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Low-degree root-MUSIC algorithm for fast DOA estimation based on variable substitution technique

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Dear editor,

U-root-MUSIC [1] is one of the most popular unitary direction of arrival (DOA) estimation methods [2], and its eigenvalue decomposition (EVD) stage involves only real-valued (RV) arithmetic. Recently, a version of RV-root-MUSIC, which allows RV computations for both EVD and polynomial rooting, was reported in [3]. Nevertheless, the RV-root-MUSIC polynomial is of a high degree $2(M-1)$, which means it has unacceptably high complexity with large arrays [4]. This study focuses on further reducing the complexity of the RV-root-MUSIC algorithm. This study's contributions are as follows: (1) we propose a variable substitution technique to reduce the degree of the RV-root-MUSIC polynomial to half its size without sacrificing accuracy, and (2) we propose a new low-degree root-MUSIC algorithm, which requires only RV computations for fast DOA estimation.

Signal model. A standard signal model for DOA estimation using a uniform linear array (ULA) composed of M sensors is given by [2–4]

$$\boldsymbol{x}(t) = \boldsymbol{A}\boldsymbol{s}(t) + \boldsymbol{n}(t), \tag{1}$$

where $\boldsymbol{x}(t) \in \mathbb{C}^{M\times 1}$ is array output, $\boldsymbol{s}(t) \in \mathbb{C}^{L\times 1}$ is signal, $\boldsymbol{n}(t) \in \mathbb{C}^{M\times 1}$ is additive white Gaussian noise (AWGN), $\boldsymbol{A} \in \mathbb{C}^{M\times L}$ is array manifold, and $L < M$ is the number of signals. Each column of $\boldsymbol{A}$ is known as a steering vector $\boldsymbol{a}(\theta_l) \triangleq [1, \mathrm{e}^{\mathrm{j}\omega_l}, \ldots, \mathrm{e}^{\mathrm{j}(M-1)\omega_l}]^{\mathrm{T}}$, where $\theta_l, l \in [1, L]$ are unknown DOAs, $\omega_l \triangleq (2\pi/\lambda)d\sin\theta_l$, λ is signal wavelength, and $d > \lambda/2$ is array interspacing. The EVD of the array covariance matrix

$$\boldsymbol{R} \triangleq E\{\boldsymbol{x}(t)\boldsymbol{x}^{\mathrm{H}}(t)\} = \boldsymbol{A}\boldsymbol{S}\boldsymbol{A}^{\mathrm{H}} + \sigma_n^2\boldsymbol{I} \tag{2}$$

can be rewritten as

$$\boldsymbol{R} = \boldsymbol{E}_s\boldsymbol{\Lambda}_s\boldsymbol{E}_s^{\mathrm{H}} + \boldsymbol{E}_n\boldsymbol{\Lambda}_n\boldsymbol{E}_n^{\mathrm{H}}, \tag{3}$$

where $\boldsymbol{S} \triangleq E\{\boldsymbol{s}(t)\boldsymbol{s}^{\mathrm{H}}(t)\}$ is source covariance matrix, σ_n^2 is AWGN power, and $\mathrm{span}(\boldsymbol{E}_s)$ and $\mathrm{span}(\boldsymbol{E}_n)$ are signal- and noise- subspaces, respectively.

Proposed algorithm. According to [3], one can exploit the intersection

$$\mathrm{span}\,(\mathbb{E}_n) \triangleq \mathrm{span}(\boldsymbol{E}_n) \cap \mathrm{span}(\boldsymbol{E}_n^*) \tag{4}$$

to obtain a real-coefficient polynomial

$$f(z) \triangleq \boldsymbol{p}^{\mathrm{T}}(z^{-1})\mathbb{E}_n\mathbb{E}_n^{\mathrm{T}}\boldsymbol{p}(z), \tag{5}$$

where $\boldsymbol{p}(z) \triangleq [1, z, \ldots, z^{M-1}]^{\mathrm{T}}$, $z \triangleq \mathrm{e}^{\mathrm{j}\omega_l}$, and $\mathbb{E}_n$ is a real noise matrix computed from the EVD of $\mathrm{Re}(\boldsymbol{R})$ (real-part of $\boldsymbol{R}$).

Observing that $f(z)$ is of a high degree $2(M-1)$, we denote the coefficient of z^k in $f(z)$ as $a_k, k \in [-(M-1), M-1]$, and from (5), we obtain that

$$\begin{aligned} a_k &= \sum_{s=1}^{M-k} \mathbb{P}_n(s, s+k) = \sum_{s=1}^{M-k} \mathbb{P}_n(s+k, s) \\ &= a_{-k}, \quad k \geqslant 0, \end{aligned} \tag{6}$$

where $\mathbb{P}_n \triangleq \mathbb{E}_n\mathbb{E}_n^{\mathrm{T}}$. Thus, we can write $f(z)$ as

$$f(z) = \sum_{k=0}^{M-1} a_k(z^k + z^{-k}). \tag{7}$$

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Define a variable ξ and a function $h(\xi,k)$ as

$$\xi \triangleq z + z^{-1}, \tag{8}$$

$$h(\xi,0)=1,\quad h(\xi,k)\triangleq z^k+z^{-k},\quad k\geqslant 1, \tag{9}$$

respectively. Clearly, for $k=1$, we have $h(\xi,1)=z+z^{-1}=\xi$, and $\forall k\geqslant 2$, we have

$$h(\xi,k)=h(\xi,k-1)\xi-h(\xi,k-2). \tag{10}$$

Using (10), we can determine the coefficients of $h(\xi,k)$ $(k\geqslant 1)$ from the following matrix:

$$\boldsymbol{D}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -2 & 0 & 0 & 0 & \cdots \\ 1 & -3 & 0 & 0 & 0 & \cdots \\ 1 & -4 & 2 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}. \tag{11}$$

The elements in $\boldsymbol{D}$ and the relationships among these elements and coefficients of factors ξ^t, $t\in[1,k]$ in $h(\xi,k)$, $k\geqslant 1$ are subject to the following rules:

(1) $\forall i\geqslant 1$, we have $\boldsymbol{D}(i,1)\equiv 1$. For $i=1$, we have $\boldsymbol{D}(1,j)=0$. $\forall j\geqslant 2$, when $i=2$, we have

$$\begin{cases}\boldsymbol{D}(2,2)=-2, & (12\text{-}1)\\ \boldsymbol{D}(2,j)=0,\quad j\geqslant 3. & (12\text{-}2)\end{cases}$$

(2) $\forall i\geqslant 3$ and $j\geqslant 2$, elements in the i-th row can be recursively computed by those in the $(i-1)$-th and $(i-2)$-th rows as

$$\boldsymbol{D}(i,j)=\boldsymbol{D}(i-1,j)-\boldsymbol{D}(i-2,j-1). \tag{13}$$

(3) Elements in the i-th row provide the coefficients of factors ξ^t, $t\in[1,k]$ in $h(\xi,k)$, $k\geqslant 1$. More specifically, $\boldsymbol{D}(i,j)$ is the coefficient of $\xi^{i-2(j-1)}$. For example, $\boldsymbol{D}(6,1)$, $\boldsymbol{D}(6,2)$, $\boldsymbol{D}(6,3)$, and $\boldsymbol{D}(6,4)$ are coefficients of ξ^6, ξ^4, ξ^2, and ξ^0 in $h(\xi,6)$, respectively, and we have

$$h(\xi,6)=z^6+z^{-6}=\xi^6-6\xi^4+9\xi^2-2.$$

Using these rules, one can obtain $h(\xi,k)$, $\forall k\geqslant 1$ immediately.

Inserting (9) into (7), we can transform $f(z)$ into a new polynomial as

$$f(z)=\sum_{k=0}^{M-1}a_k h(\xi,k)=\sum_{k=0}^{M-1}b_k\xi^k\triangleq h(\xi), \tag{14}$$

where $b_0=a_0$ and b_k, $k\in[1,M-1]$ can be computed by combining coefficients corresponding to factors ξ^k in all $h(\xi,k)$, $\forall k\geqslant 1$ as

$$b_k=a_k\left[\boldsymbol{D}(k,1)+\sum_i\sum_j\boldsymbol{D}(i,j)\right], \tag{15-1}$$

$$\text{s.t.}\quad i-2(j-1)=k. \tag{15-2}$$

As $h(\xi,k)$ has degree k, $h(\xi)$ has degree $M-1$. By rooting $h(\xi)$, we obtain $M-1$ roots ξ_k, $k\in[1,M-1]$. Inserting each root into (8), we obtain

$$z+z^{-1}=\xi_k,\quad k\in[1,M-1]. \tag{16}$$

The two roots of (16) are given by

$$z_k=\frac{\xi_k+\sqrt{\xi_k^2-4}}{2},\quad z_k^*=\frac{\xi_k-\sqrt{\xi_k^2-4}}{2}. \tag{17}$$

According to [3], by selecting among the $2(M-1)$ roots $z_k, z_k^*, k\in[1,M-1]$ for the $2L$ ones $z_l, z_l^*, l=1\in[1,L]$ that lie closest to the unit circle, we can compute $2L$ possible DOAs as

$$\theta_l=\sin^{-1}\left(\frac{\lambda}{2\pi d}\angle z_l\right),\ -\theta_l=\sin^{-1}\left(\frac{\lambda}{2\pi d}\angle z_l^*\right). \tag{18}$$

The L true DOAs can be selected among those $2L$ possible ones by maximizing $\|\boldsymbol{a}^{\mathrm{H}}(\theta)\boldsymbol{R}\boldsymbol{a}(\theta)\|_{\mathrm{F}}^2$ [3].

Summary & complexity analysis. Detailed steps for implementing the proposed algorithm are summarized in Algorithm 1. A comparison of primary computational flops required by different methods is shown in Figure 1(a), where $\mathcal{O}(M^3)$ denotes both flops of EVD on an $M\times M$ real matrix and those of rooting an M-degree real polynomial [5]. It can be seen clearly from Figure 1(a) that our proposed technique is the most efficient one, and it reduces complexity in the rooting stage by a factor of eight when when compared to RV-root-MUSIC.

Algorithm 1 Low-degree root-MUSIC algorithm

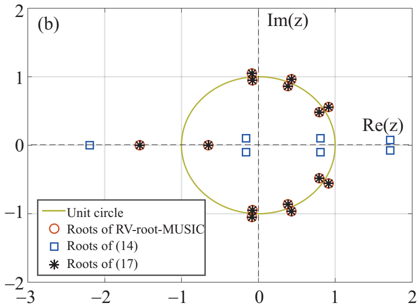
Require: $\{\boldsymbol{x}(t)\}_{t=1}^N$: N snapshots of received data.
Return: $\{\theta_l\}_{l=1}^L$: L signal DOAs.

1: Compute $\boldsymbol{R}=\frac{1}{N}\sum_{t=1}^N\boldsymbol{x}(t)\boldsymbol{x}^{\mathrm{H}}(t)$, perform EVD on $\mathrm{Re}(\boldsymbol{R})$ to obtain the real matrix $\mathbb{E}_n$;
2: Compute $\{b_k\}_{k=0}^{M-1}$ by (15), obtain $h(\xi)$ by (14);
3: Root $h(\xi)$ for $\{\xi_k\}_{k=1}^{M-1}$, get $\{z_k,z_k^*\}_{k=1}^{M-1}$ by (17);
4: Select among $\{z_k,z_k^*\}_{k=1}^{M-1}$ for $\{z_k,z_k^*\}_{k=1}^{L}$ by finding the $2L$ ones that lie closest to the unit circle;
5: Compute the $2L$ possible DOAs $\{\pm\theta\}_{l=1}^L$ by (18), select among $\{\pm\theta\}_{l=1}^L$ for the L true DOAs $\{\theta_l\}_{l=1}^L$ by maximizing $\|\boldsymbol{a}^{\mathrm{H}}(\theta)\boldsymbol{R}\boldsymbol{a}(\theta)\|_{\mathrm{F}}^2$.

Simulation results. Simulations were performed to gain more insights into the performance of the proposed methods. First, as can be seen in Figure 1(b), we compared the root distributions of RV-root-MUSIC and the proposed method, where the rooting procedure in the proposed method is considered as two separate steps ((14) and (17)). It can be seen clearly from Figure 1(b) that for $M=8$ sensors, there are $2(M-1)=14$ roots for RV-root-MUSIC. Because (14) reduces the degree by a half, it contains only $M-1=7$ roots,

Algorithm	Flops in EVD	Flops in rooting
Root-MUSIC [2]	$4\times\mathcal{O}(M^3)$	$32\times\mathcal{O}[(M-1)^3]$
U-root-MUSIC [1]	$\mathcal{O}(M^3)$	$32\times\mathcal{O}[(M-1)^3]$
RV-root-MUSIC [3]	$\mathcal{O}(M^3)$	$8\times\mathcal{O}[(M-1)^3]$
Proposed	$\mathcal{O}(M^3)$	$\mathcal{O}[(M-1)^3]$

(a)



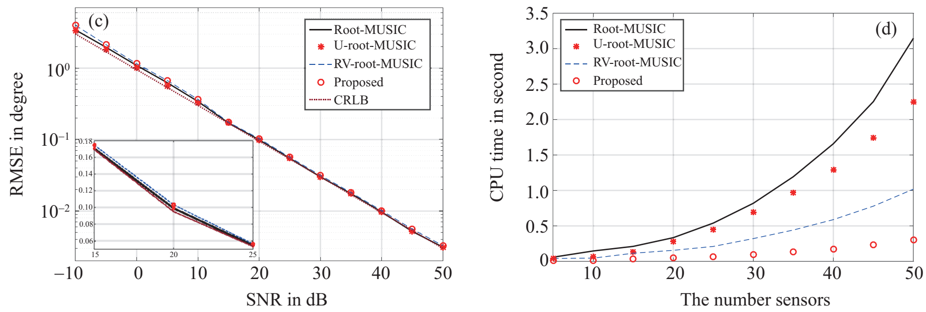


Figure 1 (Color online) (a) Comparison of primary computational flops; (b) roots distribution, 8 sensors ULA, SNR = 10 dB, 100 snapshots, 3 sources at 10°, 20°, and 30°; (c) RMSE vs. the SNR, 11 sensors ULA, 100 snapshots, 3 sources at 20°, 23°, and 30°; (d) simulation time vs. the number of sensors, ULA, 100 snapshots, 2 signals at 20° and 30°.

and each of the 7 roots leads to a quadratic equation given by (16). By rooting these quadratic equations, (17) gives exactly the same 14 roots as RV-root-MUSIC, and this verifies the correctness of our analysis.

Next, an shown in Figure 1(c), we investigated the root mean square error (RMSE) performance of our method compared to root-MUSIC [2], U-root-MUSIC [1], and RV-root-MUSIC [3], where the Cramér-Rao lower bound (CRLB) [6] was also applied. It can be seen from Figure 1(c) that both root-MUSIC and U-root-MUSIC slightly outperform RV-root-MUSIC and the proposed method. However, such disparity is negligible and acceptable because all four estimators gave satisfactory RMSEs close to the CRLB. It can also be seen from that the proposed technique provides the same performance as RV-root-MUSIC with no sacrifice of accuracy. This is because our method is based on variable substitution without approximation.

Finally, as can be seen in Figure 1(d), we examined the computational efficiency of the proposed method and compared it with root-MUSIC, U-root-MUSIC, and RV-root-MUSIC, where efficiency is equivalently evaluated in terms of CPU times by running MATLAB codes in the same environment. It can be seen from Figure 1(d) that with an obvious efficiency advantage, the proposed technique costs much less CPU time than RV-root-MUSIC, which matches our analysis in Figure 1(a).

Conclusion. We proposed a new algorithm, which reduces the complexity of RV-root-MUSIC by a factor of eight in the rooting step because it constructs an equivalent polynomial with half the degree. The new algorithm can provide similar performance to RV-root-MUSIC because it introduces no mathematical approximations.

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