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• **Supplementary File** •

Three matrix conditions for the reduction of finite automata based on the theory of semi-tensor product of matrices

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Appendix A Propositions used in the body of the letter

A.1 (Definition of STP). Consider $A \in \mathrm{M}_{m\times n}$ and $B \in \mathrm{M}_{p\times q}$. Let α be the least common multiple of n and p. The STP of A and B is defined as $A \ltimes B = (A \otimes I_{\alpha/n})(B \otimes I_{\alpha/p})$, where $\otimes$ is the Kronecker product of matrices.
A.2 (Proposition 1) [1]. Let $X \in \mathrm{R}^m$ and $Y \in \mathrm{R}^n$ be column vectors, then $W_{[m,n]} \ltimes X \ltimes Y = Y \ltimes X$ and $W_{[n,m]} \ltimes Y \ltimes X = X \ltimes Y$, where $W_{[m,n]}$ is a $mn \times mn$ matrix defined as follows. Its rows and columns are arranged by the ordered multi-index $Id[j, i; n, m]$ and $Id[i, j; m, n]$, respectively. The element at the position $[(I, J), (i, j)]$ is 1 for $I = i$ and $J = j$, otherwise 0.
A.3 (Proposition 2) [1]. Let $X \in \mathrm{R}^s$ be a column vector and $A \in \mathrm{M}_{m\times n}$ be a matrix, then $X \ltimes A = (I_s \otimes A) \ltimes X$.
A.4 (Proposition 3) [2]. For $x \in \Delta_m$ and $y \in \Delta_n$, we have $D_{[m,n]} \ltimes x \ltimes y = y$ and $D_{[n,m]} \ltimes W_{[m,n]} \ltimes x \ltimes y = x$, where $D_{[m,n]} = \mathbf{1}_m \otimes I_n$.
A.5 (Proposition 4) [3]. Consider k -valued logical variables $x_i \in \Delta_k$ $(i = 1, 2, \cdots, n)$, for some of them $x_{i_1}, x_{i_2}, \cdots, x_{i_m} (i_1 < i_2 < \cdots < i_m)$, there is a unique matrix L such that

$$x_{i_1} \ltimes x_{i_2} \ltimes \cdots \ltimes x_{i_m} = L \ltimes x_1 \ltimes x_2 \ltimes \cdots \ltimes x_n. \tag{A1}$$

A.6 (Proposition 5) [4]. Let $M_\mathrm{n} = \delta_2[2\ 1]$, then for $x \in \Delta$, we have $\bar{x} = M_\mathrm{n} \ltimes x$, where $\bar{x}$ is the negation of x.
A.7 (Proposition 6) [4]. Let $x_i \in \Delta_2$ $(i = 1, 2, \cdots, n)$ be 2-valued logical variables; then for $x_{i_1}, x_{i_2}, \cdots, x_{i_m} (i_1 < i_2 < \cdots < i_m)$ and $x_j (1 \leqslant j \leqslant n)$, there exists a unique matrix P making $\bar{x}_j \ltimes x_{i_1} \ltimes x_{i_2} \ltimes \cdots \ltimes x_{i_m} = P \ltimes x$ hold, where $x = x_1 \ltimes x_2 \ltimes \cdots \ltimes x_n$.
A.8 (Proposition 7). Consider k-valued logical variables $x_i \in \Delta_k$ $(i = 1, 2, \cdots, n)$, there is a unique matrix E satisfying that $\sum_{i=1}^n x_i = E \ltimes x$, where $x = \ltimes_{i=1}^n x_i$, $E = D_{[k^{n-1},k^1]} \sum_{i=1}^n W_{[k^i,k^{n-i}]}$, $D_{[m,n]} = \mathbf{1}_m \otimes I_n$.

Proof. (Existence). From Proposition 4, there exists a unique matrix $E_i = D_{[k^{n-1},k^1]} W_{[k^i,k^{n-i}]}$ such that $x_i = E_i \ltimes x$. We thus have $\sum_{i=1}^n x_i = \left(\sum_{i=1}^n E_i\right) \ltimes x = \left(\sum_{i=1}^n \left(D_{[k^{n-1},k^1]} W_{[k^i,k^{n-i}]}\right)\right) \ltimes x = D_{[k^{n-1},k^1]} \sum_{i=1}^n \left(W_{[k^i,k^{n-i}]}\right) \ltimes x = E \ltimes x$.

(Uniqueness). The proof is omitted here also due to the similarity to that of the uniqueness of Proposition 4.
A.9 (Proposition 8). Suppose that $x_i \in \Delta_2$ $(i = 1, 2, \cdots, n)$ are 2-valued logical variables and Q is the sum vector of x_i; then $col_j(Q) \leqslant n$ holds for all $j = 1, 2, \cdots, 2^n$.

Proof. (Using contradiction to prove). From Definition 3 in the body of the letter, we have $Q = \mathrm{row}_1(E)$. From Proposition 7, we have $\sum_{i=1}^n x_i = \mathrm{col}_k(E), 1 \leqslant k \leqslant 2^n$. Assume there is a $j (1 \leqslant j \leqslant 2^n)$ such that $col_j(Q) = n + 1$. Then there exists a $k (1 \leqslant k \leqslant 2^n)$ such that $\mathrm{col}_k(E) = \delta_2^{n+1}$. Thus there exists a set of $(x_{i_1}, x_{i_2}, \cdots, x_{i_n})$ such that $\sum_{i=1}^n x_i = \delta_2^{n+1}$. This is apparently impossible. The proof is completed.

The following result is an immediate consequence of Propositions 7 and 8.
A.10 (Proposition 9). Suppose that $x_i \in \Delta_2$ $(i = 1, 2, \cdots, n)$ are 2-valued logical variables, Q is their sum vector, and that $x = \ltimes_{i=1}^n x_i = \delta_{2^n}^l$, $x' = \ltimes_{i=1}^n x_i' = \delta_{2^n}^{l'}$; then $\mathrm{col}_l(Q) \geqslant \mathrm{col}_{l'}(Q)$ if and only if the number of δ_2^2 in $\{x_i' | i = 1, \cdots, n\}$ is larger than that of δ_2^2 in $\{x_i | i = 1, \cdots, n\}$.
A.11 (Proposition 10) [5]. Let $M_\mathrm{r} = \delta_4[1\ \ 4]$. For $x \in \Delta$, then $x^2 = M_\mathrm{r} \ltimes x$.

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Appendix B Proofs in the body of the letter.

This section gives the proofs of the three necessary and sufficient conditions in the body of the letter.

B.1 (Proof of Compatible Enclosure Condition).

(Sufficiency) If $\mathrm{col}_l(K) = 0$, it follows from $K = \sum_{i=1}^{n} K_i^{\mathcal{G}}$ that $\mathrm{col}_l(K) = \mathrm{col}_l(\sum_{i=1}^{n} K_i^{\mathcal{G}}) = \sum_{i=1}^{n} \mathrm{col}_l(K_i^{\mathcal{G}}) = 0$. Recall that $K_i^{\mathcal{G}}$ is a vector with components being 0 or 1, thus $col_l(K_i^{\mathcal{G}}) = 0$ holds for all i. Since $K_i^{\mathcal{G}} := JL_{s_i}$, where J is a $1 \times (2^{t_i} - 1)$ row vector whose first element is 1 others are 0s; L_{s_i} is described in Definition 1 of the letter, we then get

$$col_l(L_{s_i}) \neq \delta_{2^{t_i}}^1. \tag{B1}$$

From Definition 1 in the body of the letter and $x^{\mathcal{G}} = \ltimes_{i=1}^{r} x_i^{\mathcal{G}}$, we have that

$$x_{j_1}^{\mathcal{G}} \ltimes x_{j_2}^{\mathcal{G}} \ltimes \cdots \ltimes x_{j_{t_i}}^{\mathcal{G}} = col_l(L_{s_i}). \tag{B2}$$

(B1) and (B2) suggests that $x_{j_1}^{\mathcal{G}} \ltimes x_{j_2}^{\mathcal{G}} \ltimes \cdots \ltimes x_{j_{t_i}}^{\mathcal{G}} \neq \delta_{2^{t_i}}^1$ for all i. By Definition of STP, for all i there is at least one $x_{j_k}^{\mathcal{G}}$ $(1 \leqslant k \leqslant t_i)$ such that $x_{j_k}^{\mathcal{G}} = \delta_2^2$. This shows that there exists at least one $C_{j_k} \in \mathcal{C}_{s_i}$ in $\mathcal{G}$, thus $s_i \in \mathcal{G}$ holds for all i. The proof is completed.

(Necessity) If $\mathcal{G}$ contains S, then for $s_i 1 \leqslant i \leqslant n$, there is at least one $C_{j_k} \in \mathcal{C}_{s_i} = \{C_{j_1}, C_{j_2}, \cdots, C_{j_{t_i}}\}$ belonging to $\mathcal{G}$, it follows therefore from the definition of logical vector of C_{j_k} associated with $\mathcal{G}$ that $x_{j_k}^{\mathcal{G}}$ is δ_2^2. According to Definition of STP, we have $x_{j_1}^{\mathcal{G}} \ltimes x_{j_2}^{\mathcal{G}} \ltimes \cdots \ltimes x_{j_{t_i}}^{\mathcal{G}} \neq \delta_{2^{t_i}}^1$. On the other hand, By Definition 1 in the body of the letter, we see that $x_{j_1}^{\mathcal{G}} \ltimes x_{j_2}^{\mathcal{G}} \ltimes \cdots \ltimes x_{j_{t_i}}^{\mathcal{G}} = \mathrm{col}_l(L_{s_i})$. We thus have $\mathrm{col}_l(L_{s_i}) \neq \delta_{2^{t_i}}^1$. Since $K_i^{\mathcal{G}}$ is a vector of components being 0 or 1, we get $col_l(K_i^{\mathcal{G}}) = 0$. Here i is arbitrary, it follows therefore from $K = \sum_{i=1}^{n} K_i^{\mathcal{G}}$ that $\mathrm{col}_l(K) = 0$. The necessity is proved.

B.2 (Proof of Delegate Set Condition).

(Sufficiency). Suppose that $\mathrm{col}_l(H) = 0$, then $0 = \mathrm{col}_l(H) = \mathrm{col}_l(T+K) = \mathrm{col}_l(T)+\mathrm{col}_l(K) = \mathrm{col}_l(\sum_{i=1}^{r}\sum_{a\in A} T_{i_a}^{\mathcal{G}})+ \mathrm{col}_l(K) = \sum_{i=1}^{r}\sum_{a\in A} \mathrm{col}_l(T_{i_a}^{\mathcal{G}}) + \mathrm{col}_l(K)$. Since $T_{i_a}^{\mathcal{G}}$ is a vector of components being 0 or 1, we have $\mathrm{col}_l(K) = 0$, and

$$\sum\nolimits_{a\in A} \mathrm{col}_l(T_{i_a}^{\mathcal{G}}) = 0 \tag{B3}$$

for all i. It follows then from Compatible Enclosure Condition that $\mathcal{G}$ contains S, i.e., $S \subseteq \mathcal{G}$. It is suffice from the definition of DSSS to show that $\mathcal{G}$ is a closed set. Since $T_{i_a}^{\mathcal{G}}$ is a vector of components being 0 or 1, (B3) implies that $\mathrm{col}_l(T_{i_a}^{\mathcal{G}}) = 0$ for all $a \in A$. Without loss of generality, take some certain $a \in A$ for consideration, it follows from Definition 2 in the body of the letter that 1) $|C_{i_a}| \leqslant 1$ implies that a moves M to at most one state $s' \in S$ from $s \in C_i$, it is evident from $S \subseteq \mathcal{G}$ that

$$s' = f_s(a, s) \in \mathcal{G}. \tag{B4}$$

for all $s \in C_i$. 2). By Definition of STP, it follows from $x^{\mathcal{G}} = \delta_{2^r}^l$ and Definition 2 in the body of the letter that $\bar{x}_i^{\mathcal{G}} \ltimes x_{j_1}^{\mathcal{G}} \ltimes x_{j_2}^{\mathcal{G}} \ltimes \cdots \ltimes x_{j_{t_i}}^{\mathcal{G}} = \mathrm{col}_l(P_i)$. We then get $\bar{x}_i^{\mathcal{G}} \ltimes x_{j_1}^{\mathcal{G}} \ltimes x_{j_2}^{\mathcal{G}} \ltimes \cdots \ltimes x_{j_{t_i}}^{\mathcal{G}} \neq \delta_{2^{t_i+1}}^1$. If $C_i \in \mathcal{G}$, then $\bar{x}_i^{\mathcal{G}} = \delta_2^1$. Using Definition of STP again we know that there is at least one $x_{j_m}^{\mathcal{G}}$ $(1 \leqslant m \leqslant t_i)$ such that $x_{j_m}^{\mathcal{G}} = \delta_2^2$. This shows $C_{j_m} \in \mathcal{G}$; it follows then from the definition of the state transfer package of C_i that

$$f_s(a, s) \in C_{j_m} \in \mathcal{G} \tag{B5}$$

for all $s \in C_i$. From (B5), (B4) and the arbitrariness of a, we know that $f_s(a, s) \in \mathcal{G}$ for all $s \in C_i$. This means $\mathcal{G}$ is closed, thus $\mathcal{G}$ is a DSSS. The sufficiency is then proved.

(Necessity). Suppose that $\mathcal{G}$ is a DSSS, then $\mathcal{G}$ contains S, it follows from Compatible Enclosure Condition that $\mathrm{col}_l(K) = 0$. For $C_j \in \mathcal{G}$ $(1 \leqslant j \leqslant r)$ with $x_j^{\mathcal{G}} = \delta_2^2$, the closure property of $\mathcal{G}$ implies that there exists a $C_{j_k} \in \mathcal{G}$ with $x_{j_k}^{\mathcal{G}} = \delta_2^2$ such that $f_s(a, s) \in C_{j_k}$ for all $s \in C_j$ and all $a \in A$ that $f_s(a, s)$ is defined well. From the definition of the state transfer package of C_i, we see that $C_{j_k} \in \mathcal{E}_{i_a}$. Since $x_j^{\mathcal{G}} = \delta_2^2$ and $x_{j_k}^{\mathcal{G}} = \delta_2^2$, we have that (from Definition of STP) C_i. It follows then from Definition 2 in the body of the letter and $x^{\mathcal{G}} = \delta_{2^r}^l$ that $\mathrm{col}_l(P_i) \neq \delta_{2^{t_i+1}}^1$. If $|C_{i_a}| > 1$, then $\mathrm{col}_l(T_{i_a}^{\mathcal{G}}) = 0$ for all $a \in A$. For $|C_{i_a}| \leqslant 1$, we have $T_{i_a}^{\mathcal{G}} = [\underbrace{0 \ \cdots 0}_{2^r}]$. Thus $\mathrm{col}_l(T_{i_a}^{\mathcal{G}}) = 0$ for all $a \in A$. It follows that $\mathrm{col}_l(T) = \mathrm{col}_l(\sum_{i=1}^{r}\sum_{a\in A} T_{i_a}^{\mathcal{G}}) = \sum_{i=1}^{r}\sum_{a\in A} \mathrm{col}_l(T_{i_a}^{\mathcal{G}}) = 0$. Since $\mathrm{col}_l(K) = 0$, we get $\mathrm{col}_l(H) = 0$. This proves the necessity.

B.3 (Proof of Least Delegate Set Condition).

(Sufficiency). Suppose that $\mathcal{G}$ is not a DSSS, it follows from Delegate Set Condition that $col_l(H) \neq 0$. Since the components of H are nonnegative integers, we have $col_l(H) \geqslant 1$. Substituting $col_l(H) \geqslant 1$ into $N = 2rH - Q$, it follows from Proposition 8 that $col_l(N) \geqslant 2r - r = r$. Since $\mathrm{col}_l(N) = \min(\mathrm{col}(N))$, we have $0 = col_{2^r}(N) \geqslant col_l(N) \geqslant r \geqslant 1$. This is a contradiction. Thus $\mathcal{G}$ is a DSSS. If there is another DSSS $\mathcal{G}$ with $x^{\mathcal{G}'} = \ltimes_{i=1}^{r} x_i^{\mathcal{G}'} = \delta_{2^r}^{l'}$. Let Q be the sum vector of $x_i^{\mathcal{G}'}$ $(i = 1, \cdots, r)$. It follows from Delegate Set Condition that $col_{l'}(H) = 0$. Since $\mathrm{col}_l(N) = \min(\mathrm{col}(N))$, then $col_{l'}(N) \geqslant col_l(N)$, this shows from $N = 2rH - Q$ that $col_l(Q) \geqslant col_{l'}(Q)$. By Proposition 9, the number of δ_2^2 in $\{x_i^{\mathcal{G}} | i = 1, \cdots, r\}$ is smaller than that of δ_2^2 in $\{x_i^{\mathcal{G}'} | i = 1, \cdots, r\}$, that is, $|\mathcal{G}| \leqslant |\mathcal{G}'|$, which shows $\mathcal{G}$ is the LDSSS. The sufficiency is proved.

(Necessity). If $col_l(N) \neq min(col(N))$, then there exists an l':

$$col_{l'}(N) < col_l(N). \tag{B6}$$

Since $\mathcal{G}=\{C_k|1\leqslant k\leqslant r, x_k^{\mathcal{G}}=\delta_2^2\}$ with $x^{\mathcal{G}}=\delta_{2^r}^l$ is the LDSSS, it follows from Delegate Set Condition that $col_l(H)=0$. Therefore

$$col_l(N)=-col_l(Q)\leqslant 0. \tag{B7}$$

If $\mathcal{G}^{'}=\{C_k|1\leqslant k\leqslant r, x_k^{\mathcal{G}^{'}}=\delta_2^2\}$ with $x^{\mathcal{G}^{'}}=\delta_{2^r}^{l^{'}}$ is not a DSSS, it follows that $col_{l'}(H)\geqslant 1$. Thus we have $col_{l'}(N)\geqslant 2r-r=r>\mathrm{col}_l(N)$. This contradicts (B6), which shows that $\mathcal{G}^{'}$ is also a DSSS. According to Delegate Set Condition, then $col_{l'}(H)=0$. From $N=2rH-Q$, we get that

$$col_{l'}(N)=2rcol_{l'}(H)-col_{l'}(Q)=-col_{l'}(Q). \tag{B8}$$

Combining (B8) with (B7) and (B6) gives $col_{l'}(Q)>col_l(Q)$. It follows from Proposition 9 that the number of δ_2^2 in $\{x_i^{\mathcal{G}^{'}}|1\leqslant i\leqslant r\}$ is smaller than that of δ_2^2 in $\{x_i^{\mathcal{G}}|1\leqslant i\leqslant r\}$; this conflicts with the fact that $\mathcal{G}$ is an LDSSS. Therefore $\mathrm{col}_l(N)=\min(\mathrm{col}(N))$. The proof of the necessity is then proved.

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