SCIENCE CHINA
Information Sciences

• LETTER •

May 2020, Vol. 63 159205:1–159205:3
https://doi.org/10.1007/s11432-018-9570-8

Study on stability in probability of general discrete-time stochastic systems

Tianliang ZHANG[1], Feiqi DENG[1*] & Weihai ZHANG[2]

[1]*School of Automation Science and Engineering, South China University of Technology, Guangzhou* 510640*, China;*
[2]*College of Electrical Engineering and Automation, Shandong University of Science and Technology, Qingdao* 266590*, China*

Received 11 June 2018/Accepted 16 July 2018/Published online 8 October 2019

Citation Zhang T L, Deng F Q, Zhang W H. Study on stability in probability of general discrete-time stochastic systems. Sci China Inf Sci, 2020, 63(5): 159205, https://doi.org/10.1007/s11432-018-9570-8

Dear editor,
It is well-known that the stability should first be considered in system analysis and synthesis. Since Lyapunov first initiated his stability theory regarding ordinary differential equations (ODEs) in 1892, Lyapunov's stability theory has been an important research topic in both mathematics and control theory. Specifically, Lyapunov's second method was soon generalized to the study of the stability of stochastic differential equations (SDEs) from ODEs [1–3]. Unfortunately, to date, although deterministic difference equations have been systematically investigated, there are few systematic monographs on the stability of discrete-time stochastic difference systems corresponding to [1,2], which are regarding the stability of continuous-time Itô systems.

Along with the development of computer technology, the computational speed of computers has increased, and thus studies on difference systems or difference equations have becomes more important, the reasons for which are as follows. First, it is often very difficult to solve most nonlinear ODEs or SDEs analytically, and hence, numerical solutions to ODEs and SDEs should be found, which requires to turn a continuous system into a discrete system, e.g., [4]. Second, many real engineering problems can be modeled through discrete-time difference equations [5], that is, discrete systems are important models in their own right. In recent years, discrete-time stochastic systems have attracted the attention of many researchers. For example, in [6], the discrete stochastic LaSalle-type invariance principle was obtained. However, it seems that few results have been reported on the stochastic stability of the following general nonlinear discrete stochastic system:

$$x_{k+1} = F_k(x_k, \omega_k), \quad F_k(0, \cdot) \equiv 0. \tag{1}$$

Moreover, even for some special nonlinear discrete stochastic systems such as affine systems, most stability criteria were given through a conditional mathematical expectation [7, 8], which are difficult to verify in practical computations.

Among the various definitions of stochastic stability, the stability in probability is the most fundamental. Although for general continuous-time Itô systems, a well-known result on the stability in probability was given early on [1, 2] by constructing a positive definite Lyapunov function $V(t, x)$, which satisfies $LV(t, x) \leqslant 0$, to date, however there remain no efficient criteria regarding the stability in probability of the general discrete-time stochastic system (1). In this study, we apply the discrete martingale theory and generalized Chebyshev's inequality to establish a useful theorem for stability in probability of system (1), which is a discrete version of the corresponding result in [1,2].

Preliminaries. Consider the following discrete-time nonlinear stochastic difference system:

* Corresponding author (email: aufqdeng@scut.edu.cn)

$$\begin{cases} x_{k+1} = F_k(x_k, \omega_k), \\ F_k(0, y) \equiv 0, \ \forall y \in \mathcal{R}^l, \ k \in \mathcal{N} := \{0, 1, 2, \ldots\}, \\ x_0 \in \mathcal{R}^n, \end{cases} \quad (2)$$

where x_0 is the deterministic initial state, $\{x_k\}_{k\in\mathcal{N}}$ is the $\mathcal{R}^n$-valued state variable sequence, and $\{\omega_k\}_{k\in\mathcal{N}}$ is an independent $\mathcal{R}^l$-valued random variable sequence, which is defined on a given completely filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_k\}_{k\in\mathcal{N}}, \mathscr{P})$, where $\mathcal{F}_k := \sigma(\omega_s, s = 0, 1, \ldots, k-1)$, $\mathcal{F}_0 = \{\phi, \Omega\}$. $F_k : \mathcal{R}^n \times \mathcal{R}^l \mapsto \mathcal{R}^n$ is a continuous function for each $k \in \mathcal{N}$. For the system (2), we define its solution sequence as $\{x_k^{0,x_0}\}_{k\in\mathcal{N}}$ or $\{x_k\}_{k\in\mathcal{N}}$ for simplicity with the initial value $(0, x_0)$. We first introduce Definition 1 for the system (2).

Definition 1 (Stability in probability [1,2])**.** We call the trivial solution $x_k \equiv 0$ of the stochastic difference system (2) to be stable in probability, if for any $\varepsilon > 0$, the following holds:

$$\lim_{x_0 \to 0} \mathscr{P}\left(\sup_{k \geqslant 0} \|x_k\| \geqslant \varepsilon\right) = 0. \quad (3)$$

Associated with system (2), we define the following:

$$\Delta V_k(x) := \mathcal{E}V_{k+1}(F_k(x, \omega_k)) - V_k(x), \quad (4)$$

where $\mathcal{E}$ indicates the mathematical expectation. Note that $\Delta V_k(x)$ only contains a mathematical expectation of ω_k, which is easily verified as $\Delta V_k(x) \leqslant 0$ or not. In many previous references, $\Delta V_k(x_k)$ is often used instead of $\Delta V_k(x)$ to judge the stability of stochastic difference systems. However, it is difficult to test $\Delta V_k(x_k) \leqslant 0$ because the mathematical expectation of the state x_k is involved in $\Delta V_k(x_k)$.

Lemma 1 is a generalized version of Chebyshev's inequality [1].

Lemma 1 (Generalized Chebyshev's inequality [1])**.** Assume $V_k(x)$ is a nonnegative function on $\mathcal{N} \times \mathcal{R}^n$, and $\eta_k(\omega)$ is a stochastic process satisfying $\mathcal{E}V_k(\eta_k(\omega)) < \infty$. Then

$$\mathscr{P}\{|\eta_k(\omega)| \geqslant r\} \leqslant \frac{\mathcal{E}V_k(\eta_k(\omega))}{\inf_{j\in\mathcal{N}, x\in\mathcal{D}_r^c} V_j(x)},$$

where $\mathcal{D}_r^c$ is the complement set of $\mathcal{D}_r$, and $\mathcal{D}_r := \{x \in \mathcal{R}^n : \|x\| < r\}$.

Lemma 2. If $\{V_k(x)\}_{k\in\mathcal{N}}$ is a positive definite Lyapunov function sequence on $\mathcal{N} \times \mathcal{R}^n$ with $V_0(x_0) < \infty$, $\Delta V_k(x) \leqslant 0$, then $\{V_k(x_k), \mathcal{F}_k\}_{k\in\mathcal{N}}$ is a super-martingale, i.e.,

$$\mathcal{E}|V_k(x_k)| = \mathcal{E}V_k(x_k) < \infty,$$

$$\mathcal{E}[V_{k+1}(x_{k+1})|\mathcal{F}_k] \leqslant V_k(x_k), \quad \text{a.s..}$$

Proof. Based on $\Delta V_k(x) \leqslant 0$ for $(k, x) \in \mathcal{N} \times \mathcal{R}^n$, it follows from $\Delta V_k(x_k) \leqslant 0$ a.s., which yields the following:

$$\begin{aligned} \mathcal{E}\Delta V_k(x_k) &= \mathcal{E}V_{k+1}(F_k(x_k, \omega_k)) - \mathcal{E}V_k(x_k) \\ &= \mathcal{E}V_{k+1}(x_{k+1}) - \mathcal{E}V_k(x_k) \leqslant 0, \end{aligned}$$

and accordingly,

$$\mathcal{E}V_{k+1}(x_{k+1}) \leqslant \mathcal{E}V_k(x_k) \leqslant \cdots \leqslant V_0(x_0) < \infty. \quad (5)$$

In addition, based on the definition of $\mathcal{F}_k$, x_k is $\mathcal{F}_k$-measurable, and ω_k is independent of $\mathcal{F}_k$ for all $k \in \mathcal{N}$. Under the condition of $\Delta V_k(x) \leqslant 0$, through Lemma 2.1 of [6], we also have

$$\begin{aligned} &\mathcal{E}[V_{k+1}(x_{k+1})|\mathcal{F}_k] - V_k(x_k) \\ &\quad = \mathcal{E}[V_{k+1}(F_k(x_k, \omega_k)) - V_k(x_k)|\mathcal{F}_k] \\ &\quad = \mathcal{E}[V_{k+1}(F_k(x, \omega_k)) - V_k(x)]\big|_{x=x_k} \\ &\quad = \Delta V_k(x)\big|_{x=x_k} \leqslant 0, \quad \text{a.s..} \end{aligned}$$

Hence,

$$\mathcal{E}[V_{k+1}(x_{k+1})|\mathcal{F}_k] \leqslant V_k(x_k), \quad \text{a.s..} \quad (6)$$

Combining (5) and (6) shows that $\{V_k(x_k), \mathcal{F}_k\}_{k\in\mathcal{N}}$ is a nonnegative super-martingale.

The following Doob's stopping time theorem can be found in Theorem 5.3.3 of [9].

Lemma 3 ([9])**.** Suppose that $\{X_k, \mathcal{F}_k\}_{k\in\mathcal{N}}$ is a super-martingale, which dominates a regular martingale $\{\mathcal{E}(Y|\mathcal{F}_k), \mathcal{F}_k\}_{k\in\mathcal{N}}$, $\mathcal{E}|Y| < \infty$. Assuming that σ and τ are two stopping times, $\sigma \leqslant \tau$ a.s., then

$$\mathcal{E}(X_\tau|\mathcal{F}_\sigma) \leqslant X_\sigma, \quad \text{a.s..}$$

Main results.

Theorem 1. If there exists a positive definite Lyapunov sequence $\{V_k(x)\}_{k\in\mathcal{N}}$ on $\mathcal{N} \times \mathcal{R}^n$, where $V_k(x)$ is a continuous function for $k \in \mathcal{N}$ and $V_0(x_0) < \infty$, which satisfies

$$\Delta V_k(x) = \mathcal{E}V_{k+1}(F_k(x, \omega_k)) - V_k(x) \leqslant 0, \quad (7)$$

then the system (2) is stable in probability.

Proof. Let $\varepsilon > 0$ be a sufficiently small number, $\mathcal{D}_\varepsilon := \{x \in \mathcal{R}^n : \|x\| < \varepsilon\}$, which is the neighborhood of the original point. Set

$$V_\varepsilon := \inf_{x\in\mathcal{D}_\varepsilon^c, \ k\in\mathcal{N}} V_k(x).$$

Because $\{V_k(x)\}_{k\in\mathcal{N}}$ is a positive definite and continuous function sequence on $\mathcal{N} \times \mathcal{R}^n$, we have $V_\varepsilon > 0$. Define $\tau_{\mathcal{D}_\varepsilon}$ as the first exit time from $\mathcal{D}_\varepsilon$, that is, $\tau_{\mathcal{D}_\varepsilon} = \inf\{k : x_k \notin \mathcal{D}_\varepsilon\}$. Based on Lemma 2, $\{V_k(x_k), \mathcal{F}_k\}_{k\in\mathcal{N}}$ is a nonnegative super-martingale, which dominates a zero regular martingale. Hence, through Lemma 3,

$$\mathcal{E}[V_{\tau_{\mathcal{D}_\varepsilon}\wedge k}(x_{\tau_{\mathcal{D}_\varepsilon}\wedge k})|\mathcal{F}_0] \leqslant V_0(x_0), \quad \text{a.s.},$$

which leads to

$$\mathcal{E}[V_{\tau_{\mathcal{D}_\varepsilon}\wedge k}(x_{\tau_{\mathcal{D}_\varepsilon}\wedge k})] \leqslant V_0(x_0). \tag{8}$$

Combing (8) with Lemma 1 yields the following:

$$\begin{aligned}&\mathscr{P}\left(\omega : \sup_{0\leqslant i\leqslant k} \|x_i\| \geqslant \varepsilon\right)\\ &= \mathscr{P}\left(\omega : \|x_{\tau_{\mathcal{D}_\varepsilon}\wedge k}\| \geqslant \varepsilon\right)\\ &\leqslant \frac{\mathcal{E}[V_{\tau_{\mathcal{D}_\varepsilon}\wedge k}(x_{\tau_{\mathcal{D}_\varepsilon}\wedge k})]}{V_\varepsilon} \leqslant \frac{V_0(x_0)}{V_\varepsilon}.\end{aligned} \tag{9}$$

Taking $k \to \infty$ in (9), it follows that

$$\mathscr{P}\left(\omega : \sup_{k\geqslant 0} \|x_k\| \geqslant \varepsilon\right) \leqslant \frac{V_0(x_0)}{V_\varepsilon}.$$

Considering that $V_0(0) = 0$, and $V_0(x)$ is a continuous function, we thus have

$$\lim_{x_0\to 0} \mathscr{P}\left(\omega : \sup_{k\geqslant 0} \|x_k\| \geqslant \varepsilon\right) \leqslant \lim_{x_0\to 0} \frac{V_0(x_0)}{V_\varepsilon} = 0.$$

Theorem 1 is therefore proved.

Remark 1. Compared with existing researches on the stability of discrete stochastic systems, such as in [7], the contributions of this study are mainly reflected in the following two aspects. (i) The system (2) is more general than the model in [7]. (ii) The conditions in Theorem 1 are independent of the mathematical expectation or conditional mathematical expectation of the state x_k. Thus, it is more practical than Theorem 3.1 of [7].

Remark 2. The condition (7) in Theorem 1 can be weakened following the line of Theorem 3.1 and Remark 3.2 of [7].

Example 1. In system (2), we consider a one-dimensional stochastic time-invariant difference system

$$x_{k+1} = \mathrm{e}^{-|x_k|} x_k \omega_k^4, \quad x_0 \in \mathcal{R}, \tag{10}$$

where $\{\omega_k\}_{k\in\mathcal{N}}$ is an independent random variable sequence with $\mathcal{E}\omega_k^4 = 1$. We set $V_k(x) = V(x) = |x|$, and thus

$$\begin{aligned}\Delta V_k(x) &= \mathcal{E}V_{k+1}(F_k(x,\omega_k)) - V_k(x)\\ &= \mathrm{e}^{-|x|}|x| - |x| \leqslant 0, \quad \forall x \in \mathcal{R}.\end{aligned}$$

Hence, through Theorem 1, the system (10) is stable in probability.

Conclusion. In this study, based on the generalized Chebyshev's inequality and Doob's stopping time theorem, we proved a new theorem on the stability in probability, which is applicable because our given criterion does not contain a mathematical expectation of the state x_k, but only a mathematical expectation of ω_k. We believe that, following the line of Theorem 1, we are in a position to show other stability results of general discrete-time stochastic difference systems.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 61573156, 61733008), and Fundamental Research Funds for the Central Universities (Grant No. x2zdD2153620).

References

1 Has'minskii R Z. Stochastic Stability of Differential Equations. Alphen: Sijtjoff and Noordhoff, 1980
2 Mao X R. Stochastic Differential Equations and Applications. 2nd ed. Chichester: Horwood, 2007
3 Xue L R, Zhang W H, Xie X J. Global practical tracking for stochastic time-delay nonlinear systems with SISS-like inverse dynamics. Sci China Inf Sci, 2017, 60: 122201
4 Mao X R. Almost sure exponential stability in the numerical simulation of stochastic differential equations. SIAM J Numer Anal, 2015, 53: 370–389
5 Li X, Ruan X B, Jin Q, et al. Approximate discrete-time modeling of DC-DC converters with consideration of the effects of pulse width modulation. IEEE Trans Power Electron, 2018, 33: 7071–7082
6 Zhang W H, Lin X Y, Chen B S. LaSalle-type theorem and its applications to infinite horizon optimal control of discrete-time nonlinear stochastic systems. IEEE Trans Autom Control, 2017, 62: 250–261
7 Yin J L. Asymptotic stability in probability and stabilization for a class of discrete-time stochastic systems. Int J Robust Nonlinear Control, 2015, 25: 2803–2815
8 Saif M, Liu B, Fan H J. Stabilisation and control of a class of discrete-time nonlinear stochastic output-dependent system with random missing measurements. Int J Control, 2017, 90: 1678–1687
9 Wang S R. Foundation of Probability Theory and Stochastic Process (in Chinese). Beijing: Science Press, 1997