_i \geqslant 0$. The set of n dimensional probabilistic vectors is denoted by Υ_n.

Definition 1 ([9])**.** The semi-tensor product of two matrices $A\in M_{m\times n}$, $B\in M_{p\times q}$ is defined as

$$A\ltimes B=(A\otimes I_{\alpha/n})(B\otimes I_{\alpha/p}),$$

where $\alpha=\mathrm{lcm}(n,p)$ is the least common multiple of n and p, and $\otimes$ is Kronecker product.

Definition 2 ([9])**.** For $A\in M_{m\times r}$, $B\in M_{n\times r}$, the Khatri-Rao product of A and B is defined as

$$A*B=[\mathrm{Col}_1(A)\mathrm{Col}_1(B),\ldots,\mathrm{Col}_r(A)\mathrm{Col}_r(B)].$$

Lemma 1 ([9])**.** A logical function $f(x_1,\ldots,x_n)$ with logical arguments $x_1,\ldots,x_n\in\Delta_2$ can be expressed in a multi-linear form as

$$f(x_1,\ldots,x_n)=M_f x_1 x_2\cdots x_n,$$

where $M_f\in\mathcal{L}_{2\times 2^n}$ is unique, and is called the structure matrix of f.

Consider the PBN that is described as follows:

$$x_i(t+1)=L_i x(t),\quad i=1,\ldots,n, \tag{1}$$

where $x(t)=\ltimes_{i=1}^n x_i(t)$ and $L_i\in\mathcal{L}_{2\times 2^n}$, $i=1,2,\ldots,n$, which could be one of the l_i possible models $\mathcal{F}_i=\{L_i^1,L_i^2,\ldots,L_i^{l_i}\}$. Denote the

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probability as $\mathbb{P}\{L_i = L_i^j\} = \mathbb{P}_i^j$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, l_i$ and $\sum_{j=1}^{l_i} \mathbb{P}_i^j = 1$. Then, for each $i \in \{1, \ldots, n\}$, we have $Ex_i(t+1) = \hat{M}_i x(t)$, where $\hat{M}_i = \Sigma_{j=1}^{l_i} \mathbb{P}_i^j M_i^j$. Multiplying these equations together yields

$$Ex(t+1) = MEx(t), \tag{2}$$

where $M = \hat{M}_1 * \hat{M}_2 * \cdots * \hat{M}_n$.

Definition 3. For the system (2), the following definitions are adopted:

(i) $X_d \in \mathcal{X}$ is said to be reachable from X_0 in probability ρ at time t, for $0 < \rho < 1$, if we can find an integer t such that

$$\mathbb{P}\{x(t) = X_d | x(0) = X_0\} \geqslant \rho.$$

(ii) $X_d \in \mathcal{X}$ is said to be reachable in probability ρ if for every $X_0 \in \mathcal{X}$, there exists an integer t such that (i) holds.

Next, we shall design pinning feedback controllers to make the desired state reachable from every initial states of the system (1). First, we need to find some suitable nodes to be controlled. Then, state feedback controllers need to be designed for these nodes as follows:

$$u_i(t) = H_i x(t), \quad i \in \Lambda, \tag{3}$$

where nodes in Λ should be found such that the system (1) can be converted to the system (4), whose given state is globally reachable in probability ρ:

$$\begin{cases} x_i(t+1) = M_{\oplus_i} u_i(t) L_i x(t), & i \in \Lambda, \\ x_j(t+1) = L_j x(t), & j \in \{1, 2, \ldots, n\} \setminus \Lambda, \end{cases} \tag{4}$$

where $M_{\oplus_i} \in \mathcal{L}_{2\times 4}$, and $i \in \Lambda$ are logical matrices obtained for the variables $u_i(t)$ and $L_i x(t)$.

Suppose that the destination state of (2) is $\delta_{2^n}^r$ and the transition matrix M of the system can be expressed as $M = (r_1\ r_2\ \cdots\ r_{2^n})$, $r_i \in \Upsilon_{2^n}$. Next, we are going to change the columns of M to get a new matrix M', which is the transition matrix of a new system $\mathbb{E}x(t+1) = M'\mathbb{E}x(t)$, such that the state $\delta_{2^n}^r$ of this system is reachable from every initial states in probability ρ.

Given $k \in \mathbb{Z}_+$, define a series of sets inductively as follows:

$$\begin{aligned} &S_1(r) = \{\delta_{2^n}^i | \mathrm{Row}_r(M\delta_{2^n}^i) \geqslant \rho,\ i \in \Omega_n\}, \\ &\rho_{ir}^{(1)} = \mathrm{Row}_r(M\delta_{2^n}^i),\ i \in S_1(r), \\ &S_{k+1}(r) = \{\delta_{2^n}^i | \Sigma_{j\in S_k} \mathrm{Row}_j(M\delta_{2^n}^i)\rho_{jr} \geqslant \rho, \\ &\qquad\qquad i \in \Omega_n \setminus \cup_{j=1}^k S_j\}, \\ &\rho_{ir}^{(k+1)} = \Sigma_{j\in S_k} \mathrm{Row}_j(M\delta_{2^n}^i)\rho_{jr}^{(k)},\ i \in S_{k+1}(r). \end{aligned} \tag{5}$$

Lemma 2. If the set $S_k(r) = \emptyset$, then $S_{k+1}(r) = \emptyset$.

Theorem 1. If there exists a k such that $\cup_{j=1}^k S_j(r) = \Omega_n$, then the state $\delta_{2^n}^r$ of the system (1) is reachable in probability ρ.

Proof. According to Lemma 2 and the finiteness of the set Ω_n, there exists a k such that $\Omega_n = \cup_{j=1}^k S_j(r)$. Meanwhile, based on the definition of $S_j(r)$ we have $S_i(r) \cap S_j(r) = \emptyset$ while $i \neq j$. Then, for each $i \in \Omega_n$, there is a unique $S_{l_i}(r)$ such that $\delta_{2^n}^i \in S_{l_i}(r)$. Next, we prove that $\mathbb{P}\{x(l_i) = \delta_{2^n}^r | x(0) = \delta_{2^n}^i\} \geqslant \rho$.

If $l_i = 1$, then we have $\mathrm{Row}_r(M\delta_{2^n}^i) \geqslant \rho$, that is to say, $\mathbb{P}\{x(1) = \delta_{2^n}^r | x(0) = \delta_{2^n}^i\} \geqslant \rho$. If $l_i = 2$, then $i \in S_2(r)$. Furthermore, we have

$$\begin{aligned} \mathrm{Row}_r(x(2)) &= \mathrm{Row}_r(M(M\delta_{2^n}^i)) \\ &= \mathrm{Row}_r(M(\Sigma_{j\in\Omega_n} \mathrm{Row}_j(M\delta_{2^n}^i)\delta_{2^n}^j)) \\ &= \mathrm{Row}_r(\Sigma_{j\in\Omega_n} \mathrm{Row}_j(M\delta_{2^n}^i)M\delta_{2^n}^j) \\ &\geqslant \Sigma_{j\in S_1(r)} \mathrm{Row}_j(M\delta_{2^n}^i)\mathrm{Row}_r(M\delta_{2^n}^j) \\ &= \Sigma_{j\in S_1(r)} \mathrm{Row}_j(M\delta_{2^n}^i)\rho_{jr} \geqslant \rho, \end{aligned}$$

and so $\mathbb{P}\{x(2) = \delta_{2^n}^r | x(0) = \delta_{2^n}^i\} \geqslant \rho$.

Next, assume that when $l_i = k$, $i \in S_k(r)$, one has that $\mathbb{P}\{x(k) = \delta_{2^n}^r | x(0) = \delta_{2^n}^i\} \geqslant \rho$. Then, for $l_i = k+1$, $i \in S_{k+1}(r)$, it can be obtained that

$$\begin{aligned} &\mathrm{Row}_r(x(k+1)) \\ &\quad \geqslant \Sigma_{j\in S_k} \mathrm{Row}_j(M\delta_{2^n}^i)\rho_{jr} \geqslant \rho, \end{aligned}$$

which means $\mathbb{P}\{x(k+1) = \delta_{2^n}^r | x(0) = \delta_{2^n}^i\} \geqslant \rho$.

Consequently, for any initial state $\delta_{2^n}^i$, there exists a unique $S_{l_i}(r)$ such that $\delta_{2^n}^i \in S_{l_i}(r)$, and the state $\delta_{2^n}^r$ is reachable from the state $\delta_{2^n}^i$ after l_i steps in probability ρ.

Algorithm 1 gives a way to obtain the new matrix M' from M, and the complexity of the algorithm is $O(4^n)$.

Algorithm 1

Step 1. Calculate $S_1, S_2, \ldots, S_k$ as defined in (5) until $S_{k+1} = \emptyset$;
Step 2. Define $S(r) = \cup_{j=1}^k S_k(r)$ and the complementary set $S^c(r) = \Delta_{2^n} \setminus S(r)$;
Step 3. Let $M' = M$;
Step 4. For any $j, \delta_{2^n}^j \in S^c(r)$, let $\mathrm{Col}_j(M') = \delta_{2^n}^r$.

Theorem 2. For the system $\mathbb{E}x(t+1) = M'\mathbb{E}x(t)$, where M' is obtained from M by Algorithm 1, state $\delta_{2^n}^r$ is reachable in probability ρ.

According to (2), $M = \hat{M}_1 * \cdots * \hat{M}_n$, where $\hat{M}_i \in \Upsilon_{2\times 2^n}$ for $i = 1, \ldots, n$. M' can also be decomposed as $M' = M_1' * \cdots * M_n'$. Then, we have the following transformation, for $i = 1, \ldots, n$, $x_i(t+1) = \hat{M}_i x(t) \Rightarrow x_i(t+1) = M_i' x(t)$.

Let $\Lambda = \{i|\hat{M}_i \neq M'_i, i = 1, 2, \ldots, n\}$, and for every $i \in \Lambda$, controller $u_i(t)$ is supposed to be added such that

$$\mathbb{E}x_i(t+1) = M_{\oplus_i} u_i(t)\hat{M}_i \mathbb{E}x(t) = M'_i \mathbb{E}x(t), \quad (6)$$

where $M_{\oplus_i}$ has been stated in (4).

Without loss of generality, we assume that $\Lambda = \{1, \ldots, k\}$. That is to say, $\hat{M}_i$ needs to be changed to M'_i for $i = 1, \ldots, k$. Now, we shall design the feedback controllers of the corresponding nodes for the reachability of PBN (1).

Suppose that for $i \in \Lambda$, $L_i^j x(t)$ has been controlled; the following equation holds:

$$M_{\oplus_i} H_i x(t) \hat{M}_i x(t) = M'_i x(t).$$

If one can solve $M_{\oplus_i}$, H_i from the following equations:

$$M_{\oplus_i} H_i (I_{2^n} \otimes \hat{M}_i)\Phi_n = M'_i, \quad i \in \Lambda, \quad (7)$$

then $\delta_{2^n}^r$ in the PBN (1) is reachable in probability ρ by controlling the nodes in Λ.

Proposition 1. Equations in (7) are solvable.

Proof. We first prove the solvability of $M_{\oplus_1} H_1 (I_{2^n} \otimes \hat{M}_1)\Phi_n = M'_1$. Assume that

$$M_{\oplus_1} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ 1-\alpha_1 & 1-\alpha_2 & 1-\alpha_3 & 1-\alpha_4 \end{bmatrix},$$

$$H_1 = \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_{2^n} \\ 1-\beta_1 & 1-\beta_2 & \cdots & 1-\beta_{2^n} \end{bmatrix},$$

$$\hat{M}_1 = \begin{bmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_{2^n} \\ 1-\gamma_1 & 1-\gamma_2 & \cdots & 1-\gamma_{2^n} \end{bmatrix},$$

$$M'_1 = \begin{bmatrix} a_1 & a_2 & \cdots & a_{2^n} \\ 1-a_1 & 1-a_2 & \cdots & 1-a_{2^n} \end{bmatrix},$$

where $\alpha_i, \beta_j \in \mathcal{D}$, $i = 1, 2, 3, 4$, $j = 1, 2, \ldots, 2^n$, and $\gamma_i, a_i \in [0, 1] \subset R$, $i = 1, 2, \ldots, 2^n$. Then, the above equation can be converted into the following forms:

$$\begin{cases} \alpha_1\beta_1\gamma_1 + \alpha_2\beta_1(1-\gamma_1) + \alpha_3(1-\beta_1)\gamma_1 \\ \quad + \alpha_4(1-\beta_1)(1-\gamma_1) = a_1, \\ \qquad \vdots \\ \alpha_1\beta_{2^n}\gamma_{2^n} + \alpha_2\beta_{2^n}(1-\gamma_{2^n}) + \alpha_3(1-\beta_{2^n})\gamma_{2^n} \\ \quad + \alpha_4(1-\beta_{2^n})(1-\gamma_{2^n}) = a_{2^n}. \end{cases} \quad (8)$$

According to Algorithm 1 and the unique decomposition of $\delta_{2^n}^r = \ltimes_{i=1}^n \delta_2^{r_i}$, $r_i \in \{1, 2\}$, we have

$$a_i = \begin{cases} \gamma_i, & i \in S(r), \\ r_i, & i \in S^c(r). \end{cases}$$

If $r_i = 1$, then these 2^n equations can be divided to two parts as follows:

$$\begin{cases} a_i = \gamma_i, & i \in S(r), \\ a_i = 1, & i \in S^c(r). \end{cases}$$

Then, for $i \in S(r)$, let $\alpha_1 = 1$, $\alpha_2 = 0$ and $\beta_i = 1$, and for $i \in S^c(r)$, let $\alpha_3 = 1$, $\alpha_4 = 1$ and $\beta_i = 0$.

If $r_i = 2$, then these 2^n equations can be divided to two parts as follows:

$$\begin{cases} a_i = \gamma_i, & i \in S(r), \\ a_i = 0, & i \in S^c(r). \end{cases}$$

Then, for $i \in S(r)$ let $\alpha_1 = 1$, $\alpha_2 = 0$ and $\beta_i = 1$, for $i \in S^c(r)$ let $\alpha_3 = 0$, $\alpha_4 = 0$ and $\beta_i = 0$.

Hence, equation $M_{\oplus_1} H_1 (I_{2^n} \otimes \hat{M}_1)\Phi_n = M'_1$ is solvable; and similarly, equations in (7) are as well solvable.

Once (7) is solved, then for each $i \in \Lambda$, $M_{\oplus_i}$ and H_i can be obtained.

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References

1 Kauffman S A. Metabolic stability and epigenesis in randomly constructed genetic nets. J Theory Biol, 1969, 22: 437–467
2 Davidson E H. A genomic regulatory network for development. Science, 2002, 295: 1669–1678
3 Shmulevich I, Dougherty E R, Kim S, et al. Probabilistic Boolean networks: a rule-based uncertainty model for gene regulatory networks. Bioinformatics, 2002, 18: 261–274
4 Lu J Q, Li M L, Liu Y, et al. Nonsingularity of Grain-like cascade FSRs via semi-tensor product. Sci China Inf Sci, 2018, 61: 010204
5 Wang L Q, Liu Y, Wu Z G, et al. Strategy optimization for static games based on STP method. Appl Math Comput, 2018, 316: 390–399
6 Li R, Yang M, Chu T G. State feedback stabilization for probabilistic Boolean networks. Automatica, 2014, 50: 1272–1278
7 Zhu Q X, Liu Y, Lu J Q, et al. Observability of Boolean control networks. Sci China Inf Sci, 2018, 61: 092201
8 Wu Y H, Shen T L. An algebraic expression of finite horizon optimal control algorithm for stochastic logical dynamical systems. Syst Control Lett, 2015, 82: 108–114
9 Qi H S, Cheng D Z. Analysis and control of Boolean networks: a semi-tensor product approach. In: Proceedings of the 7th Asian Control Conference, 2009