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The existence of Zeno behavior and its application to finite-time event-triggered control

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Dear editor,

Event-triggered control is a control strategy that is not necessarily periodic and takes resource constraints into consideration, such as limited communication bandwidth and restricted energy resources [1]. In event-triggered control, the execution of control tasks is decided by an event, instead of a fixed period of time (known as time-triggered control). The event is generated by a triggering condition involving the state or output variables. Thus, event-triggered control relates the execution of control actions to the operation of systems; moreover, it has the potential to significantly reduce the use of communication resources without excessive loss of control performance [2].

For event-triggered control, it is not only suitable to simply study stability and control performance, but also important to examine sampling behavior. The authors in [1,3] proposed some useful tools to provide a positive lower bound of inter-event times. Ref. [3] reported on the sampling performance of several popular types of event-triggered control systems, both in the absence and presence of external disturbances. The examination of sampling behavior is very important because the event-triggered control system can be formulated as a hybrid system model [4]. In fact, when a positive lower bound of inter-event times cannot be ensured, Zeno behavior [5], a particular phenomenon of hybrid dynamical systems, might occur. Zeno behavior leads to occurrence of an infinite number of discrete transitions (events) within a finite time interval. Hence, Zeno behavior is extremely undesirable in event-triggered control.

Thus, the conditions, both necessary and sufficient, for Zeno behavior in event-triggered control systems are very important in the study of control systems. Generally, the necessary conditions for the existence of Zeno behavior can be used to eliminate it in event-triggered control systems, while sufficient conditions could assist the designer to avoid creating an event-triggered control system having Zeno behavior. Therefore, we investigated the conditions related to Zeno behavior in event-triggered control systems, and the main contributions can be described as follows.

First, some necessary and sufficient conditions for Zeno behavior are provided by limiting the triggering condition to a specific form, which can cover several types of triggering conditions in existing studies. This triggering condition is triggered when the difference, between the current and the previously sampled output signals, violates the threshold function that involves the output signals. Thus, the corresponding Zeno equilibrium [5] is created when outputs make the threshold function equal to zero. This property is quite crucial to validate the sufficiency of our results.

Second, the innate contradiction between finite-time stability and the considered event-triggered control is demonstrated. For unstable plant, it has been proven that the finite-time stability cannot be achieved by the triggering condition if the threshold function is independent of time. Our results

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infer that an enormous obstacle to exclude Zeno behavior in the finite-time event-triggered control systems may exist. Hence, it is quite difficult to achieve finite-time stability using event-triggered control. This may explain why few studies regarding this kind of event-triggered control strategy are available [6].

Problem formulation. Consider the following time-invariant nonlinear plant:

$$\begin{aligned} &\dot{x} = f(x, u, d),\ y = g(x);\\ &f(0, 0, 0) = 0,\ g(0) = 0. \end{aligned} \tag{1}$$

$x \in \mathbb{R}^n$ is the state vector and $u \in \mathbb{R}^m$ is a control input. $y \in \mathbb{R}^q$ is the measurable output and $d \in \mathbb{R}^p$ denotes the bounded external disturbance with the bound $d_0 \geqslant 0$. $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^n$ is Lipschitz continuous on compacts. $g : \mathbb{R}^n \to \mathbb{R}^q$ is differentiable and $\frac{\partial g}{\partial x}$ is semi-globally bounded[1)]. Without losing generality, the initial instant is assumed to be zero.

The controller is implemented in an event-triggered manner, i.e.,

$$u = \kappa(y(t_k)), \quad t \in [t_k, t_{k+1}), \tag{2}$$

where $\kappa : \mathbb{R}^q \to \mathbb{R}^m$ is a semi-globally bounded function with $\kappa(0) = 0$. In particular, κ is not required to be Lipschitz continuous on compacts. The monotone increasing sequence $\{t_k\}_{k=0}^{\infty}$ represents the triggering times with $t_0 = 0$, which is decided by the following triggering condition:

$$t_{k+1} = \inf\{t \geqslant t_k |\, \|e_y(t)\| \geqslant g_0(y(t))\}, \tag{3}$$

where the measurement error is defined as $e_y(t) := y(t_k) - y(t), t \in [t_k, t_{k+1})$, and the threshold function $g_0 : \mathbb{R}^q \to \mathbb{R}_{\geqslant 0}$ is nonnegative and continuous. Certainly, the triggering condition (3) is independent of time in the sense that for any triggering instant $t_a \neq t_b$ with $x(t_a) = x(t_b)$, the corresponding sets, characterized by a state that would not meet the triggering condition, are the same. Hence, triggering condition (3) is said to be time-invariant. Moreover, the following definition is introduced.

Definition 1. The function $g_0(g(x))$ in (3) is considered to be sufficiently small with respect to the origin (for simplicity, sufficiently small), if there exists $\delta > 0$ for any $\varepsilon > 0$ such that $g_0(g(x)) < \varepsilon$ holds for any $x \in \mathbb{R}^n$ satisfying $0 < \|x\| < \delta$.

Remark 1. The proposed triggering condition has been used to cover several kinds of triggering conditions in previous studies. For example, when g_0 is a positive semi-definite function and only depends on y, Eq. (3) denotes the relative triggering condition in [1], which is sufficiently small. When $g_0(y)$ is equal to a positive constant, Eq. (3) denotes the absolute triggering condition in [3], which is not sufficiently small.

The closed-loop even-triggered control system can then be formulated as

$$\dot{x} = f(x, \kappa(y + e_y), d). \tag{4}$$

A fundamental requirement of the controller and triggering condition is that the controller can ensure that state x will not diverge if the triggering conditions are not violated. This is summarized in the following assumption.

Assumption 1. The controller (2) and triggering condition (3) are supposed to be well-designed; i.e., for any bounded $d(t)$, they can lead the corresponding system (4) to admit a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ such that there exist a positive constant $M > 0$ and $\phi_1, \phi_2 \in \mathcal{K}_\infty$ satisfying

$$\phi_1(\|x\|) \leqslant V(x) \leqslant \phi_2(\|x\|),$$

$$V(x) \geqslant M \Rightarrow \dot{V} = \frac{\partial V}{\partial x} \cdot f(x, \kappa((y + e_y)), d) \leqslant 0,$$

for all $(x^{\mathrm{T}}, e_y^{\mathrm{T}})^{\mathrm{T}} \in \{(x^{\mathrm{T}}, e_y^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{n+q} |\, \|e_y\| \leqslant g_0(g(x))\}$.

In event-triggered control systems, there may exist Zeno behavior [5] defined as follows.

Definition 2. For given $x(0)$ and bound disturbance $d(t)$, χ is defined as the corresponding solution of (4). The solution is Zeno if there exists a positive constant $t_\infty < \infty$ such that

$$\lim_{k\to\infty} (t_k) = \sum_{k=0}^{\infty} (t_{k+1} - t_k) = t_\infty,$$

where t_∞ is referred as the Zeno instant. An event-triggered control system is Zeno if it possesses a Zeno solution for some initial state and disturbance signal. Moreover, for a Zeno solution χ, it is chattering Zeno if there exists a finite integer $L > 0$ such that $t_{k+1} - t_k = 0$ for all $k \geqslant L$; or genuinely Zeno if $t_{k+1} - t_k > 0$ for all $k \geqslant 0$.

This study also focuses on finite-time event-triggered control [7], where the state is expected to arrive at the origin in a finite amount of time. In particular, by referring to [8], we define the following finite-time stability.

Definition 3. The system in (4) is said to be robustly finite-time stable at the origin, if it satisfies the following statements:

(1) Robust stability. For any $\varepsilon > 0$, there exists a constant $\delta(\varepsilon) > 0$ such that if $\|[x^{\mathrm{T}}(0), d_0^{\mathrm{T}}]^{\mathrm{T}}\| < \delta(\varepsilon)$, then $\|x(t)\| < \varepsilon$, for $t \in [0, \infty)$.

1) A function $\alpha : \mathbb{R}^n \to \mathbb{R}^m$ is said to be semi-globally bounded if for any compact set $\mathcal{S} \in \mathbb{R}^n$, there exists a constant $M(\mathcal{S}) > 0$ such that $\|\alpha(x)\| < M$ for every $x \in \mathcal{S}$.

(2) Finite-time convergence. For any initial state $x(0) \in \mathbb{R}^n$, there exists a finite-time $T \geqslant 0$ such that

$$\lim_{t \to T} x(t) = 0, \text{ and } x(t) = 0, \text{ for all } t \geqslant T.$$

Note that the robust stability can be regarded as an improved version of conventional Lyapunov stability [9], because they are the same when $d_0 = 0$.

Therefore, the main purpose of this study is to (i) give conditions on the existence of Zeno behavior in (4); and (ii) study the Zeno behavior in finite-time event-triggered control systems.

Conditions on Zeno behavior. Several studies [5] have implied that Zeno behavior is closely related with a class of special state sets, which are called Zeno equilibria.

Definition 4. A Zeno equilibrium of an event-triggered control system in (4) is a set

$$\Gamma := \{x \in \mathbb{R}^n | g_0(g(x)) = 0\}.$$

Thus, the sufficient and necessary condition for the existence of Zeno behavior can be summarized as follows.

Theorem 1. Under Assumption 1, a solution χ of the event-triggered control system (4) is Zeno if and only if there exists a constant $T_0 \geqslant 0$ such that $\lim_{t \to T_0} x(t) = x_\Gamma$ with $x_\Gamma \in \Gamma$.

Finite-time event-triggered control. In this section, only the unstable open-loop plant (1) is considered because, for a stable one, the trivial controller $u = 0$ is enough to stabilize the plant without sampling requirement. Thus, the plant (1) is not bounded-input bounded-state stable, and we propose the following assumption.

Assumption 2. For any given function κ and positive constants $\delta, d_0 > 0$, there always exist $x_0 \in \{x \in \mathbb{R}^n | \, \|x\| \leqslant \delta\}$ and $d(t) \in \{d \in \mathbb{R}^p | \, \|d\| \leqslant d_0\}, t \geqslant 0$, such that the solution of $\dot{x} = f(x(t), \kappa(g(x(0))), d(t))$, with the initial state $x(0) = x_0$, is unbounded.

Then, we will show the relation between robust stability and sufficiently small $g_0 \circ g$.

Lemma 1. Under Assumption 2, if the plant (1) with the controller (2) and the triggering condition (3) satisfies the robust stability with respect to the origin, then $g_0 \circ g$ is sufficiently small.

Following Lemma 1, we propose the following lemma that can be easily proved by Definition 1 and the continuity of $g_0(g(x))$ for $x \in \mathbb{R}^n$.

Lemma 2. For the time-invariant triggering condition (3), if $g_0 \circ g$ is sufficiently small, then the origin belongs to the corresponding Zeno equilibrium.

Based on the Lemmas 1 and 2, the following results of Zeno behavior in finite-time event-triggered control systems are provided.

Theorem 2. Under Assumption 2, the event-triggered control system (4) with triggering condition (3) cannot be finite-time stable without Zeno behavior.

Theorem 2 reveals a contradiction between finite-time stability and the time-invariant triggering condition (3). It should be noted that the results above are feasible for any controller (2) aiming at achieving finite time stability by the triggering condition (3). Therefore, it is inferred that there may exist an immense obstacle in the study of Zeno-free finite-time event-triggered control.

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Supporting information Appendixes A–E. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

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