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Necessary and sufficient conditions for normalization and sliding mode control of singular fractional-order systems with uncertainties

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Abstract The sliding mode control (SMC) problem for a normalized singular fractional-order system (SFOS) with matched uncertainties was investigated. Firstly, SFOS was normalized under constrained conditions. Then, the linear sliding mode (SM) function was designed using a fractional-order (FO) positive definite matrix and a linear matrix inequality (LMI). The SM controller was subsequently constructed based on switching laws. Finally, the feasibility of the method was evaluated using a numerical example.

Keywords sliding mode control, normalization, fractional-order positive definite matrix, singular fractional-order systems, uncertainties

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1 Introduction

In recent decades, fractional-order systems (FOS) have attracted significant research attention and have been applied in mathematics, engineering, physics and other fields [1–4]. In FOS, systems have pseudo-states that can be estimated using a continuous filter and approximated using functions in Matlab Toolbox [5, 6].

Singular systems (SS) have been widely used in electrical circuits, economic systems, and several other fields, and have attracted much attention in [7–10]. The linear form of SS has more applications than the general case because it can be decomposed into differential and algebraic equations. Therefore, it is also called the descriptor system (DS). Using normalization or regularization methods, SS can be converted into normal systems [10, 11], however, the problem of stability is a topic of broad interest.

Recently, several scholars have focused on singular fractional-order systems (SFOS), descriptor FOS or FO differential-algebraic systems (FODAS) [12–16]. Their interests include regularity, impulse free, stability, admissibility, and applications. Ref. [12] outlined the conditions obtained using SFOS for electrical circuits. Ref. [13] considered the admissibility condition for SFOS with $\alpha \in (0, 1)$. Ref. [14] summarized several admissibility and stability conditions for SFOS with $\alpha \in (0, 1)$. Ref. [15] investigated the stabilization for descriptor FOS with an $\alpha \in (0, 2)$ based observer. In [16], the mathematical expression for a pendulum located in a Newtonian fluid environment was given by FODAS. There are several examples of competitive and unsolved problems of SFOS in control theory such as robustness for disturbances, controllability, and observability.

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Until now, there have been limited studies on the stability and stabilization of SFOS with uncertainties [11, 17–21]. Ref. [11] investigated robust stability for uncertain descriptor FOS with $\alpha \in (0, 2)$ via linear matrix inequality (LMI). Ref. [17] examined the control problem for SFOS with $\alpha \in (0, 1)$ via static output feedback. Ref. [18] considered output feedback normalization of SFOS with $\alpha \in (0, 2)$ based on the output differential feedback. Ref. [19] investigated asymptotic stability of delayed SFOS. In [20], the existence and uniqueness of solution for SFOS using the Caputo fractional derivative were examined. Ref. [21] investigated the output feedback normalization and stabilization problem of SFOS with FO $\alpha \in (0, 2)$ based on output feedback control. Currently, there are few published studies on sliding mode control (SMC) of SFOS [22].

As a type of nonlinear control, SMC has rapidly developed and is currently widely used [23–26]. The aim of the SMC control method is to guide a system's state trajectories to correctly select the sliding mode (SM) manifold in the process of motion, and to maintain the motion at all times. Therefore, the main advantages of SMC are insensitivity to internal and external disturbances, and the convergence of the sliding variable to zero in a finite time. There are several different SMC methods such as second-order SM [27], high order SM [28], super twisting SM [29], and integral SM [30]. Adaptive SMC, that organically combines SMC with adaptive control, is a control strategy that can be applied in an uncertain or time-varying parameter system [31].

In this article, the objective is to study the SMC of SFOS. Three important innovations are finished. Firstly, the sufficient and necessary conditions for the normalization of SFOS are given, which are the basis of the whole paper. Then, different from integral SM function in the previous FOS literature, a linear SM function which is easy to implement is designed for the normalized FOS. In addition, to ensure accessibility of SM in a finite time, the FO derivation of Lyapunov function is obtained, which is different from integer derivative in the previous FOS literature. Finally, numerical examples show the effectiveness of the proposed method.

This article is arranged as follows: the system model and some advance preparations are summarized in Section 2. The normalizable condition of a class of SFOS is presented in Section 3, and then SMC of normal FOS is used to stabilize uncertain normalizable SFOS. In Section 4, The effectiveness of the method is illustrated using a numerical example and the main conclusion is summarized in Section 5.

Notation. In the article, $\mathbb{R}$, $\mathbb{C}$ denotes the real and complex set, respectively. $I_n \in \mathbb{R}^{n\times n}$ is an identity matrix. rank(A) is the rank of matrix A. range(B) is the range of matrix B. D^α, $D^{-\alpha}$ are the respective abbreviations of ${}_0D_t^\alpha$ (Caputo derivative operator) and ${}_0D_t^{-\alpha}$ (Riemann-Liouville fractional-order integral operator) without confusion. sym(M) represents $M + M^{\mathrm{T}}$. spec(A, α) represents the spectrum of $\det(sI - A) = 0$. $\|\cdot\|$ represents the matrix norm. The FO positive definite matrices set $\mathbb{P}_\alpha^{n\times n}$ is given as

$$\mathbb{P}_\alpha^{n\times n} = \left\{ \sin\left(\frac{\alpha\pi}{2}\right) X + \cos\left(\frac{\alpha\pi}{2}\right) Y : X, Y \in \mathbb{R}^{n\times n}, \begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} > 0 \right\}.$$

2 System description and preliminaries

Considering a nonlinear SFOS,

$$ED^\alpha x(t) = \bar{f}(x(t), u(t), t), \tag{1}$$

where $E \in \mathbb{R}^{n\times n}$ is singular, $0 < \alpha \leqslant 1$, $x \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $\bar{f}(x(t), t)$ are FO of the system, pseudo semi-state, input and nonlinear function, respectively. Choose initial condition of system (1) as

$$x(0) = x_0. \tag{2}$$

For simplicity, the linear form of system (1) is discussed or the linearization technique is used such that the system (1) becomes

$$ED^\alpha x(t) = Ax(t) + Bu(t) + f(x(t), t), \tag{3}$$

where $A \in \mathbb{R}^{n\times n}$ is the system matrix, $B \in \mathbb{R}^{n\times m}$ is the control matrix, $f(x(t),t) \in \mathbb{R}^n$ (f) is the nonlinear term. In this article, it is assumed that f is bounded and E is not full rank, i.e., $\mathrm{rank}(E) = r < n$. When $f = 0$, system (3) is called a linear time-invariant SFOS. When $f = 0$ and $u = 0$, (E, A, α) represents system (3).

Remark 1. Consider systems (3) with $f = 0$.

- When E $(= I)$ is nonsingular, system (3) is a linear time-invariant FOS,

$$D^{\alpha}x(t) = Ax(t) + Bu(t); \tag{4}$$

- When $\alpha = 1$, system (3) is a linear time-invariant SS,

$$E\dot{x}(t) = Ax(t) + Bu(t). \tag{5}$$

Therefore, system (3) is a joint promotion form of a linear time-invariant FOS and SS.

Some definitions, lemmas, assumptions and properties are first outlined.

Definition 1 ([1]). The Riemann-Liouville FO integral and Caputo's FO derivative with $0 < \alpha < 1$ are alternatively defined as

$$\begin{aligned} {}_0D_t^{-\alpha}f(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t f(\tau)(t-\tau)^{\alpha-1}\mathrm{d}\tau, \\ {}_0D_t^{\alpha}f(t) &= \frac{1}{\Gamma(1-\alpha)}\int_0^t f'(\tau)(t-\tau)^{-\alpha}\mathrm{d}\tau, \end{aligned}$$

where $\Gamma(\alpha) = \int_0^{+\infty}\tau^{\alpha-1}\mathrm{e}^{-\tau}\mathrm{d}\tau$ is the Euler gamma function.

Definition 2 ([15]). The regularity and impulse free definitions of systems (3) with $f = 0$ and $u = 0$ are given as follows.

- If there exists at least one complex number $s \in \mathbb{C}$ such that the pseudo-polynomial $\det(s^{\alpha}E - A)$ is not identically zero, then the system (E, A, α) is regular.
- If $\deg(\det(sE - A)) = \mathrm{rank}(E)$ is satisfied, then the system (E, A, α) is impulse free.

The regularity ensures that the solutions of system (3) are existential and unique under $f = 0$ and $u = 0$.

Definition 3 ([32]). If all the roots of $\det(sI - A) = 0$ satisfy $|\arg(\mathrm{spec}(A, \alpha))| > \alpha\frac{\pi}{2}$, then the linear FOS (4) with $u = 0$ is stable.

Lemma 1 ([32, 33]). The linear FOS (4) with $u = 0$ is asymptotically stable if and only if a matrix $P \in \mathbb{P}_{\alpha}^{n\times n}$ can be found that satisfies

$$\mathrm{sym}(AP) < 0. \tag{6}$$

Lemma 2 ([4]). For $l_1 \in \mathbb{R}$ and $l_2 \in \mathbb{R}$, if ${}_0D_t^{\alpha}f(t)$ and ${}_0D_t^{\alpha}g(t)$ exist, then the linear operation for Caputo's FO derivative with $0 < \alpha < 1$ is given as

$$D^{\alpha}(l_1f(t) + l_2g(t)) = l_1D^{\alpha}f(t) + l_2D^{\alpha}g(t),$$

$$D^{-\alpha}D^{\alpha}f(t) = f(t) - f(0).$$

Assumption 1. Suppose $\mathrm{rank}(s^{\alpha}E - A, B) = n$ for any $s \in \mathbb{C}$.

Assumption 2. Suppose $\mathrm{rank}(E, B) = n$ and B is a column full rank.

Assumption 3. Suppose $f(t, x) \in \mathrm{range}(B)$ and $\|f(t, x)\| \leqslant \rho$, where ρ is a constant positive real number.

Ref. [34] outlines the controllable condition (Assumption 1) for singular systems. Ref. [35] has proven the controllability condition (Assumption 2) for linear FOS is the same as the normal condition. Therefore, the controllable conditions for system (3) with $f = 0$ can be similarly given as follows.

Lemma 3. If Assumptions 1 and 2 are satisfied, then system (3) with $f = 0$ is completely controllable.

Remark 2. In terms of slow-fast decomposition for system (3) with $f = 0$, Assumption 1 can ensure the controllability of a slow system, while Assumption 2 can ensure the controllability of a fast system [10].

3 Main results

3.1 Normalization

Ref. [10] has already discussed that feedback control can be used to normalize linear SS when the system meets certain conditions, so that the various performance requirements of linear SS can be discussed using a normal system approach. These results will be extended to SFOS (3) with $f = 0$.

When system (3) with $f = 0$ is singular, if the feedback control can be given as

$$u_f(t) = -K_1 D^{\alpha} x(t) + v(t), \tag{7}$$

in the preceding equation, where the first term is the FO derivative feedback control and the other one is the new input or virtual input, then system (3) with (7) can be represented as

$$(E + BK_1)D^{\alpha} x(t) = Ax(t) + Bv(t). \tag{8}$$

Furthermore, if $\det(E + BK_1) \neq 0$, system (8) is assumed to be the normal form

$$D^{\alpha} x(t) = (E + BK_1)^{-1} Ax(t) + (E + BK_1)^{-1} Bv(t). \tag{9}$$

Now, the definition of the system normalizability is given.

Definition 4. If there exists $u_f(t)$ (7) such that system (8) is normal, i.e., $\det(E + BK_1) \neq 0$, then system (3) with $f = 0$ is called normalizable.

Next, the system normalizability condition is provided, which is sufficient and necessary.

Theorem 1. The system (3) with $f = 0$ can be normalized if and only if the condition $\operatorname{rank}(E, B) = n$ is satisfied.

Proof. If system (3) with $f = 0$ is normalizable, according to Definition 4, a gain matrix $K_1 \in \mathbb{R}^{m\times n}$ can be given such that $\det(E + BK_1) \neq 0$. Then,

$$\operatorname{rank}(E + BK_1) = \operatorname{rank}[E, B] \begin{bmatrix} I \\ K_1 \end{bmatrix} = n.$$

Given that $[\begin{smallmatrix} I \\ K_1 \end{smallmatrix}]$ is a column full rank matrix, $\operatorname{rank}(E, B) = n$ is obtained.

Conversely, suppose $\operatorname{rank}(E, B) = n$ is satisfied. Two nonsingular matrices M_2, N_2 can be chosen that satisfy

$$M_2 E N_2 = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

where $B_1 \in \mathbb{R}^{r\times m}$, $B_2 \in \mathbb{R}^{(n-r)\times m}$. As such, we have

$$\operatorname{rank}(E, B) = \operatorname{rank}(M_2 E N_2, M_2 B) = \operatorname{rank} \begin{bmatrix} I_r & 0 & B_1 \\ 0 & 0 & B_2 \end{bmatrix} = n,$$

where B_2 is a row full rank matrix, i.e., $\operatorname{rank}(B_2) = n - r$. This indicates that $B_2 B_2^{\mathrm{T}} \in \mathbb{R}^{(n-r)\times(n-r)}$ is invertible. The gain matrix K_1 is then chosen as

$$K_1 = \left[0, B_2^{\mathrm{T}}\right] N_2^{-1} \in \mathbb{R}^{m\times n}. \tag{10}$$

Finally, by operating some rigorous derivations, one can obtain

$$\begin{aligned}\det(E+BK_1) &= \det(M_2^{-1}M_2EN_2N_2^{-1}+M_2^{-1}M_2BK_1)\\ &= \det\left(M_2^{-1}\begin{bmatrix} I_r & 0\\ 0 & 0\end{bmatrix}N_2^{-1}+M_2^{-1}\begin{bmatrix} B_1\\ B_2\end{bmatrix}\left[0, B_2^{\mathrm{T}}\right]N_2^{-1}\right)\\ &= \det\left(M_2^{-1}\begin{bmatrix} I_r & B_1B_2^{\mathrm{T}}\\ 0 & B_2B_2^{\mathrm{T}}\end{bmatrix}N_2^{-1}\right)\\ &= \det(M_2^{-1})\det(B_2B_2^{\mathrm{T}})\det(N_2^{-1})\\ &\neq 0.\end{aligned}$$

Therefore, system (3) with $f=0$ is normalizable.

Remark 3. Ref. [11] gives another sufficient and necessary condition in the form of LMI, in which system (3) with $f=0$ can be normalized. As such, a nonsingular matrix $X\in\mathbb{R}^{n\times n}$ and a matrix $Y\in\mathbb{R}^{m\times n}$ should be found and satisfy

$$EX+X^{\mathrm{T}}E^{\mathrm{T}}+BY+Y^{\mathrm{T}}B^{\mathrm{T}}<0. \tag{11}$$

Then one can choose $K_1=YX^{-1}$. Under the normalizability assumption, one can transform the SS's infinite poles to finite ones via control (7). Thus, closed-loop systems have impulse free solutions.

3.2 Design of sliding mode function

In Subsection 3.1, when the system satisfies $\mathrm{rank}(E,B)=n$, then a feedback matrix K_1 can be chosen such that system (3) can be normalized to (9). Let $\breve{A}=(E+BK_1)^{-1}A$, $\breve{B}=(E+BK_1)^{-1}B$, $\breve{f}=(E+BK_1)^{-1}f$. One can rewrite (9) as

$$D^{\alpha}x(t)=\breve{A}x(t)+\breve{B}v(t)+\breve{f}. \tag{12}$$

If $\mathrm{rank}(B)=m$, it is possible to choose an appropriate matrix T such that

$$T\breve{B}=\begin{bmatrix} 0\\ \breve{B}_2\end{bmatrix},$$

where $\breve{B}_2\in\mathbb{R}^{m\times m}$ is invertible. For $f\in\mathrm{range}(B)$, a vector $\breve{f}\in\mathbb{R}^m$ can be found such that $f=B\breve{f}$, that is

$$T\breve{f}=T\breve{B}\breve{f}=\begin{bmatrix} 0\\ \breve{B}_2\breve{f}\end{bmatrix}.$$

Denoting $\breve{f}_2=\breve{B}_2\breve{f}$, the following conclusion is established.

If Assumptions 2 and 3 are satisfied, given an invertible matrix T, it is possible to obtain

$$T\breve{B}=\begin{bmatrix} 0\\ \breve{B}_2\end{bmatrix},\quad T\breve{f}=\begin{bmatrix} 0\\ \breve{f}_2\end{bmatrix}. \tag{13}$$

Then, $T\breve{A}$ can be written as

$$T\breve{A}T^{-1}=\begin{bmatrix} \breve{A}_{11} & \breve{A}_{12}\\ \breve{A}_{21} & \breve{A}_{22}\end{bmatrix}. \tag{14}$$

Let $Tx(t)=z(t)=\left[\begin{smallmatrix} z_1(t)\\ z_2(t)\end{smallmatrix}\right]$; one can rewrite system (12) as

$$\begin{cases} D^{\alpha}z_1(t)=\breve{A}_{11}z_1(t)+\breve{A}_{12}z_2(t),\\ D^{\alpha}z_2(t)=\breve{A}_{21}z_1(t)+\breve{A}_{22}z_2(t)+\breve{B}_2v(t)+\breve{f}_2(t),\end{cases} \tag{15}$$

where $z_1(t) \in \mathbb{R}^{n-m}$, $z_2(t) \in \mathbb{R}^m$. The following SM function is chosen:

$$s(t) = Cz(t) = K_2 z_1(t) + z_2(t), \tag{16}$$

where matrix K_2 will be designed.

Take derivative of $s(t)$ along with system (15). According to Lemma 2,

$$\begin{aligned} D^\alpha s(t) &= K_2 D^\alpha z_1(t) + D^\alpha z_2(t) \\ &= K_2(\breve{A}_{11} z_1(t) + \breve{A}_{12} z_2(t)) + \breve{A}_{21} z_1(t) + \breve{A}_{22} z_2(t) + \breve{B}_2 v(t) + \breve{f}_2(t) \\ &= (K_2 \breve{A}_{11} + \breve{A}_{21}) z_1(t) + (K_2 \breve{A}_{12} + \breve{A}_{22}) z_2(t) + \breve{B}_2 v(t) + \breve{f}_2(t) \\ &= \Phi(t) + \breve{B}_2 v(t) + \breve{f}_2(t), \end{aligned} \tag{17}$$

where $\Phi(t) = (K_2 \breve{A}_{11} + \breve{A}_{21}) z_1(t) + (K_2 \breve{A}_{12} + \breve{A}_{22}) z_2(t)$.

If $s = 0$, then $K_2 z_1(t) + z_2(t) = 0$. So SM dynamics equation can be given as

$$\begin{cases} D^\alpha z_1(t) = \breve{A}_{11} z_1(t) + \breve{A}_{12} z_2(t), \\ K_2 z_1(t) + z_2(t) = 0. \end{cases} \tag{18}$$

From the second equation of (18), one can obtain that $z_2(t) = -K_2 z_1(t)$. Then substituting it into the first equation of (18), we have

$$D^\alpha z_1(t) = (\breve{A}_{11} - \breve{A}_{12} K_2) z_1(t). \tag{19}$$

As soon as K_2 satisfies $|\arg(\mathrm{spec}(\breve{A}_{11} - \breve{A}_{12} K_2))| > \alpha \frac{\pi}{2}$, system (19) is stable by Definition 3. $\lim_{t\to\infty} z_1(t) = 0$ can then be guaranteed. In addition, according to the second part of equation (18), $\lim_{t\to\infty} z_2(t) = 0$ can also be obtained. The upper part indicates that the SM motion is stable based on the appropriate K_2. However, the existence of such a K_2 and its selection process are not given. The following theorem is a criterion for system stability based on LMI.

Theorem 2. The system (19) is asymptotically stable, if a matrix K_2 can be found that satisfies

$$\mathrm{sym}(\breve{A}_{11} P - \breve{A}_{12} K_2 P) < 0, \tag{20}$$

where K_2 is an intermediate matrix. If one chooses $K_2 = -QP^{-1}$, then Eq. (20) can be rewritten as

$$\mathrm{sym}(\breve{A}_{11} P + \breve{A}_{12} Q) < 0, \tag{21}$$

where $P \in \mathbb{P}_\alpha^{(n-m)\times(n-m)}$ and $Q \in \mathbb{R}^{m\times(n-m)}$.

Proof. This conclusion can be easily obtained using a similar derivation to [9].

3.3 Design of sliding mode controller

Let $D^\alpha s = 0$. One can derive equivalent control

$$v_{\mathrm{eq}} = -\breve{B}_2^{-1}(\Phi(t) + \breve{f}_2); \tag{22}$$

furthermore, one chooses a switching control term

$$v_{\mathrm{sw}} = -\breve{B}_2^{-1}(\varepsilon_1 s + (\rho + \varepsilon_2)\|T(E + BK_1)^{-1}\| \mathrm{sign} s), \tag{23}$$

where ε_1, $\varepsilon_2 > 0$, sign is a signal function which is defined by

$$\mathrm{sign} s = \begin{cases} 1, & s > 0, \\ -1, & s < 0, \\ 0, & s = 0. \end{cases}$$

Theorem 3. Under Assumptions 2 and 3, if there exists an SM controller

$$v(t) = -\check{B}_2^{-1}(\varPhi(t) + \varepsilon_1 s + (\rho + \varepsilon_2)\|T(E + BK_1)^{-1}\|\text{sign}s), \tag{24}$$

then the SM surface can be reached in a finite time, where s is chosen by (16).

Proof. Let $V(t) = \frac{1}{2}s^{\mathrm{T}}s$. Using the conclusion of Lemma 1 in [36], one can take the α-order derivative of $V(t)$ along with system (15) under the SM controller $v(t)$ of (24):

$$\begin{aligned}
D^\alpha V &\leqslant s^{\mathrm{T}} D^\alpha s \\
&= s^{\mathrm{T}}(\varPhi(t) + \check{B}_2 v(t) + \check{f}_2(t)) \\
&= s^{\mathrm{T}}(\varPhi(t) + \check{B}_2(-\check{B}_2^{-1}(\varPhi(t) + \varepsilon_1 s + (\rho + \varepsilon_2)\|T(E + BK_1)^{-1}\|\text{sign}s)) + \check{f}_2(t)) \\
&= s^{\mathrm{T}}(-\varepsilon_1 s - (\rho + \varepsilon_2)\|T(E + BK_1)^{-1}\|\text{sign}s + \check{f}_2(t)) \\
&= -\varepsilon_1 s^{\mathrm{T}} s - (\rho + \varepsilon_2)\|T(E + BK_1)^{-1}\| s^{\mathrm{T}}\text{sign}s + s^{\mathrm{T}}\check{f}_2(t) \\
&< -(\rho + \varepsilon_2)\|T(E + BK_1)^{-1}\|\|s^{\mathrm{T}}\| + \|s^{\mathrm{T}}\|\|\check{f}_2\| \\
&\leqslant -\|T(E + BK_1)^{-1}\|\|s^{\mathrm{T}}\|(\rho - \|f\| + \varepsilon_2) \\
&\leqslant -\varepsilon_2\|T(E + BK_1)^{-1}\|\|s^{\mathrm{T}}\| \\
&= -\varepsilon_2\|T(E + BK_1)^{-1}\|\sqrt{2V},
\end{aligned}$$

where $\| \check{f}_2 \| \leqslant \| T(E + BK_1)^{-1} \| \| f \|$ can be obtained under Assumption 3.

According to the theory of SM, there exists an instant of time $T > 0$ such that $s(T) = 0$. By Definition 1, one can obtain

$$D^\alpha(\sqrt{2V}) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (\sqrt{2V})'(t-\tau)^{\alpha-1}\mathrm{d}\tau = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{V'}{2\sqrt{V}}(t-\tau)^{\alpha-1}\mathrm{d}\tau.$$

As $V(T) = 0$, when t is near T, $V(t)$ is close to 0. That is, $V(\tau) \geqslant V(t)$, when $0 < \tau < t$. This inequality becomes

$$D^\alpha(\sqrt{2V}) \leqslant \frac{1}{\Gamma(1-\alpha)}\frac{1}{2\sqrt{V}}\int_0^t V'(t-\tau)^{\alpha-1}\mathrm{d}\tau = \frac{1}{2\sqrt{V}}D^\alpha V.$$

Then, one obtains $D^\alpha\sqrt{2V} \leqslant \frac{1}{2\sqrt{V}}D^\alpha V < -\frac{\sqrt{2}\varepsilon_2}{2}\|T(E + BK_1)^{-1}\|$. Taking the integral from 0 to T for $D^\alpha\sqrt{2V} < -\frac{\sqrt{2}\varepsilon_2}{2}\|T(E + BK_1)^{-1}\|$ under Lemma 2,

$$\begin{aligned}
\sqrt{V(T)} - \sqrt{V(0)} &< -\frac{\sqrt{2}\varepsilon_2}{2\Gamma(\alpha)}\|T(E + BK_1)^{-1}\|\int_0^T (T-\tau)^{\alpha-1}\mathrm{d}\tau \\
&= \frac{\sqrt{2}\varepsilon_2}{2\alpha\Gamma(\alpha)}\|T(E + BK_1)^{-1}\|[(T-\tau)^\alpha]_0^T \\
&= -\frac{\sqrt{2}\varepsilon_2}{2\alpha\Gamma(\alpha)}\|T(E + BK_1)^{-1}\|T^\alpha.
\end{aligned}$$

Finally, one obtains $T < \sqrt[\alpha]{\frac{\sqrt{2V(0)}\alpha\Gamma(\alpha)}{\varepsilon_2\|T(E+BK_1)^{-1}\|}}$. This means that the SM surface can be reached in a finite time.

Finally, the main conclusion of this investigation can be summarized.

Remark 4. For SFOS (3), if the condition of $\text{rank}(E, B) = n$ is satisfied, then system (3) is normalizable. Theorem 1 and its remark give the methods of choosing matrix K_1. System (3) can be rewritten as (15) based on a reversible transformation T under the condition $\text{rank}(B) = m$. Next, one chooses the appropriate FOSM function and designs the SM controller to ensure the accessibility of SM. As such, when the Assumptions 2 and 3 are satisfied, one can design an SM controller to stabilize SFOS with matched uncertainties.

4 Numerical simulations

Consider the SMC problem of uncertain SFOS in (3) with parameters

$$E=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix},\quad A=\begin{bmatrix}1&1&0\\0&2&0\\0&0&-1\end{bmatrix},\quad B=\begin{bmatrix}0&1\\1&0\\0&1\end{bmatrix},\quad f=\begin{bmatrix}\sin wt\\\sin wt\\\sin wt\end{bmatrix}.$$

Let $\alpha=0.8$. $\mathrm{rank}(E)=\deg(\det(sE-A))=2$ is easily obtained, so system (3) with $f=0$ is regular, impulse-free and unstable. According to $\mathrm{rank}(E,B)=3$, system (3) with $f=0$ can be normalizable.

Choose $M_2=N_2=I_3$; then $K_1=\left[\begin{smallmatrix}0&1&0\\0&0&1\end{smallmatrix}\right]$ according to Theorem 1, and

$$(E+BK_1)^{-1}=\begin{bmatrix}1&0&-1\\0&0.5&0\\0&0&1\end{bmatrix}$$

can be obtained. Moreover,

$$\check{A}=\begin{bmatrix}1&1&1\\0&1&0\\0&0&-1\end{bmatrix},\quad \check{B}=\begin{bmatrix}0&0\\0.5&0\\0&1\end{bmatrix}.$$

Given a matrix $T=I_3$, system (12) can be decomposed by control separation:

$$\begin{cases}D^{0.8}z_1(t)=z_1(t)+[1,1]\,z_2(t),\\ D^{0.8}z_2(t)=\begin{bmatrix}1&0\\0&-1\end{bmatrix}z_2(t)+\begin{bmatrix}0.5&0\\0&1\end{bmatrix}v(t)+\begin{bmatrix}0.5\sin wt\\\sin wt\end{bmatrix}.\end{cases}\tag{25}$$

According to Theorem 2, choose $P=\sin 0.4\pi\in\mathbb{P}_{0.8}^{1\times1}$, $Q=\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$; then

$$K_2=-QP^{-1}=\begin{bmatrix}-\dfrac{1}{\sin 0.4\pi}\\-\dfrac{1}{\sin 0.4\pi}\end{bmatrix}.$$

Then define the sliding mode function as

$$s=\begin{bmatrix}-\dfrac{1}{\sin 0.4\pi}\\-\dfrac{1}{\sin 0.4\pi}\end{bmatrix}z_1(t)+z_2(t).$$

The SM equation can be obtained as

$$\begin{cases}D^{0.8}z_1(t)=z_1(t)+[1,1]\,z_2(t),\\ 0=\begin{bmatrix}-\dfrac{1}{\sin 0.4\pi}\\-\dfrac{1}{\sin 0.4\pi}\end{bmatrix}z_1(t)+z_2(t).\end{cases}\tag{26}$$

Finally, choose the SM controller

$$v(t)=-\begin{bmatrix}2&0\\0&1\end{bmatrix}(\varPhi(t)+0.5s+2\mathrm{sign}s),\tag{27}$$

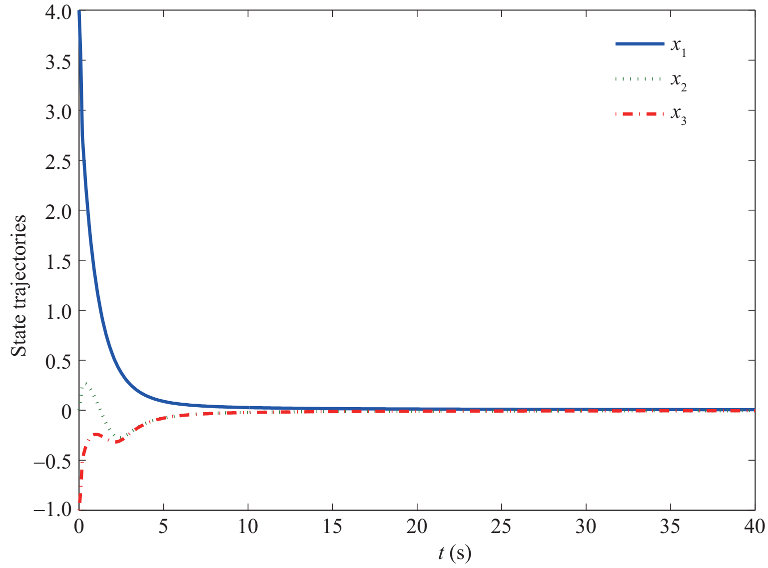

Figure 1 (Color online) State responses of SFOS.

where

$$\Phi(t)=\begin{bmatrix}-\dfrac{1}{\sin 0.4\pi}\\ -\dfrac{1}{\sin 0.4\pi}\end{bmatrix}z_1(t)+\begin{bmatrix}1-\dfrac{1}{\sin 0.4\pi} & -\dfrac{1}{\sin 0.4\pi}\\ -\dfrac{1}{\sin 0.4\pi} & -1-\dfrac{1}{\sin 0.4\pi}\end{bmatrix}z_2(t),$$

$\varepsilon=0.5$, $\rho=1$, $\|T(E+BK_1)^{-1}\|=2$. The closed system is stable. Figure 1 shows the system state trajectories and it is easy to observe that the system state responses converge to zero.

5 Conclusion

In this article, the SMC problem for a class of normalizable SFOS with external matching uncertainties was investigated. The necessary and sufficient conditions were obtained to guarantee normalization of the system. Feedback control with an FO derivative controller was designed. Linear form SMC for SFOS was initially investigated. These results will be extended to SMC of SFOS ($0<\alpha<2$) without normalization. In the future, asynchronous output feedback control based SMC for SFOS or switching SFOS will be investigated [37–39].

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