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Special Focus on Advanced Techniques for Event-Triggered Control and Estimation

Lyapunov-based event-triggered control for nonlinear plants subject to disturbances and transmission delays

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Abstract This paper studies event-triggered control for disturbed nonlinear systems. A new dual-stage Lyapunov-based event-triggering condition is proposed to cope with the time-varying transmission delays. In the first stage, the ratio of Lyapunov function values at the last two triggering instants is calculated. Then based on the ratio, the corresponding threshold function is selected from two candidate forms. It is proved that the designed event-triggered control system is input-to-state practically stable with respect to the measurement errors and disturbances. Moreover, Zeno behavior is excluded successfully by calculating the lower bound of the minimum inter-event times. Finally, a simulation example is provided to show the feasibility and the effectiveness of the proposed approach.

Keywords event-triggered control, nonlinear systems, networked control systems, disturbed systems, transmission delays

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1 Introduction

In traditional control systems, the control tasks are executed in time-triggered manner, which has been well-developed in theory. In this scheme, the sampling and updating of signals are implemented in a periodic way. However, time-triggered control may lead to wastes of communication and computation resources because the maximum allowable transfer interval (MATI) [1] is usually designed conservatively for guaranteeing the desired system performance in the worst scenarios. As an alternative way, the event-triggered sampling scheme has significant advantages in terms of reducing transmission rates and mitigating unnecessary resource consumption [2]. Under event-triggered control, control tasks are determined by checking a pre-designed event-triggering condition. Therefore, it plays the most important role to design a reasonable event-triggering condition in the study of event-triggered control.

Event-triggered control has been widely investigated for theoretical purposes since it was presented in [3]. It also has been applied to a lot of practical systems successfully, such as offshore platforms [4, 5] and delayed neural networks [6]. Generally, the occurrence of an event implies that the measurement error between the current signal and the last released one exceeds a predetermined threshold. Thus, the existing event-triggered schemes can be classified into many categories by the various forms of thresholds. The relative event-triggering condition [7] was presented for nonlinear systems that are input-to-state

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stable with respect to measurement errors. Event-triggered control [8] was combined with model-based control to study the stabilization of systems subject to quantization and time-varying network delays. The integral-based event-triggering condition which utilizes the integrals of the measurement signals was studied in [9]. Ref. [10] proved that the dynamic event-triggering condition with an internal dynamic variable was better than the static ones in [11]. Ref. [12] designed an event-triggering condition in which the threshold is adapted with the dynamic error of the system. In addition, self-triggered control and periodic event-triggered control [13] were also widely studied to overcome the drawbacks of continuous verification of events in [7]. Observed-based self-triggered control [14] was presented for certain and uncertain systems. Periodic event-triggered control [15] was designed and analyzed based on various models. Ref. [16] made a comparison between event- and self-triggered schemes and pointed out their respective advantages and disadvantages. A time-dependent event-triggering condition [17] was proposed for interconnected linear systems. Recalling the literature mentioned above, it was always assumed that the full state information is available. For going beyond this assumption, an output-based event-triggering condition for a class of nonlinear systems subject to disturbances and measurement noises was designed in [18]. In addition to the application in controller design, Refs. [19, 20] investigated the estimators and filters combined with the event-triggered communication schemes in wireless sensor networks, respectively. More specific and recent advances in event-triggered control were presented in [21–24].

Although event-triggered control can reduce the transmission rates, it also brings some difficulties in analysis and design. In event-triggered control, the time between two successive triggering instants, known as inter-event time [25], varies with the state trajectories. Thus, how to exclude Zeno behavior, that is, how to avoid an infinite number of samples occurring in a finite-time interval, is a challenging problem. Event-separation properties [25] were investigated both in the absence and presence of external disturbances or measurement noises. Zeno phenomenon [26, 27] was avoided by augmenting the event-triggering condition mixed with a time-triggering rule which is a uniform strictly positive lower bound on inter-event times. For several types of triggering conditions with a specific form, Ref. [28] provided some necessary and sufficient conditions for Zeno behavior. The other methodologies discussing Zeno behavior can be found in [29, 30].

In control theory, Lyapunov second method is one of the most fundamental and important tools to analyze the stability and performance. Because Lyapunov function can describe the variation of system states graphically, it is intuitional to choose Lyapunov function to construct the event-triggering condition. Only a few literature [31–33] investigated the events based on Lyapunov functions. Stable and unstable Lyapunov sampling [31] were both studied for the closed-loop systems. Threshold lines [32, 33] were proposed to describe the upper bound of Lyapunov function that is effected by the time delays. The simulation results of [32] indicated that this Lyapunov-based event-triggered mechanism has a much better transmission performance than the previous event- and self-triggered mechanisms. However, none of the above-mentioned literature investigated the effects of disturbances on Lyapunov-based event-triggered control. As one of the most common imperfections in many practical systems, external disturbances are unavoidable and may decrease or even deteriorate the stability of the considered systems. The effect of disturbances is always a problem worth studying in the analysis of the control and communication properties. For instance, in [34], the stability of coupled hyperbolic partial differential equation-ordinary differential equation systems with respect to external disturbances was investigated. Moreover, it was shown in [25] that the popular event-triggering condition of [7] may lead to Zeno behavior in presence of disturbances. Motivated by the discussion above, this paper focuses on an event-triggered feedback control issue for the nonlinear systems with disturbances and transmission delays.

The main contributions of this paper are summarized as follows. First, a novel Lyapunov-based event-triggering condition is proposed for the nonlinear systems that satisfy the input-to-state stability (ISS stability) with respect to measurement errors and disturbances. Based on the evolution of Lyapunov function values, two different threshold functions are designed to reduce transmission rates further. Second, two important time parameters, the maximum allowable transfer interval and the minimum inter-event time, are calculated by a classified discussion about the relationship between Lyapunov function and threshold function. In the meantime, by determining the upper bound of Lyapunov function during delays, we relax the restriction that its evolution needs decrease monotonically in the existing references. Especially, the negative correlation between disturbances and the minimum inter-event time is also verified. Finally, it is proved that this Lyapunov-based event-triggered control can ensure the practical stability and exclude the Zeno behavior. Moreover, by comparing with the existing references, it is proved that this dual-stage event-triggering condition has better communication performance.

The rest of this paper is organized as follows. The event-triggered feedback control system is formulated in Section 2. In Section 3, we discuss stabilization of systems affected by bounded disturbances and transmission delays. An illustrative example is given in Section 4 and the conclusion is presented in Section 5.

Notation. Let $\mathbb{R}$ ($\mathbb{R}^+$) be the set of real (positive) numbers. Euclidian norm of a vector $x \in \mathbb{R}^n$ is denoted by $\|x\|$. The space of all bounded signals of dimension n is represented by $\mathcal{L}_\infty^n$. A continuous function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be of class $\mathcal{K}$ if it is strictly increasing and satisfies $\eta(0) = 0$. A function $g : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz continuous if there exists a constant $L > 0$ such that $\|g(x) - g(y)\| \leqslant L\|x - y\|$ for all x, y and L is called a Lipschitz constant.

2 Problem statement

In this section, the event-triggered control with the state feedback configuration is studied, which is shown in Figure 1. Consider the following nonlinear system:

$$\begin{aligned}\dot{x}(t) &= g(x(t), u(t), d(t)),\\ u(t) &= \gamma(x(t)),\\ x(0) &= x_0,\end{aligned}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are the system's state vector and control input, respectively. $x_0 \in \mathbb{R}^n$ is the non-zero initial state. $d(t) \in \mathcal{L}_\infty^q$ is the unknown bounded disturbance with $\|d(t)\| \in [0, d_{\max}]$ for $t \in [0, \infty)$ and $d_{\max} \geqslant 0$. $g : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}^n$ and $\gamma : \mathbb{R}^n \to \mathbb{R}^m$ are supposed to be Lipschitz functions.

As shown in Figure 1, the solid lines represent the continuous-time signals and the dashed lines represent the discrete-time signals generated by the sampling. Under an event-triggered mechanism, the system state $x(t)$ is sampled only when some event-triggering conditions are violated. In this paper, r_k denotes the instant when the k-th event occurs. Because the transmission delays are inevitable in many practical implementations, we consider the impact of delays on the systems. Let f_k represent the instant when the sampled signal $x(r_k)$ arrives at the controller. Once the controller side receives a data packet or a signal, it will send back a one-bit ACK signal [35] to the trigger. Generally, this kind of ACK signals do not take up too much transmission resources because they contain very little data, and thus, it is assumed that the transmission delay of ACK can be negligible. By this way, the trigger begins judging the event-triggering condition at f_k, i.e., $r_{k+1} > f_k$. The state feedback controller is implemented in a zero-order-hold (ZOH) manner, i.e.,

$$u(t) = \gamma(x(r_k)), \quad t \in [f_k, f_{k+1}).$$

Without loss of generality, we assume $k = 0, \ldots, \infty$ and $r_0 = f_0 = 0$. Let $T_k = r_{k+1} - r_k$ denote the k-th interval time between two consecutive triggering instants, and $D_k = f_k - r_k$ denote the k-th transmission delay. Generally, it is bounded. Thus, we make Δ represent the maximum allowed transmission delay.

By defining the measurement error as $e_k(t) = x(r_k) - x(t)$ for $t \in [f_k, f_{k+1})$, the controller can be given by $u(t) = \gamma(x(t) + e_k(t))$. Thus, the closed-loop system with the event-triggered controller becomes

$$\dot{x}(t) = g(x(t), \gamma(x(t) + e_k(t)), d(t)), \quad t \in [f_k, f_{k+1}). \tag{1}$$

Most existing studies on event-triggered controllers are based on the emulation-based method [36]. To be specific, it is assumed that a stabilizing feedback law is constructed in continuous time firstly. Then

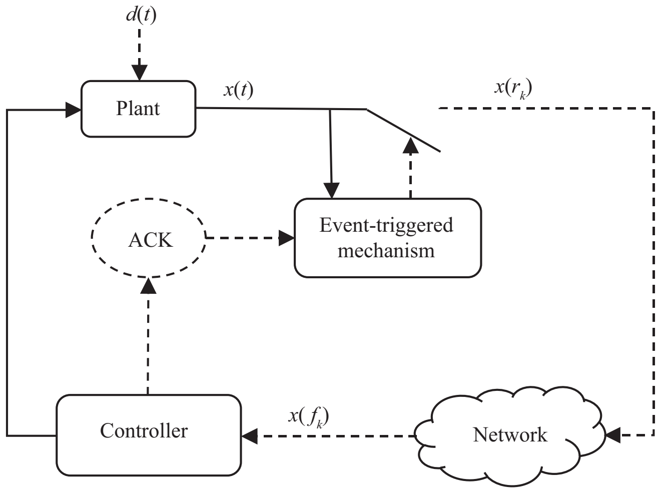

Figure 1 Configuration of the event-triggered control with disturbances and transmission delays.

an event-triggering condition is designed to preserve relevant stability under the influence of sampling. According to [18, 32, 33, 37], the controller gain function γ is assumed to be designed rendering the continuous closed-loop system (1) input-to-state stable with respect to the measurement error $e_k(t)$ and disturbance $d(t)$. Accordingly, we propose the following assumption.

Assumption 1. For the continuous closed-loop system (1), there exist positive constants $L, \alpha, \beta, \mu, L_1$, a positive definite, $\mathbb{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}^+$, and $\alpha_1, \alpha_2 \in \mathcal{K}$ such that

$$\|g(x, \gamma(x+e), d)\| \leqslant L\|x\| + L\|e\| + L\|d\|, \tag{2}$$

$$\alpha_1(\|x\|) \leqslant V(x(t)) \leqslant \alpha_2(\|x\|), \tag{3}$$

$$\frac{\partial V(x)}{\partial x} f(x, \gamma(x+e), d) \leqslant -\alpha V(x) + \beta\|e\| + \mu\|d\|, \tag{4}$$

$$\alpha_1^{-1}(\|x\|) \leqslant L_1\|x\| \tag{5}$$

hold for all x, e, d.

Based on the above assumption, Ref. [7] proposed the following relative event-triggering condition:

$$\beta(\|e(t)\|) \leqslant \sigma\alpha(\|x(t)\|),$$

where $0 < \sigma < 1$. Under this event-triggering condition, an event occurs when the ratio of the measurement error $\|e(t)\|$ to the state $\|x(t)\|$ exceeds a predetermined threshold. However, the positive minimum inter-event time cannot be guaranteed in the presence of disturbances generally. To solve this problem, the relative event-triggering condition is modified by adding a positive constant to the threshold [25] or adding a time-trigger value to the inter-event time [26]. On the other hand, although the Lyapunov-based event-triggering condition in [32] can yield better triggering performance than the relative one, it cannot be applied directly to the systems with external disturbances. Therefore, we expect to obtain better sampling performance by designing a new Lyapunov-based event-triggering strategy for the systems subject to disturbances. Specifically, this paper considers the following Lyapunov-based event-triggering condition:

$$r_{k+1} = \inf\{t > f_k | V(t) > \phi(V(r_k), V(r_{k-1}), t)\}, \tag{6}$$

where $\phi : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is the threshold function and $V(\cdot) = V(x(\cdot))$.

Remark 1. The reason why we make $t > f_k$ in (6) is to guarantee that neither the sequence of sampling nor the sequence of executing is out of order, i.e., the next triggering instant r_{k+1} occurs after the last transmitted signal has arrived at the controller. Thus, this event-triggering condition can satisfy

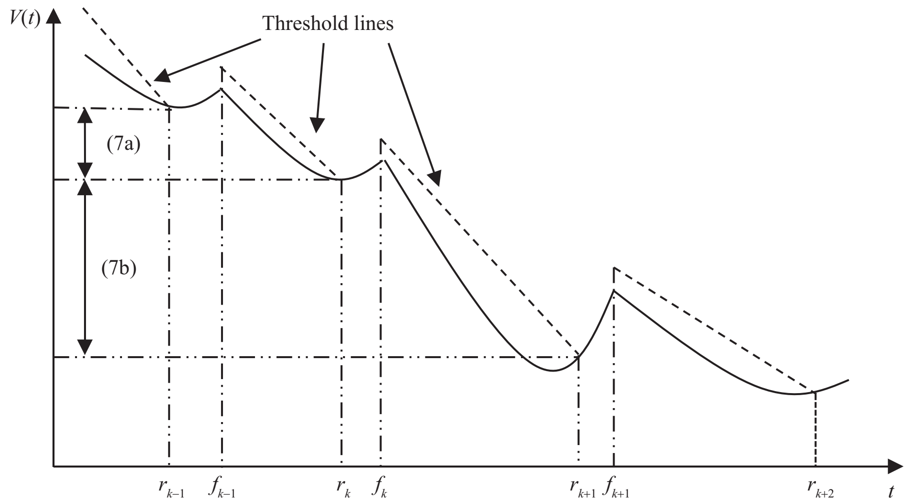

Figure 2 Two cases in the trajectory of $V(t)$.

the requirement $r_{k+1} > f_k$. It also implies that the event-triggering condition (6) do not need to be monitored continuously during delays (r_k, f_k).

The main purpose of this paper is to design the threshold function in event-triggering strategy to ensure the practical stability of the event-triggered closed-loop systems in the presence of disturbances and transmission delays. Moreover, the lower bound of the inter-event times needs to be positive to exclude Zeno behavior. The definition of practical stability can be described as follows [38].

Definition 1. Consider the differential system $\dot{x} = g(t, x), x(t_0) = x_0, t_0 \geqslant 0$, where $g : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$. The system is said to be practically stable if, given (c, M) with $0 < c < M$, we have $\|x_0\| < c$ implies $\|x(t)\| < M, t \geqslant t_0$ for some $t_0 \in \mathbb{R}^+$.

3 Main results

In this section, the threshold function ϕ in (6) will be designed to ensure the practical stability of event-triggered feedback system (1) with bounded disturbances and non-zero delays. In general, there is no system that can operate normally with infinite transmission delays. Therefore, the maximum allowable transmission delay needs to be determined. Firstly, owing to the existence of delays, the values of $V(t)$ over $t \in [r_k, f_k)$ must be bounded. Meanwhile, the time interval between f_k and r_{k+1} should be always bounded form below by a positive constant.

To reduce transmission costs while guaranteeing the practical stability of system (1), we propose the following dual-stage Lyapunov-based threshold function:

$$\phi(V(r_k), V(r_{k-1}), t) = \begin{cases} V(r_k)(1 + \rho\Delta - \delta\alpha(t - f_k)) + C\Delta + C_1, & \text{if } V(r_{k-1}) \leqslant \lambda V(r_k); \quad \text{(7a)} \\ V(r_{k-1}) \left(\dfrac{1}{\lambda} + \rho'\Delta - \delta\alpha(t - f_k)\right) + C\Delta + C_1, & \text{if } V(r_{k-1}) > \lambda V(r_k). \quad \text{(7b)} \end{cases}$$

Δ represents the maximum allowed transmission delay. $\rho > \beta L_1(1 + \lambda), \rho' = \frac{1}{\lambda}\rho, C > \mu d_{\max}$, where β, μ, L_1 are defined in Assumption 1. $\lambda > 1, \delta \in (0, 1), C_1 > 0$ and α are the parameters to be designed.

For the further illustration, Figure 2 provides an example of $V(t)$ and the threshold function ϕ in (7), where the continuous solid curve and the piecewise continuous dashed lines are the trajectory of $V(t)$ and the threshold lines under event-triggered mechanism (6), respectively. Eqs. (7a) and (7b) represent two different relationships between the values of Lyapunov functions at the last two triggering instants, i.e., $V(r_{k-1}) \leqslant \lambda V(r_k)$ and $V(r_{k-1}) > \lambda V(r_k)$. And they are used as the judgments for the selection of threshold functions. When the trajectory of $V(t)$ intersects with the triggered threshold lines, the event occurs and the measurement signal is released and sent to the controller.

Based on the above analysis, the following lemma shows the upper bound of $V(t)$ for $t \in [r_k, f_k)$.

Lemma 1. Consider the event-triggered feedback system (1) satisfying Assumption 1. The delay D_k satisfies $D_k \leqslant \Delta = \min\{\Delta_1, \Delta_2\}$, and Δ_1, Δ_2 are the smallest positive solutions to the following equations:

$$1 + \frac{\beta L_1(1+\lambda)}{\alpha} \mathrm{e}^{2L\Delta_1}(1-\mathrm{e}^{-\alpha\Delta_1}) + \frac{\beta L_1 \lambda}{2\alpha}(\mathrm{e}^{2L\Delta_1} - 1)(1-\mathrm{e}^{-\alpha\Delta_1}) = 1 + \rho\Delta_1, \tag{8}$$

$$\frac{\beta d_{\max}}{2\alpha}(\mathrm{e}^{2L\Delta_2} - 1)(1-\mathrm{e}^{-\alpha\Delta_2}) + \frac{\mu d_{\max}}{\alpha}(1-\mathrm{e}^{-\alpha\Delta_2}) = C\Delta_2, \tag{9}$$

respectively. Then the upper bound of $V(t)$ for all $t \in [r_k, f_k)$ can be determined by the following inequalities:

$$V(t) \leqslant \begin{cases} V(r_k)(1+\rho D_k) + CD_k, & \text{if } V(r_{k-1}) \leqslant \lambda V(r_k); \\ V(r_{k-1})(\frac{1}{\lambda} + \rho' D_k) + CD_k, & \text{if } V(r_{k-1}) > \lambda V(r_k). \end{cases} \tag{10}$$

Proof. According to the definitions of e_k and r_k, the measurement error $e_{k-1}(t)$ is calculated by $x(r_{k-1})$, i.e., $e_{k-1}(t) = x(t) - x(r_{k-1})$. First, determine the upper bound of the $\|e_{k-1}(t)\|$ over $t \in [r_k, f_k)$. According to (2) in Assumption 1,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|e_{k-1}(t)\| &\leqslant \|\dot{e}_{k-1}(t)\| \\ &\leqslant L\|x(t)\| + L\|e_{k-1}(t)\| + L\|d(t)\| \\ &\leqslant 2L\|e_{k-1}(t)\| + L\|x(r_{k-1})\| + L\|d(t)\|, \end{aligned}$$

for all $t \in [r_k, f_k)$. By integrating the above differential inequality with the initial condition $\|e_{k-1}(r_k)\| \neq 0$, it follows that

$$\|e_{k-1}(t)\| \leqslant \mathrm{e}^{2LD_k}\|e_{k-1}(r_k)\| + \frac{\|x(r_{k-1})\| + d_{\max}}{2}(\mathrm{e}^{2LD_k} - 1), \tag{11}$$

for $t \in [r_k, f_k)$. Then, by combining (11) and (4), we obtain

$$\begin{aligned} \dot{V}(t) &\leqslant -\alpha V(t) + \beta\|e_{k-1}(t)\| + \mu\|d(t)\| \\ &\leqslant -\alpha V(t) + \beta\|e_{k-1}(r_k)\|\mathrm{e}^{2LD_k} + \beta\frac{\|x(r_{k-1})\| + d_{\max}}{2}(\mathrm{e}^{2LD_k} - 1) + \mu\|d(t)\|, \end{aligned}$$

for $t \in [r_k, f_k)$. By integrating this differential inequality with the initial condition $V(r_k)$, it holds that

$$\begin{aligned} V(t) \leqslant{}& \mathrm{e}^{-\alpha(t-r_k)}V(r_k) - \frac{\beta\|e_{k-1}(r_k)\|}{\alpha}\mathrm{e}^{2LD_k}(\mathrm{e}^{-\alpha(t-r_k)} - 1) \\ &- \frac{\beta(\|x(r_{k-1})\| + d_{\max})}{2\alpha}(\mathrm{e}^{2LD_k} - 1)(\mathrm{e}^{-\alpha(t-r_k)} - 1) - \frac{\mu d_{\max}}{\alpha}(\mathrm{e}^{-\alpha(t-r_k)} - 1), \end{aligned} \tag{12}$$

for $t \in [r_k, f_k)$. According to (3) and (5) in Assumption 1, we can derive

$$\|x(r_k)\| \leqslant \alpha_1^{-1}(V(r_k)) \leqslant L_1 V(r_k). \tag{13}$$

Then the rest analysis is made in two cases.

Case 1. $V(r_{k-1}) \leqslant \lambda V(r_k)$. By applying (13) and $\|e_{k-1}(r_k)\| \leqslant \|x(r_k)\| + \|x(r_{k-1})\|$ into (12), we have

$$\begin{aligned} V(t) \leqslant{}& V(r_k) + \frac{\beta L_1(1+\lambda)V(r_k)}{\alpha}\mathrm{e}^{2LD_k}(1-\mathrm{e}^{-\alpha D_k}) \\ &+ \frac{\beta L_1 \lambda V(r_k) + \beta d_{\max}}{2\alpha}(\mathrm{e}^{2LD_k} - 1)(1-\mathrm{e}^{-\alpha D_k}) + \frac{\mu d_{\max}}{\alpha}(1-\mathrm{e}^{-\alpha D_k}) \\ \triangleq{}& V(r_k)p(D_k), \end{aligned}$$

for $t \in [r_k, f_k)$, where $p(D_k) = 1+\frac{\beta L_1(1+\lambda)}{\alpha}\mathrm{e}^{2LD_k}(1-\mathrm{e}^{-\alpha D_k})+\frac{\beta L_1\lambda}{2\alpha}(\mathrm{e}^{2LD_k}-1)(1-\mathrm{e}^{-\alpha D_k})+\frac{\beta d_{\max}}{2\alpha V(r_k)}(\mathrm{e}^{2LD_k}-1)(1-\mathrm{e}^{-\alpha D_k}) + \frac{\mu d_{\max}}{\alpha V(r_k)}(1-\mathrm{e}^{-\alpha D_k})$. Let $q(D_k) \triangleq 1 + \rho D_k + \frac{CD_k}{V(r_k)}$. Obviously, $p(0) = q(0) = 1$, $\dot{p}(0) = \beta L_1(1+\lambda) + \frac{\mu d_{\max}}{V(r_k)}$, $\dot{q}(0) > \dot{p}(0)$.

Owing to the existence of the unknown positive value $V(r_k)$ in $p(D_k)$ and $q(D_k)$, we further have the following three subcases.

Subcase (a). $V(r_k)\to\infty$. Δ_1 is the minimum positive solution of $\lim_{V(r_k)\to\infty}p(t)=\lim_{V(r_k)\to\infty}q(t)$, which corresponds to (8). Because Eq. (8) is a transcendental equation, the analytic solution is not available generally. The approximate numerical value of Δ_1 can be solved graphically. By the continuity of p,q and $D_k\leqslant\Delta_1$, the inequality $p(D_k)\leqslant q(D_k)$ holds.

Subcase (b). $V(r_k)\to 0$. Δ_2 is the minimum positive solution of $\lim_{V(r_k)\to 0}p(t)=\lim_{V(r_k)\to 0}q(t)$, which is given by (9). By the continuity of p,q and $D_k\leqslant\Delta_2$, the inequality $p(D_k)\leqslant q(D_k)$ holds.

Subcase (c). $V(r_k)$ is an arbitrary constants belonging to $(0,\infty)$. Based on Subcase (a), Subcase (b) and $D_k\leqslant\min\{\Delta_1,\Delta_2\}$, the inequality $p(D_k)\leqslant q(D_k)$, i.e.,

$$\begin{aligned}1&+\frac{\beta L_1(1+\lambda)}{\alpha}\mathrm{e}^{2LD_k}(1-\mathrm{e}^{-\alpha D_k})+\frac{\beta L_1\lambda}{2\alpha}(\mathrm{e}^{2LD_k}-1)(1-\mathrm{e}^{-\alpha D_k})\\&+\frac{\beta d_{\max}}{2\alpha V(r_k)}(\mathrm{e}^{2LD_k}-1)(1-\mathrm{e}^{-\alpha D_k})+\frac{\mu d_{\max}}{\alpha V(r_k)}(1-\mathrm{e}^{-\alpha D_k})\leqslant 1+\rho D_k+\frac{CD_k}{V(r_k)}\end{aligned}$$

holds.

Thus, when $V(r_{k-1})\leqslant\lambda V(r_k)$, we can obtain that $V(t)\leqslant V(r_k)q(D_k)=V(r_k)(1+\rho D_k)+CD_k$ always holds for $t\in[r_k,f_k)$.

Case 2. $V(r_{k-1})>\lambda V(r_k)$. By applying (13) and $\|e_{k-1}(r_k)\|\leqslant\|x(r_k)\|+\|x(r_{k-1})\|$ into (12), it follows that

$$\begin{aligned}V(t)&\leqslant\frac{1}{\lambda}V(r_{k-1})+\frac{\beta L_1(1+\frac{1}{\lambda})V(r_{k-1})}{\alpha}\mathrm{e}^{2LD_k}(1-\mathrm{e}^{-\alpha D_k})\\&\quad+\frac{\beta L_1V(r_{k-1})+\beta d_{\max}}{2\alpha}(\mathrm{e}^{2LD_k}-1)(1-\mathrm{e}^{-\alpha D_k})+\frac{\mu d_{\max}}{\alpha}(1-\mathrm{e}^{-\alpha D_k})\\&\triangleq V(r_{k-1})p'(D_k),\end{aligned}$$

for $t\in[r_k,f_k)$. Let $q'(D_k)\triangleq\frac{1}{\lambda}+\rho'D_k+\frac{CD_k}{V(r_{k-1})}$. Obviously, $p'(0)=q'(0)=\frac{1}{\lambda},\dot{p}'(0)=\frac{1}{\lambda}\beta L_1(1+\lambda)+\frac{\mu d_{\max}}{V(r_{k-1})},\dot{q}'(0)>\dot{p}'(0)$.

Similarly, we also consider the three subcases for the value of $V(r_{k-1})$. In the first subcase, Δ_1' is the minimum positive solution of $\lim_{V(r_{k-1})\to\infty}p(t)=\lim_{V(r_{k-1})\to\infty}q(t)$. It can be easily verified that $\Delta_1'=\Delta_1$. By the continuity of p,q and $D_k\leqslant\Delta_1$, the inequality $p'(D_k)\leqslant q'(D_k)$ holds. And then by using similar technique to the last two subcases, we can conclude that when $D_k\leqslant\min\{\Delta_1,\Delta_2\},p'(D_k)\leqslant q'(D_k)$ also holds.

Thus, when $V(r_{k-1})>\lambda V(r_k)$, it holds that $V(t)\leqslant V(r_{k-1})q'(D_k)=V(r_{k-1})(\frac{1}{\lambda}+\rho'D_k)+CD_k$ for $t\in[r_k,f_k)$.

Based on the above two cases, inequalities (10) hold for all $t\in[r_k,f_k)$, which completes the proof.

Then, we propose the following lemma to guarantee that the time between f_k and the triggering instant r_{k+1} is bounded from below by a positive constant depending on time delays D_k.

Lemma 2. Under Assumption 1, if the delay D_k satisfies $D_k\leqslant\Delta=\min\{\Delta_1,\Delta_2,\Delta_3\}$, where Δ_3 is the smallest positive solution to the following equation:

$$\beta L_1\frac{2+\lambda}{2}(\mathrm{e}^{2L\Delta_3}-1)-\alpha(1+\rho\Delta_3-\delta)=0,$$

then $r_{k+1}-f_k\geqslant\xi_d(\Delta)>0$ holds, where r_{k+1} is the instant triggered by the violation of the event-triggering condition (6).

Proof. First, consider the derivative of $\|e_k(t)\|$ for all $t\in[r_k,f_k)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|e_k(t)\|\leqslant 2L\|e_k(t)\|+L\|x(r_k)\|+L\|e_{k-1}(r_k)\|+L\|d(t)\|.$$

By integrating this differential inequality with the initial condition $e_k(r_k)=0$, it holds that

$$\|e_k(t)\|\leqslant\frac{\|x(r_k)\|+\|e_{k-1}(r_k)\|+d_{\max}}{2}(\mathrm{e}^{2LD_k}-1),$$

for $t \in [r_k, f_k)$. Then, consider the derivative of $\|e_k(t)\|$ for all $t \in [f_k, r_{k+1})$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|e_k(t)\| \leqslant 2L\|e_k(t)\| + L\|x(r_k)\| + L\|d(t)\|.$$

Similarly, by using the initial state $\|e_k(f_k)\|$, it yields that

$$\begin{aligned}\|e_k(t)\| \leqslant& \frac{\|x(r_k)\| + \|e_{k-1}(r_k)\| + d_{\max}}{2}(\mathrm{e}^{2LD_k} - 1)\mathrm{e}^{2L(r_{k+1}-f_k)} \\ &+ \frac{\|x(r_k)\| + d_{\max}}{2}(\mathrm{e}^{2L(r_{k+1}-f_k)} - 1),\end{aligned} \tag{14}$$

for $t \in [f_k, r_{k+1})$. By combining (4) and (14), we obtain that

$$\begin{aligned}\dot{V}(t) \leqslant& -\alpha V(t) + \beta\|e_k(t)\| + \mu\|d(t)\| \\ \leqslant& -\alpha V(t) + \beta\frac{\|x(r_k)\| + \|e_{k-1}(r_k)\|}{2}(\mathrm{e}^{2LD_k} - 1)\mathrm{e}^{2L(r_{k+1}-f_k)} + \beta\frac{\|x(r_k)\|}{2}(\mathrm{e}^{2L(r_{k+1}-f_k)} - 1) \\ &+ \frac{\beta d_{\max}}{2}(\mathrm{e}^{2LD_k} - 1)\mathrm{e}^{2L(r_{k+1}-f_k)} + \frac{\beta d_{\max}}{2}(\mathrm{e}^{2L(r_{k+1}-f_k)} - 1) + \mu d_{\max},\end{aligned} \tag{15}$$

for $t \in [f_k, r_{k+1})$. For the sake of convenience, define $\Theta(\tau, \Delta) = \frac{\beta d_{\max}}{2}[(\mathrm{e}^{2L\Delta} - 1)\mathrm{e}^{2L\tau} + \mathrm{e}^{2L\tau} - 1 + \frac{2\mu}{\beta}]$ and $g(\tau, \Delta) = \beta L_1[\frac{2+\lambda}{2}(\mathrm{e}^{2L\Delta} - 1)\mathrm{e}^{2L\tau} + \frac{1}{2}(\mathrm{e}^{2L\tau} - 1)]$. By applying (13) and $\|e_{k-1}(r_k)\| \leqslant \|x(r_k)\| + \|x(r_{k-1})\|$ into (15), one has that

$$\begin{aligned}\dot{V}(t) \leqslant& -\alpha V(t) + \frac{\beta(2L_1V(r_k) + L_1V(r_{k-1}) + d_{\max})}{2}(\mathrm{e}^{2LD_k} - 1)\mathrm{e}^{2L(r_{k+1}-f_k)} \\ &+ \frac{\beta L_1V(r_k) + d_{\max}}{2}(\mathrm{e}^{2L(r_{k+1}-f_k)} - 1) + \mu d_{\max} \\ \leqslant& -\alpha V(t) + \left[\beta L_1(\mathrm{e}^{2LD_k} - 1)\mathrm{e}^{2L(r_{k+1}-f_k)} + \frac{\beta L_1}{2}(\mathrm{e}^{2L(r_{k+1}-f_k)} - 1)\right]V(r_k) \\ &+ \frac{\beta L_1}{2}(\mathrm{e}^{2LD_k} - 1)\mathrm{e}^{2L(r_{k+1}-f_k)}V(r_{k-1}) + \Theta(r_{k+1} - f_k, D_k),\end{aligned}$$

for $t \in [f_k, r_{k+1})$. By integrating this differential inequality with the initial condition $V(f_k)$, it holds that

$$\begin{aligned}V(r_{k+1}) \leqslant& \mathrm{e}^{-\alpha(r_{k+1}-f_k)}V(f_k) - \frac{g(r_{k+1} - f_k, D_k)}{\alpha}V(r_k)(\mathrm{e}^{-\alpha(r_{k+1}-f_k)} - 1) \\ &- \frac{\Theta(r_{k+1} - f_k, D_k)}{\alpha}(\mathrm{e}^{-\alpha(r_{k+1}-f_k)} - 1).\end{aligned} \tag{16}$$

Then, the rest analysis is made in two cases.

Case 1. $V(r_{k-1}) \leqslant \lambda V(r_k)$. From Lemma 1, we know that $V(f_k) \leqslant V(r_k)(1 + \rho\Delta) + C\Delta$. Applying this inequality into (16) provides

$$\begin{aligned}V(r_{k+1}) \leqslant& V(r_k)\left[(1 + \rho\Delta)\mathrm{e}^{-\alpha(r_{k+1}-f_k)} - \frac{g(r_{k+1} - f_k, D_k)}{\alpha}(\mathrm{e}^{-\alpha(r_{k+1}-f_k)} - 1)\right. \\ &\left.+\frac{C\Delta}{V(r_k)}\mathrm{e}^{-\alpha(r_{k+1}-f_k)} - \frac{\Theta(r_{k+1} - f_k, D_k)}{\alpha V(r_k)}(\mathrm{e}^{-\alpha(r_{k+1}-f_k)} - 1)\right] \\ \triangleq& V(r_k)q(r_{k+1} - f_k).\end{aligned}$$

Based on (7a), let $p(r_{k+1} - f_k) = 1 + \rho\Delta - \delta\alpha(r_{k+1} - f_k) + \frac{C\Delta + C_1}{V(r_k)}$. Obviously, $p(r_{k+1} - f_k) \leqslant q(r_{k+1} - f_k)$, $q(0) = 1 + \rho\Delta + \frac{C\Delta}{V(r_k)}$, $\dot{q}(0) = \beta L_1\frac{2+\lambda}{2}(\mathrm{e}^{2L\Delta} - 1) - \alpha(1 + \rho\Delta) + \frac{\beta d_{\max}}{2V(r_k)}(\mathrm{e}^{2L\Delta} - 1 + \frac{2\mu}{\beta})$.

Then, we consider the following three subcases.

Subcase (a). $V(r_k) \to \infty$. $\lim_{V(r_k)\to\infty} p(0) = \lim_{V(r_k)\to\infty} q(0)$. By the continuity of $\dot{p}, \dot{q}$, $\Delta \leqslant \Delta_3$ implies $\dot{p}(0) > \dot{q}(0)$. Then by the continuity of p, q, $\xi_1(\Delta)$ is the minimum positive solution of $\lim_{V(r_k)\to\infty} p(r_{k+1} - f_k) = \lim_{V(r_k)\to\infty} q(r_{k+1} - f_k)$, i.e., $1 + \rho\Delta - \delta\alpha\xi_1(\Delta) = (1 + \rho\Delta)\mathrm{e}^{-\alpha\xi_1(\Delta)} - \frac{g(\xi_1(\Delta), D_k)}{\alpha}(\mathrm{e}^{-\alpha\xi_1(\Delta)} - 1)$.

Subcase (b). $V(r_k) \to 0$. $p(0) > q(0)$. $\xi_2(\Delta)$ is the minimum positive solution of $\lim_{V(r_k)\to 0} p(r_{k+1} - f_k) = \lim_{V(r_k)\to 0} q(r_{k+1} - f_k)$, i.e., $C\Delta + C_1 = \frac{\Theta(\xi_2(\Delta), D_k)}{\alpha}(1 - \mathrm{e}^{-\alpha\xi_2(\Delta)}) + C\Delta\mathrm{e}^{-\alpha\xi_2(\Delta)}$.

Subcase (c). $V(r_k)$ is an arbitrary constant belonging to $(0, \infty)$. $\xi_3(\Delta)$ is the minimum positive solution of $p(t) = q(t)$.

Combine the above three subcases and define $\xi_d(\Delta) = \min\{\xi_1(\Delta), \xi_2(\Delta), \xi_3(\Delta)\}$. The time interval between f_k and r_{k+1} is bounded from below by a positive constant $\xi_d(\Delta)$.

Case 2. $V(r_{k-1}) > \lambda V(r_k)$. From Lemma 1, we know that $V(f_k) \leqslant V(r_{k-1})(\frac{1}{\lambda} + \rho'\Delta) + C\Delta$. Applying this inequality into (16) provides

$$\begin{aligned} V(r_{k+1}) \leqslant V(r_{k-1}) \bigg[&(\frac{1}{\lambda} + \rho'\Delta)\mathrm{e}^{-\alpha(r_{k+1}-f_k)} - \frac{g(r_{k+1} - f_k, D_k)}{\lambda\alpha}(\mathrm{e}^{-\alpha(r_{k+1}-f_k)} - 1) \\ &+ \frac{C\Delta}{V(r_{k-1})}\mathrm{e}^{-\alpha(r_{k+1}-f_k)} - \frac{\Theta(r_{k+1} - f_k, D_k)}{\alpha V(r_{k-1})}(\mathrm{e}^{-\alpha(r_{k+1}-f_k)} - 1)\bigg] \\ \triangleq\ & V(r_{k-1})q'(r_{k+1} - f_k). \end{aligned}$$

Based on (7b), let $p'(r_{k+1}-f_k) = \frac{1}{\lambda} + \rho'\Delta - \delta\alpha(r_{k+1}-f_k) + \frac{C\Delta + C_1}{V(r_{k-1})}$. Obviously, $p'(r_{k+1}-f_k) \leqslant q'(r_{k+1}-f_k)$ holds.

Using the similar technical method of Case 1, we obtain that the inequality $r_{k+1} - f_k \geqslant \xi_d(\Delta) > 0$ holds.

Remark 2. Different from the proposed dual-stage event-triggering condition (6), $V(r_{k-1}) \leqslant \lambda V(r_k)$ is a precondition of the event-triggered control in [32]. That is, the event would be immediately triggered if $\lambda V(t) = V(r_k)$. Clearly, the mechanism in [32] would cause a number of unnecessary events because the case $\lambda V(t) \leqslant V(r_k)$ is beneficial to guarantee the desirable system performance. In this paper, $V(r_{k-1}) \leqslant \lambda V(r_k)$ and $V(r_{k-1}) > \lambda V(r_k)$ are just considered as judgement conditions to decide which kind of threshold functions should be used. Thus, this approach can avoid efficiently transmitting all the signals that satisfy $V(r_k) \geqslant \lambda V(t)$.

Remark 3. Observe the calculation of $\xi_d(\Delta)$, when the external disturbance $d_{\max}$ increases, $p(r_{k+1} - f_k)$ remains unchanged and $q(r_{k+1} - f_k)$ increases. This will lead to an decrease in the values of $\xi_1(\Delta), \xi_2(\Delta), \xi_3(\Delta)$. Thus, the minimum time-interval $\xi_d(\Delta)$ is a positive constant related to the external disturbance $d_{\max}$ negatively.

Theorem 1. Under Assumption 1, if the delay D_k satisfies $D_k \leqslant \Delta = \min\{\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5\}$, where Δ_4 and Δ_5 are, respectively, the smallest positive solutions to

$$\begin{aligned} \rho\Delta_4 - \xi_d(\Delta_4)\delta\alpha = 0, \\ \rho'\Delta_5 - \xi_d(\Delta_5)\delta\alpha = 0, \end{aligned} \tag{17}$$

with ξ_d defined in Lemma 2, or Δ_4 and Δ_5 can be arbitrary large positive constants when there are no positive solutions in (17), then the event-triggered system is practically stable under the event-triggering condition (6). In addition, the ultimate bound of $\|x(t)\|$ satisfies

$$\|x(t)\| \leqslant \alpha_1^{-1}(\max\{M_1, M_2\}),$$

where $M_1 = (1 + \rho\Delta)[(1 + \rho\Delta - \delta\alpha\xi_d(\Delta))\frac{C\Delta + C_1}{\delta\alpha\xi_d(\Delta) - \rho\Delta} + C\Delta + C_1] + C\Delta + C_1$, $M_2 = (1 + \rho\Delta)[(\frac{1}{\lambda} + \rho'\Delta - \delta\alpha\xi_d(\Delta))\frac{(C\Delta + C_1)\lambda}{\delta\alpha\xi_d(\Delta) - \rho\Delta} + C\Delta + C_1] + C\Delta + C_1$.

Proof. Construct a piecewise continuous function $h(t)$ in the following way:

$$h(t) = \begin{cases} h_1(t), & \text{if } V(r_{k-1}) \leqslant \lambda V(r_k), \\ h_2(t), & \text{if } V(r_{k-1}) > \lambda V(r_k), \end{cases}$$

where

$$h_1(t) = \begin{cases} (1 + \rho\Delta - \delta\alpha(t - f_k))V(r_k) + C\Delta + C_1, & t \in [f_k, r_{k+1}), \\ (1 + \rho\Delta)V(r_{k+1}) + C\Delta + C_1, & t \in [r_{k+1}, f_{k+1}), \end{cases}$$

and

$$h_2(t) = \begin{cases} (\frac{1}{\lambda} + \rho'\Delta - \delta\alpha(t - f_k))V(r_{k-1}) + C\Delta + C_1, & t \in [f_k, r_{k+1}), \\ (\frac{1}{\lambda} + \rho'\Delta)V(r_k) + C\Delta + C_1, & t \in [r_{k+1}, f_{k+1}). \end{cases}$$

Based on the event-triggering condition (6), it follows that $V(t) \leqslant h(t)$ obviously. Next, the analysis for the relation between $h(f_k)$ and $h(f_{k+1})$ can be made in four cases.

Case 1. If the values of Lyapunov function at r_{k-1}, r_k, r_{k+1} satisfy that $V(r_{k-1}) \leqslant \lambda V(r_k)$ and $V(r_k) \leqslant \lambda V(r_{k+1})$, then the following equations:

$$\begin{aligned} &h_1(f_k) = (1 + \rho\Delta)V(r_k) + C\Delta + C_1, \\ &h_1(f_{k+1}) = (1 + \rho\Delta)V(r_{k+1}) + C\Delta + C_1, \\ &V(r_{k+1}) = (1 + \rho\Delta - \delta\alpha T_k)V(r_k) + C\Delta + C_1, \\ &h_1(f_{k+1}) - h_1(f_k) = (1 + \rho\Delta)[(\rho\Delta - \delta\alpha T_k)V(r_k) + C\Delta + C_1] \end{aligned}$$

hold.

It is easy to show that $\xi_d(0) > 0$. Based on the continuity of $\rho\Delta_4 - \xi_d(\Delta_4)\delta\alpha$ and the definition of Δ_4, we have $\rho\Delta - \xi_d(\Delta)\delta\alpha < 0$ for $D_k < \Delta_4$. Then, we consider the following two subcases:

Subcase (a). When $V(r_k) \geqslant \frac{C\Delta + C_1}{\delta\alpha\xi_d(\Delta) - \rho\Delta}$,

$$h_1(f_{k+1}) - h_1(f_k) < 0;$$

Subcase (b). When $V(r_k) < \frac{C\Delta + C_1}{\delta\alpha\xi_d(\Delta) - \rho\Delta}$,

$$h_1(f_{k+1}) < (1 + \rho\Delta)\left[(1 + \rho\Delta - \delta\alpha\xi_d(\Delta))\frac{C\Delta + C_1}{\delta\alpha\xi_d(\Delta) - \rho\Delta} + C\Delta + C_1\right] + C\Delta + C_1.$$

When $V(r_k) > \frac{C\Delta + C_1}{\delta\alpha\xi_d(\Delta) - \rho\Delta}$, the values of $h(t)$ at f_k and f_{k+1} are decreasing; when $V(r_k) < \frac{C\Delta + C_1}{\delta\alpha\xi_d(\Delta) - \rho\Delta}$, there exists an upper bound depending on $\Delta, \xi_d(\Delta)$, C and C_1 for $h(t)$. Owing to $V(t) \leqslant h(t)$, we can conclude that $V(t)$ is bounded in this case.

Case 2. If the values of Lyapunov function at r_{k-1}, r_k, r_{k+1} satisfy that $V(r_{k-1}) \leqslant \lambda V(r_k)$ and $V(r_k) > \lambda V(r_{k+1})$, then the following equations:

$$\begin{aligned} &h_1(f_k) = (1 + \rho\Delta)V(r_k) + C\Delta + C_1, \\ &h_2(f_{k+1}) = \left(\frac{1}{\lambda} + \rho'\Delta\right)V(r_k) + C\Delta + C_1, \\ &h_2(f_{k+1}) - h_1(f_k) = \left(\frac{1}{\lambda} - 1\right)(1 + \rho\Delta)V(r_k) \end{aligned}$$

hold.

Obviously, the values of $h(t)$ at f_k and f_{k+1} are decreasing.

Case 3. If the values of Lyapunov function at r_{k-1}, r_k, r_{k+1} satisfy that $V(r_{k-1}) > \lambda V(r_k)$ and $V(r_k) \leqslant \lambda V(r_{k+1})$, then the following equations:

$$\begin{aligned} &h_2(f_k) = \left(\frac{1}{\lambda} + \rho'\Delta\right)V(r_{k-1}) + C\Delta + C_1, \\ &V(r_{k+1}) = \left(\frac{1}{\lambda} + \rho'\Delta - \delta\alpha T_k\right)V(r_{k-1}) + C\Delta + C_1, \\ &h_1(f_{k+1}) = (1 + \rho\Delta)\left[\left(\frac{1}{\lambda} + \rho'\Delta - \delta\alpha T_k\right)V(r_{k-1}) + C\Delta + C_1\right] + C\Delta + C_1, \\ &h_1(f_{k+1}) - h_2(f_k) = (1 + \rho\Delta)[(\rho'\Delta - \delta\alpha T_k)V(r_{k-1}) + C\Delta + C_1] \end{aligned}$$

hold.

Then, we consider the following two subcases:

Subcase (a). When $V(r_{k-1}) \geqslant \frac{(C\Delta+C_1)\lambda}{\delta\alpha\xi_d(\Delta)-\rho\Delta}$,

$$h_1(f_{k+1}) - h_2(f_k) < 0;$$

Subcase (b). When $V(r_{k-1}) < \frac{(C\Delta+C_1)\lambda}{\delta\alpha\xi_d(\Delta)-\rho\Delta}$,

$$h_1(f_{k+1}) < (1+\rho\Delta)\left[\left(\frac{1}{\lambda}+\rho'\Delta-\delta\alpha\xi_d(\Delta)\right)\frac{(C\Delta+C_1)\lambda}{\delta\alpha\xi_d(\Delta)-\rho\Delta}+C\Delta+C_1\right]+C\Delta+C_1.$$

Similar with Case 1, the upper bound of $h(t)$ exists.

Case 4. If the values of Lyapunov function at r_{k-1}, r_k, r_{k+1} satisfy that $V(r_{k-1}) > \lambda V(r_k)$ and $V(r_k) > \lambda V(r_{k+1})$, then the following equations:

$$\begin{aligned}
h_2(f_k) &= \left(\frac{1}{\lambda}+\rho'\Delta\right)V(r_{k-1})+C\Delta+C_1,\\
h_2(f_{k+1}) &= \left(\frac{1}{\lambda}+\rho'\Delta\right)V(r_k)+C\Delta+C_1,\\
h_2(f_{k+1})-h_2(f_k) &= \left(\frac{1}{\lambda}+\rho'\Delta\right)(V(r_k)-V(r_{k-1}))
\end{aligned}$$

hold.

Obviously, the values of $h(t)$ at f_k and f_{k+1} are decreasing.

Based on all of the analysis above, $h(t)$ is decreasing at each executing instant or bounded over time. The upper bound of $h(t)$ is shown as

$$h(t) \leqslant \max\{M_1, M_2\}.$$

Thus, based on (3), we can conclude that this event-triggered mechanism can ensure the practical stability of the feedback system (1).

Remark 4. The reason for introducing the constants C and C_1 in the threshold function ϕ (7) is the presence of disturbances. Therefore, for the disturbance-free systems in [32], the threshold function ϕ (7) of event-triggering condition (6) can be reduced to the following form:

$$\phi(V(r_k),V(r_{k-1}),t)=\begin{cases} V(r_k)(1+\rho\Delta-\delta\alpha(t-f_k)), & \text{if } V(r_{k-1})\leqslant\lambda V(r_k); \quad (18\text{a})\\ V(r_{k-1})\left(\frac{1}{\lambda}+\rho'\Delta-\delta\alpha(t-f_k)\right), & \text{if } V(r_{k-1})>\lambda V(r_k). \quad (18\text{b})\end{cases}$$

From the theoretical point of view, we make a comparison about the two different event-triggering strategies of [32] and (18). In [32], r_{k+1} is triggered when $V(t) > (1+\rho\Delta-\delta\alpha(t-f_k))V(r_k)$ or $\lambda V(t) \leqslant V(r_k)$. This means that all signals satisfying $\lambda V(t) \leqslant V(r_k), k = 1,\ldots,\infty$ will be triggered. However, in this paper, whether these signals will be transmitted still needs to be determined based on the two cases of threshold function ϕ (18). Therefore, the transmission rates can be reduced further. This analysis will be also verified in Section 4 from the simulation point of view.

For the delay-free systems, the dual-stage form is no longer needed. We can directly rewrite the event-triggering condition (6) into the following form:

$$r_{k+1} = \inf\{t > r_k | V(t) > V(r_k)(1-\delta\alpha(t-r_k)) + C_1\}. \tag{19}$$

Then, we propose the following corollary about stability of delay-free systems, where some necessary constants are defined as follows. $\delta \in (0,1)$ and C_1 is a parameter related to disturbances.

Corollary 1. Suppose that Assumption 1 holds for the event-triggered feedback system (1) without transmission delays. Under the event-triggering condition (19), the system is practically stable and there does not exist Zeno behavior. More specifically, the ultimate bound of $V(t)$ satisfies $V(t) \leqslant \frac{C_1}{\delta\alpha\xi_d} + C_1$.

And the sample period satisfies $T_k \geqslant \xi_d$, $\xi_d = \min\{\xi_1, \xi_2, \xi_3\}$, where ξ_1 is the smallest positive solution to

$$1 - \delta\alpha t = \mathrm{e}^{-\alpha t} + \frac{\beta L_1(\mathrm{e}^{2Lt} - 1)}{2\alpha}(1 - \mathrm{e}^{-\alpha t});$$

ξ_2 is the smallest positive solution to

$$C_1 = \frac{\beta d_{\max}(\mathrm{e}^{2Lt} - 1) + 2\mu d_{\max}}{2\alpha}(1 - \mathrm{e}^{-\alpha t});$$

ξ_3 is the smallest positive solution to

$$1 - \delta\alpha t + \frac{C_1}{V(r_k)} = \mathrm{e}^{-\alpha t} + \frac{\beta L_1(\mathrm{e}^{2Lt} - 1)}{2\alpha}(1 - \mathrm{e}^{-\alpha t}) + \frac{\beta d_{\max}(\mathrm{e}^{2Lt} - 1) + 2\mu d_{\max}}{2\alpha V(r_k)}(1 - \mathrm{e}^{-\alpha t}),$$

where $V(r_k)$ is an arbitrary constant belonging to $(0, \infty)$.

4 Simulations

Consider the following linearized state equation of the inverted pendulum in [39]:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g}{l} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{-1}{Ml} \end{bmatrix} u + d, \tag{20}$$

where $M = 10, m = 1, l = 3$ and $g = 10$. The system's initial state is the vector $x_0 = [0.98\ 0\ 0.2\ 0]^{\mathrm{T}}$. The state feedback controller is $u = Kx$, where feedback gain is $K = [2\ 12\ 378\ 210]$. It is easily checked that $A + BK$ is Hurwitz. The external disturbance is $d(t) = [0.01\ 0.01\ 0.01\ 0.01]^{\mathrm{T}} \sin(t)$. The Lyapunov function for the system (20) is $V(x) = \sqrt{x^{\mathrm{T}} P x}$, where

$$P = \begin{bmatrix} 7 & 21 & 222 & 127 \\ 21 & 106 & 1180 & 675 \\ 222 & 1180 & 26578 & 14873 \\ 127 & 675 & 14873 & 8327 \end{bmatrix}.$$

Then, the parameters in Assumption 1 can be computed as $\beta = 1042.8, L = 45.58, L_1 = 1.91$.

First, the delay-free case is considered. The other parameters in event-triggering condition (19) are calculated as $\alpha = 0.75, \delta = 0.2, C_1 = 0.3$. From the trajectory of Lyapunov function in Figure 3, the event-triggered feedback system is practically stable. The state trajectories of the event-triggered system are shown at the top of Figure 4, while the sampling periods are shown at the bottom. The average sampling period and the minimum sampling period are 0.36 s and 0.33 s, respectively. Then we make a comparison about the average periods with some prior work in [1, 37, 39]. The result is shown in Table 1 that the designed event-triggering condition (19) can reduce transmission rates further.

Then, we introduce random non-zero delay $D_k \leqslant 10^{-3}$ s to the above system. The other parameters in (6) are calculated as $\lambda = 1.2, \alpha = 0.75, \delta = 0.8, C\Delta + C_1 = 0.4, \rho = 5993$. It is noted that $D_k \leqslant 10^{-8}$ s is the result of calculation according to the constraint condition $D_k \leqslant \Delta$ in Theorem 1. However, based on the simulation result, the time delays can be at least equal to or bigger than 10^{-3} s under the premise of ensuring system performance. The trajectory of Lyapunov function is shown in Figure 5. Figure 6 provides the state trajectories of the event-triggered closed-loop system and the evolution of the sampling periods. The average sampling period and the minimum sampling period are 0.328 s and 0.036 s, respectively. The numbers of occurrence for (7a) $V(r_{k-1}) \leqslant \lambda V(r_k)$ and (7b) $V(r_{k-1}) > \lambda V(r_k)$ are 38534 and 1430, respectively. And the triggering numbers of two cases are 116 and 5.

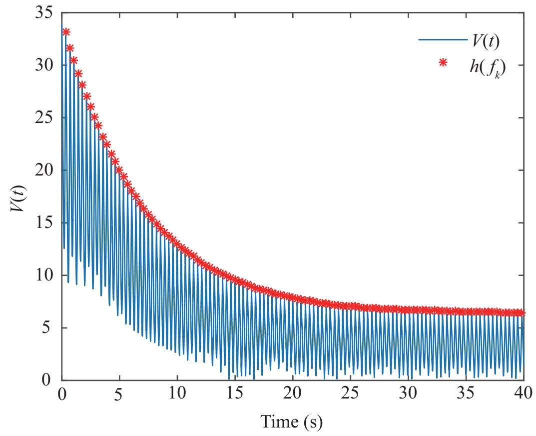

Figure 3 (Color online) The trajectory of $V(t)$ with disturbances.

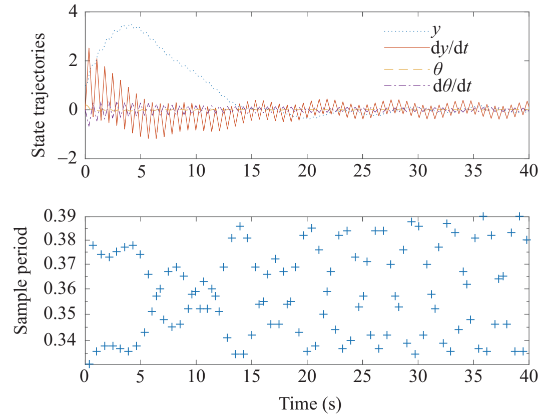

Figure 4 (Color online) The event-triggered feedback system with disturbances.

Table 1 Comparison of different schemes

Scheme	Eq. (19)	MATI [1]	Event-triggering [37]	Self-triggering [39]
Average periods (s)	0.36	0.0169	$< 10^{-5}$	0.1782

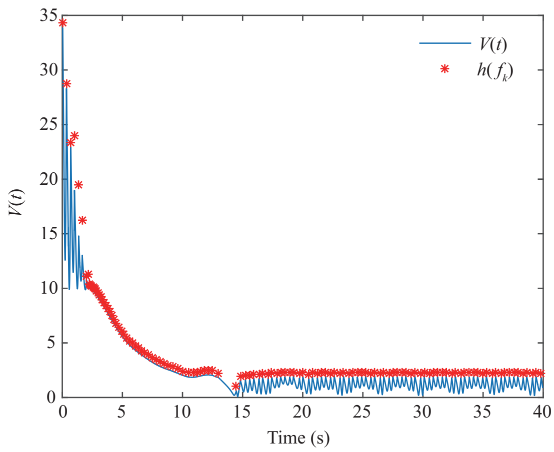

Figure 5 (Color online) The trajectory of $V(t)$ with disturbances and transmission delays.

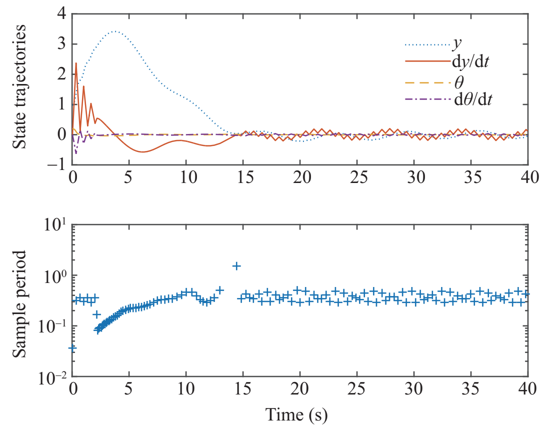

Figure 6 (Color online) The event-triggered feedback system with disturbances and transmission delays.

Finally, we compare the event-triggering condition (6) in this paper with the one in [32]. As shown in Table 2, the average periods obtained by the event-triggering condition (6) are always larger than those in [32] for different values of λ. Moreover, for the schemes in [32], the number of events in the case of (7b) increases, i.e., the transmission performance degrades as λ decreases. Thus, it can be concluded that this dual-stage Lyapunov-based event-triggering condition (6) can obtain better communication properties while ensuring the practical stability.

5 Conclusion

This paper has investigated the event-triggered control problem for nonlinear systems subject to bounded disturbances and transmission delays. By the emulation-based method, the continuous-time system was assumed input-to-stable with respect to measurement errors and disturbances. Lyapunov-based event-triggered sampling was introduced to reduce transmission rates. For constructing the new dual-stage Lyapunov-based event-triggering condition, we divided the trajectory of Lyapunov function into

Table 2 The impact of ratio λ on the method of (6) and [32]

Scheme	λ	Average periods (s)	Events of (7a)	Events of (7b)
[32]	2	0.307	116	15
(6)	2	0.333	119	1
[32]	1.2	0.272	107	39
(6)	1.2	0.328	116	5
[32]	1.1	0.221	106	76
(6)	1.1	0.345	108	8

two cases respectively for two kinds of threshold functions. Then, two important time parameters, the maximum allowable transmission delay and the minimum time-interval, were determined by analyzing the relationship between Lyapunov function and the threshold function. Finally, the proposed techniques ensured the practical stability of the closed-loop system and excluded Zeno behavior. The simulation results illustrated that the event-triggered mechanism in this paper is better than the existing mechanism. In the future study, as an important and challenging topic, the Lyapunov-based event-triggered control problem with output-based feedback is worth considering.

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