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• **Supplementary File** •

On existence of Zeno behavior and its application to finite-time event-triggered control

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Appendix A Proof of Theorem 1

Sufficiency: Consider two cases: (i) the initial state belongs to the Zeno equilibrium, and (ii) the initial state is outside the Zeno equilibrium. For convenience, in the sequel, we use $x(t,j)$ to represent the state of the system after t time units and j samplings [1].

Case I: $x(0) \in \Gamma$. Then one has $g_0(g(x(0,0))) = 0$. From the triggering condition (3), $t = 0$ must be a triggering instant. Therefore $x(0,1)$ must belong to the solution of the event-triggered control system (4). Since x is continuous, $x(0,1)$ also belongs to Γ, which means that $x(0,2)$ belongs to the solution. Repeatedly, one has that for any $i \geqslant 0$, $x(0,i)$ belongs to the solution. Namely, $\lim_{i\to\infty} t_i = 0$ and $t_{k+1} - t_k = 0$ for all $k \geqslant 0$. Therefore, this solution is Chattering Zeno and $t = 0$ is the Zeno instant.

Case II: $x(0) \notin \Gamma$. Since $x(0,0) \notin \Gamma$, without loss of generality, we assume T_0 is the first time that the state of x arrivals in the Zeno equilibrium, i.e., $x(t) \notin \Gamma$ for all $t \in [0, T_0)$ and $\lim_{t\to T_0} x(t) = x_\Gamma \in \Gamma$. We prove the conclusion by contradiction. Assume that this solution is not Zeno. Then there must exist finite triggering instants during the range $[0, T_0)$. Define $K_0 = \arg\max_k \{t \in \{t_k\}_{k=0}^{\infty} | t < T_0\}$, then there exists a positive constant $\delta > 0$ such that $T_0 - t_{K_0} = \delta$. From the triggering condition (3), one has that $\|y(t_{K_0}) - y(t)\| < g_0(g(x(t))$ holds for any $t \in [t_{K_0}, T_0)$. Due to the continuity of x, g and g_0, it yields that

$$\begin{aligned}
\|g_2(x((t_{K_0})) - g_2(x_\Gamma)\| &= \left\| y(t_{K_0}) - \lim_{t\to T_0} y(t) \right\| \\
&= \lim_{t\to T_0} \|y(t_{K_0}) - y(t)\| \\
&\leqslant \lim_{t\to T_0} g_0(g(x(t))) \\
&= g_0(g((\lim_{t\to T_0} x(t))) \\
&= g_0(g((x_\Gamma)) \\
&= 0.
\end{aligned}$$

This implies that $g_2(x(t_{K_0})) = g_2(x_\Gamma)$. Since $x_\Gamma \in \Gamma$, it holds that $x(t_{K_0}) \in \Gamma$ as well. This contradicts the fact that $x(t) \notin \Gamma$ for all $t \in [0, T_0)$. Therefore the sufficiency is proved.

Necessity: The proof is divided into two cases: the solution is (i) Chattering Zeno or (ii) Genuinely Zeno.

Case I: Chattering Zeno. Since the solution is Chattering Zeno, there exists a Zeno instant $t_\infty \geqslant 0$ and a constant $L \geqslant 0$ such that $\lim_{i\to\infty} t_{L+i} = t_\infty$ and $t_{k+1} - t_k = 0$ for all $k \geqslant L$, which yields that $t_{L+i} = t_\infty$ for all $i \geqslant 0$. Thus,

$$\begin{aligned}
g_0(g(x(t_{L+i+1}, L+i))) &= g_0(g(x(t_\infty, L+i))) \\
&= \|g(x(t_{L+i}, L+i)) - g(x(t_{L+i+1}, L+i))\| \\
&= \|g(x(t_\infty, L+i)) - g(x(t_\infty, L+i))\| \\
&= 0, \text{ for all } i \geqslant 0.
\end{aligned}$$

This implies that $x(t_{L+i}, L+i+1) \in \Gamma$ for all $i \geqslant 0$. Since x does not jump, $x(t_\infty, L) = x(t_\infty, L+i+1) \in \Gamma$, for all $i \geqslant 0$. Hence, in this case one has $T_0 = t_\infty$.

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Case II: Genuinely Zeno. Since the solution is Genuinely Zeno, there exists a Zeno instant $t_\infty > 0$ such that $\lim_{k\to\infty} t_k = t_\infty$. From the triggering condition (3), it yields that for any consecutive triggering instants t_k and t_{k+1},

$$\|y(t_k) - y(t_{k+1})\| = \|e_y(t_{k+1}, k)\| = g_0(g(x(t_{k+1}))). \tag{A1}$$

Note that $e_y(t_{k+1}, k)$ denotes the the measurement error just before the next triggering instant, and hence, it is different from $e_y(t_{k+1}, k+1) = 0$ which is the measurement error just after a triggering instant.

Then we studied the following two sequences:

$$\begin{aligned}\mathcal{T}_1 :=& \{x(t_1), x(t_2), \ldots, x(t_k)\ldots\},\\ \mathcal{T}_2 :=& \{\|e_y(t_1,0)\|, \|e_y(t_2,1)\|, \ldots, \|e_y(t_k,k-1)\|, \ldots\}.\end{aligned}$$

Since Assumption 1 holds, for any solution with $x(0) \notin \Gamma$ and bounded disturbance $d(t)$, there exists compact set $\mathcal{S} \in \mathbb{R}^n$ such that $x(t) \in \mathcal{S}$ for all $t \geqslant 0$. Note that if a function is Lipschitz continuous or differentiable then it is semi-globally bounded. Hence, due to the fact that f, g and κ are semi-globally bounded, there must exists a positive constant $M_1 > 0$ such that $\|f(x, \kappa(y + e_y), d)\| < M_1$ for all $t \geqslant 0$. Similarly, since $\frac{\partial g}{\partial x}$ is semi-globally bounded, there also exists a positive constant $M_2 > 0$ such that $\|\dot{e}_y\| < M_2$ for all $t \geqslant 0$, where $\dot{e}_y = \frac{\mathrm{d}}{\mathrm{d}t} g(x(t)) = \frac{\partial g}{\partial x} \cdot f(x, \kappa(y + e_y), d)$.

Since $\lim_{k\to\infty} t_k = t_\infty$, for any given $\varepsilon > 0$, there must exist positive integer $H(\varepsilon) > 0$ such that $\left|t_{k_1} - t_{k_2}\right| < \varepsilon$ for any integers $k_1, k_2 > H(\varepsilon)$.

Furthermore, for any $\varepsilon_x > 0$, let $H_x = H(\frac{\varepsilon_x}{M_1})$. Then for any integers $m > n \geqslant H_x$,

$$\|x(t_m) - x(t_n)\| \leqslant (t_m - t_n) M_1 \leqslant \frac{\varepsilon_x}{M_1} M_1 = \varepsilon_x. \tag{A2}$$

According to **Cauchy Convergence Criterion**, (A2) implies that $x(t_k)$ converges as $k \to \infty$, i.e., there exists $x_\Gamma \in \mathbb{R}^n$ such that $\lim_{k\to\infty} x(t_k) = x_\Gamma$.

Similarly, for any $\varepsilon_e > 0$, let $H_e = H(\frac{\varepsilon_e}{M_2})$. Then for any integer $k \geqslant H_e$,

$$\|e_y(t_{k+1}, k)\| = \left\| \int_{t_k}^{t_{k+1}} \dot{e}_y \mathrm{d}t \right\| \leqslant (t_{k+1} - t_k) M_2 \leqslant \frac{\varepsilon_e}{M_2} M_2 = \varepsilon_e,$$

which implies that $\lim_{k\to\infty} \|e_z(t_k, k-1)\| = 0$.

Then we consider the following sequence.

$$\mathcal{T}_3 := \{g_0(g(x(t_1))), g_0(g(x(t_2))), \ldots, g_0(g(x(t_k))), \ldots\}.$$

Since t_k is the triggering instant, one has $\mathcal{T}_2 = \mathcal{T}_3$ due to (A1). Hence, $\lim_{k\to\infty} g_0(g(x(t_k))) = 0$. Due to the continuity of g and x, one has that

$$\lim_{k\to\infty} g_0(g(x(t_k))) = g_0(g(\lim_{k\to\infty} x(t_k))) = g_0(g(x_\Gamma)) = 0,$$

which yields $x_\Gamma \in \Gamma$. Thus, we prove that the states at the triggering instants will approach the Zeno equilibrium.

Next, we study the behavior of x during triggering instants. For any $t \in [t_k, t_{k+1})$,

$$\|x(t) - x_\Gamma\| = \left\| \int_{t_k}^{t} \dot{x} \mathrm{d}t + x(t_k) - x_\Gamma \right\| \leqslant M_1(t_{k+1} - t_k) + \|x(t_k) - x_\Gamma\|.$$

Since $\lim_{k\to\infty}(t_{k+1} - t_k) = 0$ and $\lim_{k\to\infty} \|x(t_k) - x_\Gamma\| = 0$, one has that $\lim_{t\to t_\infty} \|x(t) - x_\Gamma\| = 0$ according to the well known **Squeeze Theorem**. Therefore the proof is completed by letting $T_0 = t_\infty$.

Appendix B Proof of Lemma 1

We prove this lemma by contradiction. Assume that the closed-loop system satisfies the robust stability and the function $g_0 \circ g$ is not sufficiently small. Hence, there exist constants $\eta > 0$ and $\delta_0 > 0$ such that $g_0(g(x)) \geqslant \eta$ for $x \in \mathbb{R}^n$ satisfying $0 < \|x\| < \delta_0$.

Since the system satisfies the robust stability, for any $L_1 > 0$, there exists $\delta_1 > 0$ such that for any initial state $x(0)$ and disturbance bound d_0 with $\left\|[x^{\mathrm{T}}(0), d_0^{\mathrm{T}}]^{\mathrm{T}}\right\| < \delta_1$, the corresponding solution $x(t)$ satisfies

$$\|x(t)\| < \min\{\frac{\eta}{3L_1}, \delta_0\},$$

for $t \in [0, \infty)$.

Suppose that the Lipschitz constant of $g_2(x)$ is L_2 in the set $\{x \in \mathbb{R}^n | \, \|x\| < \frac{\eta}{3L_1}\}$, and define $L_g = \max\{L_1, L_2\}$. Then similarly, there exists $\delta_g > 0$ such that for any $x(0)$ and d_0 with $0 < \left\|[x^{\mathrm{T}}(0), d_0^{\mathrm{T}}]^{\mathrm{T}}\right\| < \delta_g$, the corresponding solution $x(t)$ satisfies

$$\|x(t)\| < \max\{\delta_0, \frac{\eta}{3L_g}\} \leqslant \max\{\delta_0, \frac{\eta}{3L_1}\},$$

for $t \in [0, \infty)$.

Furthermore, for the above solution $x(t)$, one has that

$$\|e_y\| = \|g(x(t_k)) - g(x(t))\| \leqslant L_g \|x(t_k) - x(t)\| \leqslant L_g \frac{2\eta}{3L_g} = \frac{2}{3}\eta < \eta,$$

for $t \in [0, \infty)$. Namely, $\|e_y\| < \eta \leqslant g_0(g(x(t)))$ for $t \in [0, \infty)$. This means that the event would not be triggered anymore. Define $\delta_2 = \min\{\delta_0, \delta_1, \delta_g\}$. Then, the analysis above implies that the controller $u = \kappa(g(x(0)))$ could lead the state to be bounded for $t \in [0, \infty)$ and $x(0) \in \{x \in \mathbb{R}^n | \, \|x\| < \delta_2\}$, which contradicts Assumption 2. Therefore the proof is completed.

Appendix C Proof of Theorem 2

We prove this theorem by contradiction. Assume that the system is finite time stable without Zeno behavior under some time-invariant triggering condition (3). Then from Lemma 1, if the system is finite-time stable, the triggering condition must be sufficiently small. And from Lemma 2, if the triggering condition is sufficiently small, then the origin belongs to the Zeno equilibrium. Since finite-time stability means that the solution converges to the origin in finite time, one has $\Gamma \neq \mathbb{R}^n$. In fact, if $\Gamma = \mathbb{R}^n$, then the system would always be Chattering Zeno and the state cannot converge to the origin obviously. As a result, there always exist initial state $x(0) \notin \Gamma$ such that the corresponding solution arrives in the Zeno equilibrium in finite time. Further, finite-time stability implies that Assumption 1 holds [3]. Then according to Theorem 1, this solution must be Genuinely Zeno, which contradicts the assumption of no Zeno behavior. Therefore the proof is completed.

Remark 1. Theorem 2 proves that the considered time-invariant triggering condition cannot guarantee the finite-time stability without Zeno behavior. Hence, to solve this problem, some more complicated cases may need to be considered, such as the time-varying triggering condition, and non-Lipschitz function g. However, the main challenge is that the finite-time stability are generally achieved by some non-Lipschitz controllers [2]. Hence, according to [3], [4], the closed-loop system must not be Lipschitz continuity at the origin. Since how to ensure the positive minimum inter-event times for the systems without Lipschitz continuity is an open problem so far, it is still very difficult to achieve the finite-time stability by event-triggered control even in these more complicated cases, although they are not completely impossible.

Appendix D Simulation example of systems with Zeno behavior

In this part, an example is provided to illustrate the feasibility of Theorem 1. Consider the following linear plant

$$\dot{x} = Ax + Bu, y = Cx,$$

with the matrices

$$A = \begin{bmatrix} 0 & 1.8 \\ 0 & -1.8 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The feedback controller is $u = -1.8y$ which leads the eigenvalues of $A + BKC$ to be $-1.8 + 1.8i$ and $-1.8 - 1.8i$. Referring to [5], we propose the following triggering condition

$$t_{k+1} = \inf\{t \geqslant t_k | \, |e_y| > 0.1 \, |y|\}.$$

where $e_y = y(t) - y(t_k), t \in [t_k, t_{k+1})$. It can be checked that the considered system satisfies the hypotheses of Theorem 1 based on the results in [5].

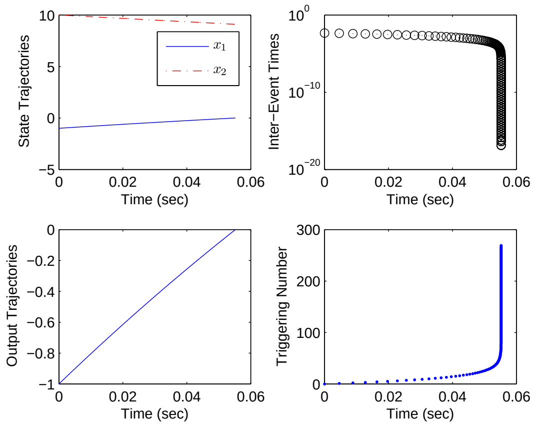

Figure D1 The simulation results of the output feedback event-triggered control system with relative triggering condition.

Figure D1 provides the state and output trajectories, the evolution of inter-event times, and the triggering numbers. The simulation results show that the solution of the system is Zeno. And the Zeno instant is about $0.055s$. The output is equal to zero (i.e., the state arrives at the Zeno equilibrium) at the Zeno instant as well. Thus, the simulation results agree with Theorem 1. Moreover, Figure D1 provides some distinct features of Zeno behaviors, that is, the inter-event times decrease sharply around the Zeno instant and the asymptotic line of the points in the subfigure about 'Triggering Number' would become a vertical line as t approaches the Zeno instant.

Appendix E Numerical example of finite-time event-triggered control

Consider the following scalar system,

$$\dot{x} = u, x(0) = 1, u = -x^{\frac{1}{2}}(t_k), \tag{E1}$$

which satisfies Assumption 2. Define the triggering condition as

$$t_{k+1} = \inf\{t \geqslant t_k \,|\, |e| \geqslant 0.5\,|x|\}, \tag{E2}$$

where $e = x(t) - x(t_k), t \in [t_k, t_{k+1})$.

First we prove that Assumption 1 holds for the above system.

Define $V = \frac{1}{2}x^2$, then the derivative of V along the solution of the system (E1) is

$$\dot{V} = x\dot{x} = -x\sqrt{|x(t_k)|} = -x\sqrt{|x+e|}.$$

Since the event condition yields $|e| < \frac{1}{2}|x|$, the signs of x and $x+e$ are the same. Then from $x(0) = 1 > 0$, one has $x(t) \geqslant 0, t \in [0, \infty)$. According to the inequality $\sqrt{|x+e|} \geqslant \sqrt{|x|} - \sqrt{|e|} > 0$,

$$\dot{V} \leqslant -x(\sqrt{|x|} - \sqrt{|e|}) \leqslant -x\sqrt{|x|} + \frac{\sqrt{2}}{2}x\sqrt{|x|} \leqslant -\frac{2-\sqrt{2}}{2}V^{\frac{3}{4}}.$$

Referring to Theorem 4.2 in [3], we have that the system is finite-time stable and Assumption 1 holds. Then we calculate the triggering instants. Let $T_k = t_k - t_{k-1}, k = 1, 2, 3 \ldots$.

When $t \geqslant 0$, one has $\dot{x} = -1$, which enforces that $x = 1 - t$ and $e = t$. From (E2), $T_1 = \frac{1}{2}(1 - T_1)$. As a result, $t_1 = T_1 = \frac{1}{3}$ and $x(t_1) = \frac{2}{3}$.

Then consider $t \geqslant t_1 = \frac{1}{3}$. We have $\dot{x} = -\sqrt{\frac{2}{3}}$, which yields $x = \frac{2}{3} - \sqrt{\frac{2}{3}}(t - t_1)$ and $e = \sqrt{\frac{2}{3}}(t - t_1)$. Thus, $\sqrt{\frac{2}{3}}T_2 = \frac{1}{2}(\sqrt{\frac{2}{3}}T_2)$. Consequently, $T_2 = \frac{2}{9}\sqrt{\frac{3}{2}}$, $t_2 = t_1 + T_2 = \frac{1}{3} + \frac{2}{9}\sqrt{\frac{3}{2}}$ and $x(t_2) = \frac{4}{9}$.

Repeating the above processes, one has that

$$T_k = \frac{1}{3}(\frac{2}{3})^{\frac{k-1}{2}}, t_k = \sum_{j=1}^{j=k} T_j = \frac{\frac{1}{3}(1 - (\frac{2}{3})^{\frac{k}{2}})}{1 - \sqrt{\frac{2}{3}}} = \frac{\sqrt{3}(1 - \sqrt{\frac{2}{3}}^{k})}{3\sqrt{3} - 3\sqrt{2}},$$

and $x(t_k) = (\frac{2}{3})^k$ for $k = 1, 2, 3 \ldots$. Obviously, $\lim_{k\to\infty} t_k = t_\infty = \frac{1}{3-\sqrt{6}} < \infty$, $\lim_{k\to\infty} x(t_k) = 0$ and $T_k > 0$ for $k = 1, 2, 3 \ldots$. Hence, the corresponding solution is Genuinely Zeno.

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