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Exponentially convergent angular velocity estimator design for rigid body motion: a singular perturbation approach

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Dear editor,

The exponentially convergent angular velocity estimator on SO(3) is of great importance for the control of rigid body [1]. Based on the idea of adaptive technique, the angular velocity observer on SO(3) was firstly proposed in [2]. Refs. [3, 4] and references therein mainly focus on constructing different attitude error functions to improve convergence rate. In those studies, based on the Lyapunov stability method, the final form of time derivative of Lyapunov function, consisting of quadratic terms of attitude and angular velocity estimation errors, does not contain the quadratic angular velocity estimation error terms, and the asymptotic convergence is obtained based on Barbalat's lemma. Although the exponential convergence can be obtained by Lyapunov function which consists of quadratic and cross terms of estimation errors, we expect that the exponential convergence rate is obtained by only quadratic terms of estimation errors in a Lyapunov function. One of the feasible approaches is that the convergence rate of estimation errors can be speeded up by designing different estimator structures. As well known, the rotation matrix SO(3) is a nonlinear manifold, and the corresponding operational rules are group operations rather than linear operations. Therefore, the properties of SO(3) and corresponding extension matrices play an important role in designing a novel estimator. Specifically, the attitude error function proposed in [3] provides us with the foundation to design a novel estimator structure.

Inspired by the form of disturbance observers proposed in [5, 6], a novel estimator structure is proposed in this study, whose main contributions are twofold. Firstly, a novel angular velocity estimator scheme on $SO(3) \times \mathbb{R}^3$ is proposed. An intermediate variable introduced in the estimator design makes the exponential convergence rate of estimation errors be achieved through Lyapunov functions in quadratic form, which is more concise than those proposed in [2–4, 7]. Different from the estimators which improve the convergence rate by constructing different attitude error functions in previous studies, the estimator proposed in this study is realized by using a new estimator structure. Secondly, by studying the properties of SO(3) and corresponding extension matrices, the singular perturbation approach is used in stability analysis of proposed estimator and it proves that the convergence rate of estimated angular velocity mainly depends on a single gain.

Problem formulation. The equation of rigid body's rotation is given by

$$\dot{R} = R\hat{\Omega}, \tag{1}$$

$$\dot{\Omega} = J^{-1}((J\Omega)^{\wedge}\Omega + \tau) \triangleq f(\Omega) + J^{-1}\tau, \tag{2}$$

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where $R \in \mathrm{SO}(3)$ is the rigid body's attitude, $\Omega \in \mathbb{R}^3$ is the angular velocity expressed in the body frame $\mathcal{F}_B$, $J = \mathrm{diag}\{J_1, J_2, J_3\} \in \mathbb{R}^{3\times3}$ is the inertia matrix, and $\tau \in \mathbb{R}^3$ is the control torque. Some notions and mathematical preliminaries are provided in Appendix A. Regarding the angular velocity, we give the following assumption.

Assumption 1. The angular velocity Ω is bounded; namely, $\|\Omega\| \leqslant \Omega_{\max}$ holds all the time.

Based on rotation matrix directly, the objective of this study is to design a novel estimator to estimate angular velocity Ω.

Main results. As the rate gyroscope measurement is unavailable, the attitude estimator needs to be designed. The conventional nonlinear angular velocity estimator is based on adaptive methods. In this study, from the view of estimator structures, a novel estimator is designed as follows:

$$\dot{\tilde{R}} = \tilde{R}(\tilde{\Omega} - k_2 E_e)^\wedge, \tag{3a}$$

$$\tilde{\Omega} = z - k_1 E_e, \tag{3b}$$

$$\dot{z} = -k_1 \Phi_e^{\mathrm{T}} \tilde{\Omega} + k_1 \Phi_e(\tilde{\Omega} - k_2 E_e) + f(\tilde{\Omega}) + J^{-1}\tau, \tag{3c}$$

where $\tilde{R} \in \mathrm{SO}(3)$ is the estimation of attitude rotation matrix R, $\tilde{\Omega} \in \mathbb{R}^3$ is the estimation of the real angular velocity $\Omega \in \mathbb{R}^3$, $z \in \mathbb{R}^3$ is an intermediate variable and $z(t_0) = [0, 0, 0]^{\mathrm{T}}$, $k_2 > 0$, k_1 is a positive gain to be determined, $E_e \in \mathbb{R}^3$ is the attitude error vector and $E = \tilde{R}^{\mathrm{T}} R$ is the attitude error matrix, and $\Phi_e \in \mathbb{R}^{3\times3}$ is a matrix associated with the attitude error and is defined as

$$E_e = \frac{-1}{2[1 + \mathrm{tr}(E)]}(E - E^{\mathrm{T}})^\vee, \tag{4}$$

$$\Phi_e = \frac{4[1 + \mathrm{tr}(E)]E_e E_e^{\mathrm{T}} + [\mathrm{tr}(E) I_3 - E]}{2[1 + \mathrm{tr}(E^{\mathrm{T}})]}. \tag{5}$$

In this study, the properties of E_e will be used for the estimator design and stability analysis. The properties are given in the following proposition and its proof is provided in Appendix B.

Proposition 1. Let $D = \{E \in \mathrm{SO}(3) | \mathrm{tr}(E) > -1\}$ be a subset of $\mathrm{SO}(3)$ and assume that the trajectory $E(t) \in D$ for all time $t \geqslant 0$. The following statements about attitude error vector E_e in (4) hold.

(i) In the set D, $\|E_e\|_2^2 = \frac{1}{4}\tan^2(\frac{\theta_e}{2})$.

(ii) In the set D, $E_e E_e^{\mathrm{T}} = \frac{1}{4}\tan^2(\frac{\theta_e}{2})(n_e n_e^{\mathrm{T}})$.

(iii) In the set D, the time derivative of E_e satisfies

$$\dot{E}_e = -\Phi_e \Omega + \Phi_e^{\mathrm{T}}(\tilde{\Omega} - k_2 E_e). \tag{6}$$

(iv) In the set D, Φ_e can be rewritten as

$$\Phi_e = \frac{E^{\mathrm{T}} + I_3}{8\cos^2(\frac{\theta_e}{2})}, \tag{7}$$

the spectrum of symmetric matrix $\Phi_e^{\mathrm{T}} + \Phi_e$ is located in the set $\{\frac{1}{2}, \frac{1}{2}, 1/(2\cos^2(\frac{\theta_e}{2}))\}$, and $\lambda_{\min}(\Phi_e^{\mathrm{T}} + \Phi_e) = \frac{1}{2}$, where $|\theta_e| \in [0, \pi)$.

Based on the fact that the symmetric matrix $\Phi_e^{\mathrm{T}} + \Phi_e$ is positive definite in the set D, the motivation of estimator (3) can be discussed as follows.

Proposition 2. Consider the estimator (3) and the attitude kinematics (1). The dynamics of attitude estimation error E and angular velocity estimation error Ω_e are given by

$$\dot{E} = -(\tilde{\Omega} - k_2 E_e)^\wedge E + E\hat{\Omega}, \tag{8a}$$

$$\dot{\Omega}_e = -k_1 \Phi_e^{\mathrm{T}} \Omega_e + f(\tilde{\Omega}) - f(\Omega), \tag{8b}$$

where $\Omega_e = \tilde{\Omega} - \Omega \in \mathbb{R}^3$ is the angular velocity estimation error, and $f(\tilde{\Omega}) - f(\Omega) \in \mathbb{R}^3$ can be viewed as a disturbance term.

Proof. Using (1) and (3a), the time derivative of E is given as

$$\begin{aligned}\dot{E} &= -\tilde{R}^{\mathrm{T}} \dot{\tilde{R}} \tilde{R}^{\mathrm{T}} R + \tilde{R}^{\mathrm{T}} \dot{R} \\ &= -(\tilde{\Omega} - k_2 E_e)^\wedge E + E\hat{\Omega}.\end{aligned} \tag{9}$$

Then, using (2), (3b) and (3c), the time derivative of Ω_e is given as

$$\begin{aligned}\dot{\Omega}_e =& \dot{z} - k_1 \dot{E}_e + J^{-1}\tau - \dot{\Omega} \\ =& -k_1 \Phi_e^{\mathrm{T}} \tilde{\Omega} + k_1 \Phi_e(\tilde{\Omega} - k_2 E_e) + f(\tilde{\Omega}) + J^{-1}\tau \\ &+ k_1 \Phi_e^{\mathrm{T}} \Omega - k_1 \Phi_e(\tilde{\Omega} - k_2 E_e) - \dot{\Omega} \\ =& -k_1 \Phi_e^{\mathrm{T}} \Omega_e + f(\tilde{\Omega}) - f(\Omega).\end{aligned} \tag{10}$$

Remark 1. The dynamics equation (8b) plays a crucial role in the behavior of estimated angular velocity. Without the perturbation term $f(\tilde{\Omega}) - f(\Omega)$, the dynamics of Ω_e is a linear time-varying system. Further assume that the matrix Φ_e^{T} is slow-varying and can be seen as a constant matrix; then the dynamics of Ω_e is a linear time invariant system, and the convergence rate can be controlled by the gain k_1 because the real part of eigenvalues of matrix Φ_e^{T} are positive in the set D. This is the basic idea why we design such a novel estimator.

For a rigid body which is only installed with direction sensors, Proposition 1 shows that the symmetric matrix $\Phi_e^{\mathrm{T}} + \Phi_e$ is positive definite in the set D, and this indicates that $x^{\mathrm{T}} \Phi_e x > 0$ for all nonzero vector x (the matrix is not necessarily symmetric). Throughout this study, the matrix Φ_e and intermediate variable z allow the estimator to have an exponential convergence, and the convergence rate of estimated angular velocity mainly depends on a gain. Thus, we present the following theorem.

Theorem 1. Consider the rigid body (1) and (2) without the gyroscopes. For $E(t_0) \in D$, there exists a small positive constant $\varepsilon^* \in (0,1)$ such that for any $\varepsilon \in (0,\varepsilon^*)$ and the gain k_1 satisfying (11), the almost global estimator (3) can exponentially estimate the attitude R and angular velocity Ω, and the convergence rate of the estimated angular velocity mainly depends on k_1, that is

$$\frac{k_1}{4} - d((\sqrt{2}+1)\Omega_{\max} + k_1\|E_e(t_0)\|) \geqslant \frac{1}{\varepsilon}, \quad (11)$$

where d is defined as

$$d \triangleq \max\left\{\left|\frac{J_3-J_2}{J_1}\right|, \left|\frac{J_1-J_3}{J_2}\right|, \left|\frac{J_2-J_1}{J_3}\right|\right\}. \quad (12)$$

The proof of Theorem 1 will be solved in a two-time-scale frame, where Ω_e is the fast variable and E is the slow variable. As for singularly perturbed approach, it is required that the fast varying Ω_e converges to a positively invariant set in a short time so that the slow varying E can always stay in the set D all the time. Then, the exponential convergence is obtained by Lyapunov analysis for overall closed loop system. The proof process is provided in Appendix C. Besides, the numerical simulation is provided in Appendix D.

Remark 2. Following the adaptive approach, an angular velocity estimator can be designed as

$$\dot{\tilde{R}} = \tilde{R}(\tilde{\Omega} - \alpha_2 E_e)^\wedge, \quad (13a)$$

$$\dot{\tilde{\Omega}} = EJ^{-1}(\tau - (E^{\mathrm{T}}\tilde{\Omega})^\wedge JE^{\mathrm{T}}\tilde{\Omega} - \alpha_1 E_e + \alpha_2 \hat{E}_e E^{\mathrm{T}}\tilde{\Omega}), \quad (13b)$$

where $\alpha_1, \alpha_2 > 0$ are positive gains, and $\bar{\Omega}_e = \Omega - E^{\mathrm{T}}\tilde{\Omega}$. Consider a Lyapunov function which consists of quadratic terms of estimation errors such that $V_0 = \Psi + \overline{W}$, where $\Psi = \ln(2) - \frac{1}{2}\ln(1+\mathrm{tr}(E))$, and $\overline{W} = \frac{1}{2\alpha_1}\bar{\Omega}_e^{\mathrm{T}} J \bar{\Omega}_e$. Then, its time derivative is $\dot{V}_0 = -\alpha_2\|E_e\|^2$, and the asymptotic stability can be obtained by Barbalat's lemma. Furthermore, consider the Lyapunov function which consists of quadratic and cross terms of estimation errors such that $\overline{V} = V_0 + pE_e^{\mathrm{T}}\bar{\Omega}_e$ for some scalar $p > 0$. It follows that $x^{\mathrm{T}}W_1x \leqslant \overline{V} \leqslant x^{\mathrm{T}}W_2x$, and $\dot{\overline{V}} \leqslant -x^{\mathrm{T}}W_3x$, where $x = [\|E_e\|, \|\bar{\Omega}_e\|]^{\mathrm{T}}$, W_1, W_2, W_3 are positive definite constant matrix, and the exponential stability is obtained. It shows that the convergence rate of estimated angular velocity is controlled by a coordinated selection of two gains α_1, α_2. In this study, the exponential stability of estimator (3) can be achieved with only quadratic terms of estimation errors in Lyapunov function so that it has desired convergence properties. By introducing an intermediate variable z in the design process of the estimator, the convergence rate of estimated angular velocity mainly depends on the gain k_1. The singularly perturbed method is used in the stability analysis, and the estimator (3) can be seen as a high gain observer [8]. Based on singular perturbation approach, the gain k_1 exists and depends on the initial condition $\Phi_e^{\mathrm{T}} + \Phi_e$.

Conclusion. A nonlinear angular estimator for rigid body whose kinematics evolves on SO(3), which can describe the rigid body's rotation globally and uniquely, is studied. By introducing an intermediate variable, a novel estimator framework is proposed. The singular perturbation method, which divides the closed loop systems into a fast system and a slow system, proves a rapid convergence properties of angular velocity estimation error, and the convergence rate of estimated angular velocity is mainly determined by a single control gain.

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Supporting information Appendixes A–D. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

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