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• **Supplementary File** •

On the conceptualization of "total disturbance" and its profound implications

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The supplementary file has the following organization. The mathematical notations are introduced in Appendix A. In Appendix B, the motivation of the work is introduced. In Appendix C, the main contributions of this work are described. The model description of the two-mass-spring (TMS) system is given in Appendix D. In Appendix E, the difficulties of dealing with unobservable and mismatched disturbances are rigorously studied. In Appendix F, the active disturbance rejection control (ADRC) design for uncertain systems with multiple disturbances via the conceptualization of integrator chain and total disturbance is proposed. Some extended discussions on ADRC designs are given in Appendix G. The experimental verification on a TMS system is presented in Appendix H.

Appendix A Mathematical notations

The following mathematical notations are introduced. $y^{(k)}$ is the k-th derivative of y for $k \geqslant 1$ and $y^{(0)} \triangleq y$. $\mathcal{C}^-$ is the left plane in the complex plane. $\lambda(M)$ is the set of the eigenvalues of the matrix M. $\lambda_{\min}(M)$ ($\lambda_{\max}(M)$) is the minimal (maximum) value of the real part of M's eigenvalues. $|\cdot|$ and $\|\cdot\|$ are the absolute value of a scalar and the 2-norm of a vector or a matrix, respectively. $C^\infty(\Omega_1, \Omega_2)$ represents a functional space in which the functions mapping from Ω_1 to Ω_2 have any order derivatives. $\boldsymbol{I}$ and $\boldsymbol{0}$ are unit matrix and null matrix, respectively. Moreover, the following notations are commonly used in this study.

$$A_e = \begin{bmatrix} A & B_f \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \quad b_{ue} = \begin{bmatrix} b_u \\ \boldsymbol{0} \end{bmatrix}, \quad c_e = \begin{bmatrix} c \\ \boldsymbol{0} \end{bmatrix}. \tag{A1}$$

Appendix B Motivation of the work

How to deal with uncertainties and/or disturbances is a central issue pushing the development of both control science and control technology [1–4]. Plenty of methods with the aim to ensure normal operation of systems against uncertainties and/or disturbances have been substantially developed, such as proportional-integral-derivative (PID) control [5, 6], feedforward control via measured disturbances [7], adaptive control [8], robust control [9], sliding-mode control [10], and various disturbance rejection methods based on estimating disturbances including disturbance accommodation control [11], proportional integral observer (PIO) based control [12], disturbance observer based control [13–17], ADRC [18], etc.

Among these approaches, ADRC has been successfully implemented in various industrial practices [19–23] due to its uniqueness in concepts, simplicity in engineering implementation, and superior performance. The ideology of ADRC was sprouted in [24, 25], which insightfully proposed the integrator chain of the controlled variable as the most fundamental structure for control system, linear or nonlinear, and went further to creatively generalize the concept of disturbance to the "total disturbance". In control theory, disturbance is usually referred to a disruption caused by an external force, such as wind gust acting on an airplane. Uncertainty is a much broader notion, which includes unknowns from both interior (e.g. parametric perturbation or structured uncertainty) and exterior (e.g. external disturbance) of the plant. Different from the concepts of disturbance and uncertainty, the "total disturbance" is the equivalent total effect of various uncertainties and/or disturbances on the controlled variable. The conceptualization of total disturbance attempts to grasp the key factor that influences the purpose of controlling a certain transitory process of the controlled variable, no matter it is caused by

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uncertainty or disturbance, from interior or exterior. Through such insightful and innovative conception of total disturbance, the extended state observer (ESO), a core of ADRC, was proposed to timely estimate the total disturbance. Finally, the ADRC design containing the compensation for the total disturbance and the control law for the system of the simple ideal integrator chain form was developed [18]. Due to its unique attitude to control design for systems with uncertainties and/or disturbances, ADRC provides a breakthrough for this challenging problem. Ever since ADRC was proposed by Jingqing Han in 1990s, it has become quite a valuable asset in engineering practice.

In many literatures, ADRC is usually illuminated by the control problem of nonlinear uncertain systems in the following form:

$$\dot{x}_i = x_{i+1} \quad (i = 1, 2, ..., n-1), \quad \dot{x}_n = f(x_1, x_2, ..., x_n, u, w(t), t) + bu(t), \quad y = x_1, \tag{B1}$$

where $(x_1, x_2, ..., x_n)$ are the system states, y is the output, measured and to be controlled, u is the control input, and $f(x_1, x_2, ..., x_n, u, w(t), t)$ is a multivariable function of the system states $(x_1, x_2, ..., x_n)$, control input u, external disturbances $w(t)$ and time t. In the framework of ADRC, $f(x_1, x_2, ..., x_n, u, w(t), t)$ is regarded as the total disturbance which lumps parametric perturbation, unmodeled nonlinear dynamics of the plant and external disturbance from the environment. By treating the total disturbance as an extended state of the system (B1), the ADRC controller can be constructed via the estimations of the system states and the total disturbance obtained from ESO [26].

Since the system (B1) is an integrator chain system with the total disturbance appearing in the same channel with that of the control input, some doubts about the capability of ADRC are raised.

1. The integrator chain system is mistaken as a simple and special kind of systems. Since many practical systems are not modeled as such special form, this misunderstanding raises a question: Can ADRC handle uncertain systems in more general form?

2. It seems that the total disturbance in the system (B1) satisfies the matched condition (MC), while the disturbances and uncertainties may appear from various sources in practice. Then, one more question about ADRC is: Can ADRC tackle mismatched uncertainties and/or disturbances?

These doubts are caused by the misunderstanding of the crucial concepts of integrator chain form and the "total disturbance" in ADRC. This missing link motivates this study. Some references have devoted to the ADRC design for a class of lower-triangular uncertain systems [26–29], which reveals that ADRC is capable of dealing with a wider class of uncertain plants not in the integrator chain form with the matched uncertainties and/or disturbances. Via the profound implications of integrator chain and total disturbance, this study further illuminates the capability of ADRC for a more general kind of uncertain systems, where there exist multiple external disturbances and uncertainties in the plant model, not assumed to be in a lower-triangular form. In order to study the possibility of dealing with uncertainties via the way of disturbance rejection, both of uncertainty and external disturbance are termed as disturbance in this study, for the sake of simplicity and clarity.

Appendix C Main contributions of this work

Firstly, the problem of multiple disturbances, which might be unobservable and/or mismatched, is systematically studied. The observability for the system states and the disturbances is rigorously studied. Furthermore, the biased estimation error of a recklessly designed observer, which may be caused by the unobservable disturbances, is quantitatively analyzed. The essence of MC and the difficulty of handling the mismatched disturbance are also discussed. The rigorous studies of multiple disturbances show that, for systems with multiple disturbances, it is usually impossible to deal with each disturbance individually. Then, to handle this challenging control problem, some detailed ADRC designs with the conceptualization of integrator chain and total disturbance are proposed. Taking a wide class of disturbances in engineering practice into consideration, the performance analysis of the closed-loop system in the whole time domain is made. The capability of ADRC for tackling the multiple unobservable and mismatched disturbances is tested via the experiments on a TMS system, which represents a typical vibration system in a variety of industry sectors.

Compared with the previous methods, this study discusses a novel way to deal with the multiple disturbances via the conceptualization of the integrator chain of the controlled variable and the total disturbance, whose essences can be described as follows.

1). The dynamics described by the integrator chain model from the control input to controlled variable is the kernel of most dynamical systems associated with control engineering practice, rather than a special and simple case;

2). The total disturbance corresponds to the difference between the dynamics of the physical plant and its idealization, i.e. the chain of cascade integrators from the control input to controlled variable. It is the lumped effects of all disturbances, internal and external, projected at the control input side in this integrator chain. This concept proves crucial in vast amount of real applications because such total disturbance can be estimated in real time and counteracted by the control force so that the controlled variable, i.e. the plant output, is shielded from the impact of various disturbances.

The analyses in the study indicate that even if some specific disturbances are unobservable and mismatched, the total disturbance is definitely observable and matched. Thus, ESO is not necessarily designed for estimating a specific disturbance. What ESO provides is the estimations of the derivatives of the controlled variable and the total disturbance.What ADRC compensates for is the total disturbance, rather than a specific disturbance. This is the core of ADRC. In other words, with the conceptualization of integrator chain and total disturbance, ADRC provides a general solution for controlling uncertain systems no matter what description of the physical plant is.

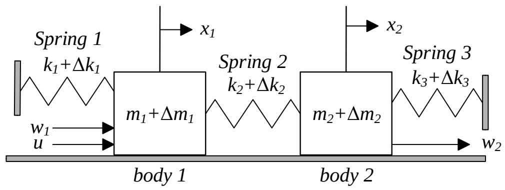

Figure D1 Two-mass-spring system with multiple disturbances.

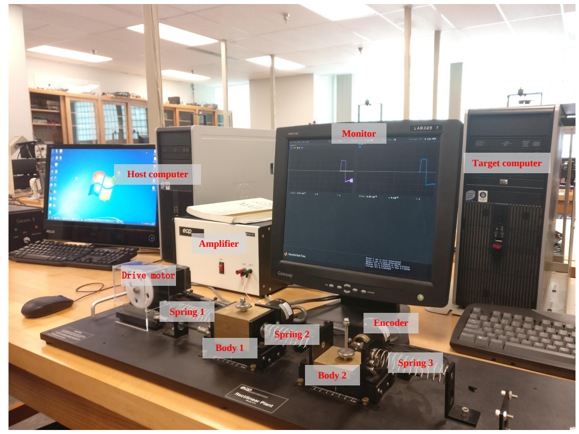

Figure D2 Photo of the test setup.

Appendix D Model description of TMS systems

A variety of practical systems can be described by the model (1), such as the TMS system shown in Fig. D1 and Fig. D2, which is regarded as an equivalent system for most typical vibration systems in a diversity of industry sectors [30]. According to Newton's second law and Hooke's law, this TMS system can be represented by the model (1) with

$$A=\begin{bmatrix}0&0&1&0\\0&0&0&1\\-\frac{k_1+k_2}{m_1}&\frac{k_2}{m_1}&0&0\\\frac{k_2}{m_2}&-\frac{k_2+k_3}{m_2}&0&0\end{bmatrix},\quad b_u=\begin{bmatrix}0\\0\\\frac{1}{m_1}\\0\end{bmatrix},\quad B_f=\begin{bmatrix}b_{f,1}&b_{f,2}\end{bmatrix}=\begin{bmatrix}0&0\\0&0\\1&0\\0&1\end{bmatrix}\tag{D1}$$

and

$$\begin{cases}f_1(x,u,t)=\dfrac{m_1w_1-\Delta m_1u+((k_1+k_2)\Delta m_1-(\Delta k_1+\Delta k_2)m_1)x_1+(\Delta k_2m_1-k_2\Delta m_1)x_2}{m_1(m_1+\Delta m_1)},\\ f_2(x,u,t)=\dfrac{m_2w_2+(\Delta k_2m_2-k_2\Delta m_2)x_1+((k_2+k_3)\Delta m_2-(\Delta k_2+\Delta k_3)m_2)x_2}{m_2(m_2+\Delta m_2)},\end{cases}\tag{D2}$$

where

$x_1,\ x_3$: the displacement and the velocity of the body 1,
$x_2,\ x_4$: the displacement and the velocity of the body 2,
$m_i,\ \Delta m_i$: the known and the unknown parts of the mass of the body i, $i=$ 1,2,
$k_i,\ \Delta k_i$: the known and the unknown parts of the spring constants of the spring i, $i=$ 1,2,3,
$w_1,\ w_2$: the disturbances affecting the body 1 and 2, respectively,
$f_1,\ f_2$: the disturbances affecting the body 1 and 2, respectively.

The measured output could be $y(t)=x_1(t)$ or $y(t)=x_2(t)$, according to different control objectives that let the body 1 or body 2 track the designed reference signal [30].

The main challenge of the control design for the TMS system (1) and (D1) is to achieve non-oscillatory tracking performance against multiple disturbances f_1 and f_2 via one control signal u. As shown in (D2), $f_i(x,u,t)$ contains both the parametric uncertainties and the external disturbances. However, neither f_1 nor f_2 is the total disturbance from the point of ADRC. This will be discussed in details in Appendix F. Throughout this supplementary file, the TMS system (1) and (D1) will be used as an example to illuminate the meanings of the theoretical studies in this work.

Appendix E Unobservable and mismatched disturbances

The difficulties for controlling the uncertain system (1) with multiple disturbances, which might be unobservable and/or mismatched, will be discussed in this appendix.

Appendix E.1 Unobservable disturbances

One solution for controlling the uncertain system (1), which may easily come to mind, is to try to estimate the disturbances f and then reject it by the control input. Nevertheless, a prerequisite for such approach is that f should be observable. The definition of observability for the system state x and the disturbances f is presented as follows.

Definition 1. The state x and the disturbances f of the uncertain system (1) are said to be observable, if for any $x(t_0)\in R^n$, $f\in C^\infty(R^{n+2},R^p)$ and $t_T>t_0$, the values of x and f on $[t_0,t_T]$ can be uniquely determined from the output $y(t)$ and the control input $u(t)$ on $[t_0,t_T]$.

Remark 1. If $f \equiv \boldsymbol{0}$, that is, the system (1) is reduced to a linear time-invariant system, then Definition 1 is the same as the observability of linear time-invariant system [31].

The observability of x and f means that the system state and the disturbances can be reconstructed by the knowledge of the input signal u and the output signal y.

The following proposition provides the necessary and sufficient condition for the observability of x and f, which is a generalization of Lemma 2 in [32].

Proposition 1. Assume $n \geqslant 2$, the state x and the disturbances f of the system (1) are observable, if and only if the pair (A_e, c_e) is observable, and

$$c^T A^i B_f = \boldsymbol{0}, \quad 0 \leqslant i \leqslant n-2, \quad c^T A^{n-1} B_f \neq \boldsymbol{0}. \tag{E1}$$

Proof of Proposition 1: (Sufficiency) Since the pair (A_e, c_e) is observable, it can be deduced that $p = 1$. Moveover, the matrix

$$T_Q \triangleq \begin{bmatrix} c_e & (c_e^T A_e)^T & \cdots & (c_e^T A_e^n)^T \end{bmatrix}^T \tag{E2}$$

is nonsingular. From the condition (E1), the following equation can be obtained.

$$\begin{bmatrix} x \\ f \end{bmatrix} = T_Q^{-1} \left(\begin{bmatrix} y \\ y^{(1)} \\ \vdots \\ y^{(n)} \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & 0 \\ c^T b_u & 0 & \cdots & 0 \\ c^T A b_u & c^T b_u & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ c^T A^{n-1} b_u & \cdots & \cdots & c^T b_u \end{bmatrix} \begin{bmatrix} u \\ u^{(1)} \\ \vdots \\ u^{(n-1)} \end{bmatrix} \right). \tag{E3}$$

From (E3), it is concluded that the state x and the disturbance f on $[t_0, t_T]$ can be uniquely determined by the output y and the control input u on $[t_0, t_T]$ for any $(x(t_0), f, t_T)$.

(Necessity) First, consider a special case that the disturbance f is a constant such that $f^{(i)} = 0$ $(i \geqslant 1)$. Then the system (1) is reduced to a linear time-invariant system with a constant disturbance. Owing to the observability of linear time-invariant system [31], the observability condition for such special case is that the pair (A_e, c_e) is observable. Therefore, the number of disturbances equals one, i.e., $p = 1$.

Next, we will proof that (E1) is satisfied. For a general disturbance $f \in C^\infty(R^{n+2}, R^p)$, by denoting

$$\bar{i} \triangleq \min_{i \geqslant 0} \left\{ i | \ c^T A^i B_f \neq 0 \right\}, \tag{E4}$$

the following equation is obtained.

$$\begin{bmatrix} y \\ y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(\bar{i}+l_f)} \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & 0 \\ c^T b_u & 0 & \cdots & 0 \\ c^T A b_u & c^T b_u & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ c^T A^{\bar{i}+l_f-1} b_u & \cdots & \cdots & c^T b_u \end{bmatrix} \begin{bmatrix} u \\ u^{(1)} \\ u^{(2)} \\ \vdots \\ u^{(\bar{i}+l_f-1)} \end{bmatrix}$$
$$= \begin{bmatrix} c^T & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c^T A^{\bar{i}} & 0 & 0 & \vdots & \vdots \\ c^T A^{\bar{i}+1} & c^T A^{\bar{i}} B_f & 0 & \ddots & \vdots \\ c^T A^{\bar{i}+2} & c^T A^{\bar{i}+1} B_f & c^T A^{\bar{i}} B_f & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ c^T A^{\bar{i}+l_f} & c^T A^{\bar{i}+l_f-1} B_f & c^T A^{\bar{i}+l_f-2} B_f & \cdots & c^T A^{\bar{i}} B_f \end{bmatrix} \begin{bmatrix} x \\ f \\ f^{(1)} \\ \vdots \\ f^{(l_f-1)} \end{bmatrix}, \tag{E5}$$

where l_f is any positive integer and $(x, f, f^{(1)}, \ldots, f^{(l_f-1)})$ are unknown variables to be determined. To get the unique (x, f) from the equation (E5), it is necessary to request the number of the equations in (E5) to be larger than the number of unknown variables, which means that $\bar{i} \geqslant n-1$. On the other hand, since the matrix T_Q in (E2) is nonsingular, it can be deduced that $\bar{i} \leqslant n-1$. Consequently, we conclude that $\bar{i} = n-1$. Hence, (E1) is satisfied. ∎

Remark 2. If $n = 1$, both the state x and the disturbances f of the system (1) are definitely observable.

Proposition 1 presents an algebraic criteria for the observability of x and f, which is only related to the structure of the system matrices (A, B_f, c) and is independent with the form of the disturbances f.

Remark 3. The disturbance resistibility of the uncertain system (1) has been fully studied in [33], where the necessary and sufficient condition for disturbance resistibility is proposed as follows.

$$c^T A^i B_f = \boldsymbol{0}, \quad i \geqslant 0. \tag{E6}$$

Based on Proposition 1, the condition (E6) indicates that the disturbances f are unobservable. Moreover, under the disturbance resistibility condition (E6), the output of the system (1) satisfies that $y^{(i)}(t) = c^T A^i x(t) + \sum_{k=1}^{i} c^T A^{k-1} b_u u^{(i-k)}(t)$, which implies that the output is invariant to the multiple disturbances f. In other words, the multiple disturbances f will not influence the output. However, if there exists a positive $k < n-1$ such that $c^T A^k B_f \neq \boldsymbol{0}$, according to (E1) and (E6), the disturbances f are unobservable but they will affect the system output.

According to Proposition 1, the condition that the pair (A_e, c_e) is observable implies that $p \leqslant 1$, i.e., the number of the disturbances should be less than one. Thus, if $p > 1$, it is impossible to construct any observer to estimate the multiple disturbances f via one measurement. However, even if there is only one disturbance in the system (1), i.e., $f \in R$, and the pair (A_e, c_e) is observable, the disturbance f still might be unobservable, which may result in the biased estimation for the state x and the disturbance f. In [34], the estimation error for x and f by the following observer

$$\dot{\hat{x}}_e(t) = A_e \hat{x}_e(t) + b_{ue} u(t) + l_e c_e^T (x_e(t) - \hat{x}_e(t)), \quad t \geqslant t_0, \tag{E7}$$

where $l_e \in R^{n+1}$ is the observer gain to be designed and $\hat{x}_e(t) = [\hat{x}(t)^T \ \hat{f}(t)]^T$ is the estimation for x and f, is quantitatively studied and is presented in the following proposition.

Proposition 2. Consider the uncertain system (1) and the observer (E7). Assume that the pair (A_e, c_e) is observable and the disturbance $f \in R$ has the bounded first n-th order derivatives. For any given symmetric set (symmetric with respect to the real axis) $\Lambda = \{\lambda_1, \cdots, \lambda_{n+1}\}$, $\lambda_i \in \mathcal{C}^-$, and any $\beta > 0$, there exists l_e such that $\lambda(A_e - l_e c_e^T) = \beta\Lambda$ and

$$\left\| \begin{bmatrix} x(t) \\ f(x,u,t) \end{bmatrix} - \hat{x}_e(t) - e_b(t) \right\| \leqslant \gamma_1 \left(e^{-\gamma_2 \beta (t-t_0)} + \frac{1}{\beta} \right), \quad t \geqslant t_0, \tag{E8}$$

where γ_1 and γ_2 are positive constants independent with β and

$$e_b(t) = -\begin{bmatrix} c_e^T \\ c_e^T A_e \\ \vdots \\ c_e^T A_e^n \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ c^T B_f & 0 & \cdots & 0 \\ c^T A B_f & c^T B_f & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ c^T A^{n-2} B_f & \cdots & & c^T B_f \end{bmatrix} \begin{bmatrix} f^{(1)} \\ f^{(2)} \\ \vdots \\ f^{(n-1)} \end{bmatrix}. \tag{E9}$$

Proof of Proposition 2: Since the pair (A_e, c_e) is observable, the matrix T_Q in (E2) is invertible. Denote

$$v(t) = T_Q(x_e(t) - \hat{x}_e(t) - e_b(t)), \quad x_e(t) = \begin{bmatrix} x(t) \\ f(x,u,t) \end{bmatrix}. \tag{E10}$$

From (1) and (E7), the dynamics of $v(t)$ is shown as follows.

$$\dot{v}(t) = (A_{e,v} - l_{e,v} c_{e,v}^T) v(t) - H F_\Delta, \tag{E11}$$

where $l_{e,v} = T_Q l_e$,

$$A_{e,v} = T_Q A_e T_Q^{-1} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ a_1 & a_2 & \cdots & a_{n+1} \end{bmatrix}, \ c_{e,v}^T = c_e^T T_Q^{-1} = \begin{bmatrix} 1 \\ \boldsymbol{0} \end{bmatrix}^T, \ H = \begin{bmatrix} \boldsymbol{0} \\ 1 \end{bmatrix} \tag{E12}$$

and

$$F_\Delta = -c_e^T A_e^{n+1} e_b - \sum_{j=1}^{n} c^T A^{n-j} B_f f^{(j)}. \tag{E13}$$

Due to the assumption that the disturbance $f \in R$ has the bounded first n-th order derivatives, F_Δ is bounded.

Since the pair $(A_{e,v}, c_{e,v})$ is observable, there exist $l_{e,v}$, $\tilde{l}_{e,v}$ and the invertible matrices P_L and $\tilde{P}_L$ such that

$$A_{e,v} - l_{e,v} c_{e,v}^T = \beta P_L^{-1} M_\Lambda P_L, \quad A_{e,v} - \tilde{l}_{e,v} c_{e,v}^T = \tilde{P}_L^{-1} M_\Lambda \tilde{P}_L, \tag{E14}$$

where

$$M_\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_{n+1} \end{bmatrix}. \tag{E15}$$

The relationship between P_L and $\tilde{P}_L$ is obtained as follows.

$$P_L = \tilde{P}_L T_P \tag{E16}$$

where $T_P = M_P^{-1} M_\beta M_P$ and

$$M_P = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -a_{n+1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & 0 \\ -a_2 & \cdots & -a_{n+1} & 1 \end{bmatrix}, \quad M_\beta = \begin{bmatrix} \beta^n & 0 & \cdots & 0 \\ 0 & \beta^{n-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{E17}$$

Denote $\tilde{v}(t) = T_P v(t)$, from (E11), (E14) and (E16), there is

$$\dot{\tilde{v}}(t) = \beta(A_{e,v} - \tilde{l}_{e,v} c_{e,v}^T)\tilde{v}(t) - HF_\Delta. \tag{E18}$$

Since the matrix $(A_{e,v} - \tilde{l}_{e,v} c_{e,v}^T)$ is Hurwitz, there exists a positive definite matrix Q_L such that

$$(A_{e,v} - \tilde{l}_{e,v} c_{e,v}^T)^T Q_L + Q_L(A_{e,v} - \tilde{l}_{e,v} c_{e,v}^T) = -\boldsymbol{I}. \tag{E19}$$

Consider the Lyapunov function $\tilde{V}(t) = \tilde{v}(t)^T Q_L \tilde{v}(t)$. Then, we have

$$\lambda_{\min}(Q_L)\left\|\tilde{v}(t)\right\|^2 \leqslant \tilde{V}(t) \leqslant \lambda_{\max}(Q_L)\left\|\tilde{v}(t)\right\|^2, \tag{E20}$$

and

$$\frac{d\sqrt{\tilde{V}(t)}}{dt} = \frac{-\beta\left\|\tilde{v}(t)\right\|^2 - 2\tilde{v}(t)^T Q_L H F_\Delta}{2\sqrt{\tilde{V}(t)}} \leqslant -\frac{\beta}{2\lambda_{\max}(Q_L)}\sqrt{\tilde{V}(t)} + \frac{\|Q_L\||F_\Delta|}{\sqrt{\lambda_{\min}(Q_L)}}. \tag{E21}$$

Since F_Δ is bounded, i.e. $|F_\Delta|^2 < \eta_{F_\Delta}$ for some positive η_{F_Δ}, with the help of Gronwall Lemma [35], it is indicated from (E21) that

$$\sqrt{\tilde{V}(t)} \leqslant e^{-\frac{\beta}{2\lambda_{\max}(Q_L)}(t-t_0)}\sqrt{\tilde{V}(t_0)} + \frac{2\|Q_L\|\eta_{F_\Delta}\lambda_{\max}(Q_L)}{\beta\sqrt{\lambda_{\min}(Q_L)}}. \tag{E22}$$

Due to (E20), (E22) and

$$\|x_e(t) - \hat{x}_e(t) - e_b(t)\| \leqslant \|T_Q^{-1}\|\|T_P^{-1}\|\|\tilde{v}(t)\|, \tag{E23}$$

(E8) is obtained. ■

Remark 4. Actually, the observer (E7) is a PIO, which a little easily comes to mind by simply regarding the disturbances f as extended states of the system (1).

From Proposition 2, the estimation error $\left([x(t)^T\ f(x,u,t)]^T - \hat{x}_e(t)\right)$ of the observer (E7) is determined by two parts. One part is $\gamma_1\left(e^{-\gamma_2\beta(t-t_0)} + 1/\beta\right)$, which can be tuned rapidly decreased to a small bound γ_1/β by the observer gain l_e. Another part is $e_b(t)$, which is related to the inherent system matrices (A, B_f, c) and the derivatives of the disturbance f. Hence, $e_b(t)$ is the bias of the estimation error, which can not be tuned smaller by the observer gain l_e. Even if the pair (A_e, c_e) is observable, the unobservable disturbances will lead to nonzero $e_b(t)$, which results in biased estimation for x and f.

With the help of Proposition 1 and Proposition 2, the observability of the TMS example system (1) and (D1) can be clearly demonstrated. If the measured output is x_1 and the disturbances only exist on the body 1, which implies $c = [1\ 0\ 0\ 0]^T$, $B_f f = b_{f,1} f_1$, and $c^T A b_{f,1} = 1 \neq 0$, then the disturbance f_1 is unobservable according to Proposition 1. Hence, the estimation for x and f from the observer (E7) is not reliable. Moreover, according to Proposition 2, the biased estimation error is

$$e_b(t) = \left[0\ \ \tfrac{m_2}{(k_2+k_3)k_2} f_1^{(2)}\ \ 0\ \ -\tfrac{1}{k_2} f_1^{(1)}\ \ -\tfrac{m_2}{k_2+k_3} f_1^{(2)}\right]^T, \tag{E24}$$

which illustrates that the first and second order derivatives of f_1 will influence the estimation for x_2, x_4 and f_1. On the other hand, since (E6) is not satisfied, the disturbance f_1 does affect the output y and needs to be dealt with by control design. Hence, how to handle the unobservable disturbance is a challenging and practical control problem.

Another difficulty for controlling the uncertain system (1) is that the disturbances f may be mismatched, which will be discussed in Appendix E.2.

Appendix E.2 Mismatched disturbances

For the system (1), the MC for the disturbances f is

$$\text{Rank}(b_u, B_f) = \text{Rank}(b_u). \tag{E25}$$

Otherwise, if there exists i such that $\text{Rank}(b_u, b_{f,i}) > \text{Rank}(b_u)$, where $b_{f,i}$ is the i-th column of the matrix B_f, the disturbance f_i is called mismatched [36–40].

The MC (E25) guarantees that there exists a constant matrix $A_{fu} \in R^{1\times p}$ such that $B_f = b_u A_{fu}$ [36–40]. Then, if the estimation of the disturbances f, denoted as $\hat{f}$, can be obtained, the compensation for the disturbances f can be easily achieved by $u = -A_{fu}\hat{f}$.

In many existing literatures on control of uncertain systems, MC is usually used as a prerequisite. For example, in the theory of linear output regulation [36, 39], MC is necessary for asymptotical stabilization of the state. In sliding model control theory [37, 38], MC, which is also named as the invariance condition, is assumed in order to achieve the desired tracking performance. In the robust control theory, MC is also a key assumption [40], while, for systems not satisfying MC, the control problem is only studied for a limited class of uncertain systems.

However, mismatched disturbances are ubiquitous in engineering practice. Hence, many recent researches are focusing on the control design for the mismatched disturbances. In [27–29], the ADRC design for a class of lower-triangular uncertain system with mismatched disturbances is proposed. In [41], by utilizing the estimation for the observable but mismatched disturbances via ESO, the control approach based on modified sliding model is proposed.

In Appendix F, the possibility of ADRC design for the more general uncertain system (1), which is not lower-triangular and contains unobservable and mismatched disturbances, will be discussed. The TMS system (1) and (D1) is such an example, which is not in the lower-triangular form. Additionally, if both the disturbances f_1 and f_2 exist, they are unobservable. Moreover, since $\text{Rank}(b_u, b_{f,2}) > \text{Rank}(b_u)$, the disturbance f_2 is mismatched. Therefore, the TMS system (1) and (D1) is a representation of uncertain systems with unobservable and mismatched disturbances.

Appendix F ADRC design via conceptualization of total disturbance

From Appendix E, dealing with multiple disturbances individually seems impossible since the disturbances might be unobservable and mismatched. ADRC, however, provides a different ideology of dealing with disturbances that focuses on the integrator chain from the control input to the controlled variable and seizes the total disturbance, which is the difference between the real physical plant and the ideal integrator chain. This appendix will uncover the integrator chain and the equivalent "total disturbance" of the uncertain system (1). Then the essence of the total disturbance, which is both observable and matched, will be clearly illuminated. Finally, some detailed ADRC design for the uncertain system (1) with multiple disturbances will be proposed.

Firstly, the total disturbance of the uncertain system (1) will be analyzed.

Appendix F.1 Conceptualization of total disturbance

For the sake of simplicity and clarity, assume that the uncertain system (1) satisfies the following assumption.

Assumption 1. For every $f \in \Omega_f$, the relative degree from u to y is n.

The relative degree is the the minimum number of integrators from the control input to the controlled output, which can be definitely determined by the control mechanism of physical plant whatever the model description is. Assumption 1 is a regular assumption. For example, the uncertain systems with lower-triangular form in [26–29], which is a special case of the uncertain system (1), satisfies Assumption 1.

Denote the controlled variable y and its up to $(n-1)$-th derivatives as a new state vector as follows.

$$\tilde{x}(t) = \begin{bmatrix} \tilde{x}_1(t) & \tilde{x}_2(t) & \cdots & \tilde{x}_n(t) \end{bmatrix}^T, \quad \tilde{x}_k(t) = y^{(k-1)}(t), \quad 1 \leqslant k \leqslant n. \tag{F1}$$

Actually, the state vector $\tilde{x}$ is an integrator chain of the controlled variable. However, due to the existence of the disturbances f, the relationship between $\tilde{x}$ and the state x of the uncertain system (1) becomes unclear. The following theorem describes the relationship between $\tilde{x}$ and x and further presents the connection between the control input u and the integrator chain $\tilde{x}$.

Theorem 1. If the uncertain system (1) satisfies Assumption 1, then

(1). For all $x_0 \in R^n$, there exists a neighborhood of x_0, $U(x_0)$, and a function $g_{f,x_0}(\tilde{x}, t)$ dependent on (f, x_0), such that $x = g_{f,x_0}(\tilde{x}, t)$ for $x \in U(x_0)$.

(2). For $x \in U(x_0)$,the integrator chain form of the uncertain system (1) is

$$\begin{cases} \dot{\tilde{x}}_i(t) = \tilde{x}_{i+1}(t), \quad 1 \leqslant i \leqslant n-1, \\ \dot{\tilde{x}}_n(t) = c^T A^{n-1} b_u u(t) + c^T A^n g_{f,x_0}(\tilde{x}, t) + \sum\limits_{k=0}^{n-1} c^T A^{n-k-1} B_f f^{(k)}(g_{f,x_0}(\tilde{x}, t), u, t), \end{cases} \quad y(t) = \tilde{x}_1(t). \tag{F2}$$

To prove Theorem 1, the notations of Lie derivatives are introduced as follows.

$$\begin{cases} L^1_{f(x,t)} h(x,t) = \dfrac{\partial h(x,t)}{\partial x} f(x,t), \quad \bar{L}^0_{f(x,t)} h(x,t) = h(x,t), \quad \bar{L}^1_{f(x,t)} h(x,t) = \dfrac{\partial h(x,t)}{\partial x} f(x,t) + \dfrac{\partial h(x,t)}{\partial t}, \\ \bar{L}^1_{g(x,t)} \bar{L}^1_{f(x,t)} h(x,t) = \bar{L}^1_{g(x,t)} (\bar{L}^1_{f(x,t)} h(x,t)), \quad \bar{L}^k_{f(x,t)} h(x,t) = \bar{L}^1_{f(x,t)} \bar{L}^{k-1}_{f(x,t)} h(x,t), \quad k \geqslant 1, \end{cases} \tag{F3}$$

Remark 5. $L^1_{f(x,t)} f(x,t)$ represents the well-known Lie derivative and $\bar{L}^1_{f(x,t)} h(x,t)$ is the corresponding generalization for time-varying function such that $\bar{L}^1_{f(x,t)} h(x,t) \triangleq L^1_{f(x,t)} h(x,t) + \frac{\partial h(x,t)}{\partial t}$ [42].

Then, the following helpful lemma is presented to prove Theorem 1.

Lemma 1: Consider the system (1). Then Assumption 1 is satisfied if and only if

$$\begin{cases} c^T A^k b_u = 0, \quad \dfrac{\partial \left(\bar{L}^{k+1}_{(Ax+b_u u+B_f f)} (c^T x) \right)}{\partial u} = \boldsymbol{0}, \quad 0 \leqslant k \leqslant n-2, \\ c^T A^{n-1} b_u \neq 0, \quad \dfrac{\partial \left(\bar{L}^{n}_{(Ax+b_u u+B_f f)} (c^T x) \right)}{\partial u} \neq \boldsymbol{0}. \end{cases} \tag{F4}$$

With the help of Lemma 1, the proof of Theorem 1 is presented as follows.

Proof of Theorem 1: Since the uncertain system (1) satisfies Assumption 1, according to Lemma 1, the state $\tilde{x}_i$ can be further expressed as

$$\tilde{x}_i(t) = c^T A^{i-1} x + \sum_{k=0}^{i-2} c^T A^{i-2-k} B_f f^{(k)}(x, u, t) = \bar{L}^{i-1}_{(Ax+b_u u+B_f f)} (c^T x), \quad 1 \leqslant i \leqslant n, \tag{F5}$$

where $\bar{L}^{i-1}_{(Ax+b_u u+B_f f)} (c^T x)$ $(1 \leqslant i \leqslant n)$ is only related to x and t. In addition, we have

$$\dot{\tilde{x}}_n(t) = c^T A^{n-1} b_u u(t) + c^T A^n x(t) + \sum_{k=0}^{n-1} c^T A^{n-k-1} B_f f^{(k)}(x, u, t). \tag{F6}$$

Moreover, according to Lemma 2 in [42], the rows of the transformation

$$\tilde{x} = \begin{bmatrix} c^T x \\ \bar{L}^1_{(Ax+b_u u+B_f f)}(c^T x) \\ \vdots \\ \bar{L}^{n-1}_{(Ax+b_u u+B_f f)}(c^T x) \end{bmatrix} \tag{F7}$$

are linearly independent. By using the differential manifold theory [43], the transformation (F7) is invertible. Moreover, for all $x_0 \in R^n$, there exists a neighborhood of x_0, $U(x_0)$, and a function $g_{f,x_0}(\tilde{x},t)$, such that $x = g_{f,x_0}(\tilde{x},t)$ for $x \in U(x_0)$. Therefore, the results of Theorem 1 can be obtained. ■

Theorem 1 illuminates that the transformation from $\tilde{x}$ to the state x exists and dependeds on the disturbances f. Via the transformation function g_{f,x_0}, the meaningful integrator chain form (F2) is obtained, which further reveals that:

(i). If the condition of disturbance resistibility (E6) is satisfied, then (F2) becomes

$$\begin{cases} \dot{\tilde{x}}_i(t) = \tilde{x}_{i+1}(t), \quad 1 \leqslant i \leqslant n-1, \\ \dot{\tilde{x}}_n(t) = c^T A^{n-1} b_u u(t) + c^T A^n T^{-1}\tilde{x}(t), \end{cases} \quad y(t) = \tilde{x}_1(t), \quad T = [c\ (c^T A)^T\ \cdots\ (c^T A^{n-1})^T]^T, \tag{F8}$$

which shows that the disturbances f do not influence the physical plant. Therefore, it is unnecessary to consider disturbance rejection in this case.

(ii). If the disturbances f are observable, according to the condition (E1), (F2) can be further expressed as follows.

$$\begin{cases} \dot{\tilde{x}}_i(t) = \tilde{x}_{i+1}(t), \quad 1 \leqslant i \leqslant n-1, \\ \dot{\tilde{x}}_n(t) = c^T A^{n-1} b_u u(t) + c^T A^n T^{-1}\tilde{x}(t) + c^T A^{n-1} B_f f(g_{f,x_0}(\tilde{x},t),u,t), \end{cases} \quad y(t) = \tilde{x}_1(t), \tag{F9}$$

which implies that the output is directly influenced by the observable disturbances f.

(iii). If there exists a positive $k < n-1$ such that $c^T A^{n-k-1} B_f \neq 0$, then according to Proposition 1, the disturbances f are unobservable. However, (F2) indicates that in this case, the k-th derivative of f will influence the controlled variable although f is unobservable.

Therefore, under both the situations (ii) and (iii), the control design of compensating for the effects of the observable or unobservable disturbances is required for achieving the desired tracking performance.

Next, we will discuss how, via the conceptualization of the total disturbance in the frame of ADRC, the disturbance rejection can be realized for the uncertain system (1), which may contain multiple unobservable and mismatched disturbances.

The integrator chain form (F2) discloses that the only known information of the physical plant is $c^T A^{n-1} b_u u$ and the rest is the equivalent total influence of the multiple disturbances f on the controlled variable, which is

$$c^T A^n g_{f,x_0}(\tilde{x},t) + \sum_{k=0}^{n-1} Q_k(g_{f,x_0}(\tilde{x},t),u,t), \tag{F10}$$

where

$$Q_k(x,u,t) \triangleq \begin{cases} 0, \ (\text{if } \ c^T A^{n-k-1} B_f = 0), \\ c^T A^{n-k-1} B_f f^{(k)}(x,u,t), \ (\text{otherwise}), \end{cases} \quad 0 \leqslant k \leqslant n-1. \tag{F11}$$

Actually, (F10) is exactly the difference between the ideal integrator chain form and the real physical plant. By conceptualizing this equivalent total effect of the multiple disturbances f on the controlled variable as the total disturbance

$$f_{total}(\tilde{x},u,t) = c^T A^n g_{f,x_0}(\tilde{x},t) + \sum_{k=0}^{n-1} Q_k(g_{f,x_0}(\tilde{x},t),u,t), \tag{F12}$$

the integrator chain form of the uncertain system (1) can be further reformulated as

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{b}_u u(t) + \tilde{b}_{f_t} f_{total}(\tilde{x},u,t), \quad y(t) = \tilde{c}^T \tilde{x}(t), \quad t \geqslant t_0, \tag{F13}$$

with the corresponding system matrices being

$$\tilde{A} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{n\times n}, \quad \tilde{b}_{f_t} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{n\times 1}, \quad \tilde{c} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1}, \quad \tilde{b}_u = \bar{b}\tilde{b}_{f_t}, \quad \bar{b} = c^T A^{n-1} b_u. \tag{F14}$$

(F12) shows that the total disturbance is not only related to the disturbances f but also contains the effects of the derivatives of the disturbances. Moreover, the new formulation (F13) further illuminates the following significance of the conceptualization of total disturbance:

(i): It can be verified that the corresponding system matrices $(\tilde{A}, \tilde{b}_{f_t}, \tilde{c})$ of the new formulation (F13) satisfy the conditions of Proposition 1. Therefore, the total disturbance f_{total} is observable. This is the root why ESO can be designed to observe the total disturbance.

(ii): Since $\tilde{b}_u$ and $\tilde{b}_{f_t}$ are linear dependent, the total disturbance f_{total} is matched with the control input. This feature guarantees the possibility that the total disturbance can be compensated for via the control design.

In conclusion, although the multiple disturbances f might be unobservable and mismatched, the "total disturbance" f_{total} for the uncertain system (1), i.e., the equivalent total effect of the multiple disturbances f, is not only observable but also matched. The significance of the conceptualization of the total disturbance is summed up in Theorem 2.

Theorem 2. Consider the uncertain system (1) satisfying Assumption 1. Its total disturbance f_{total} (F12) is both observable and matched.

Study the integrator chain form and the total disturbance of the TMS example system (1) and (D1). Assume the control objective is forcing the body 2 to track the desired reference signal. If there exist multiple disturbances (f_1, f_2) acting on the body 1 and body 2, respectively, both f_1 and f_2 are unobservable and f_2 is mismatched. According to Newton's second law and Hooke's law, the relative degree of the TMS example system (1) and (D1) is 4. In other words, there exist 4 integrators between the control input to the controlled variable. This feature is definite and will not be influenced by the disturbances f_1 and f_2. Then, according to (F12) and (F13), the integrator chain form of the TMS system (1) and (D1) is

$$\begin{bmatrix}\dot{\tilde{x}}_1\\ \dot{\tilde{x}}_2\\ \dot{\tilde{x}}_3\\ \dot{\tilde{x}}_4\end{bmatrix} = \begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}\begin{bmatrix}\tilde{x}_1\\ \tilde{x}_2\\ \tilde{x}_3\\ \tilde{x}_4\end{bmatrix} + \begin{bmatrix}0\\0\\0\\ \frac{k_2}{m_1 m_2}\end{bmatrix} u + \begin{bmatrix}0\\0\\0\\ f_{total}\end{bmatrix}, \quad y(t) = \tilde{x}_1(t) = x_2(t), \tag{F15}$$

where its total disturbance is

$$\begin{aligned} f_{total} =& -\frac{m_2k_2(k_1+k_2)+m_1k_2(k_2+k_3)}{m_1m_2^2}x_1 + \frac{m_2k_2^2+m_1(k_2+k_3)^2}{m_1m_2^2}x_2 + \frac{k_2}{m_2}f_1 - \frac{k_2+k_3}{m_2}f_2 + f_2^{(2)} \\ =& -\frac{(k_1+\Delta k_1)(k_2+\Delta k_2)+(k_1+\Delta k_1)(k_3+\Delta k_3)+(k_2+\Delta k_2)(k_3+\Delta k_3)}{(m_1+\Delta m_1)(m_2+\Delta m_2)}\tilde{x}_1 \\ &- \frac{(m_2+\Delta m_2)(k_1+\Delta k_1+k_2+\Delta k_2)+(m_1+\Delta m_1)(k_2+\Delta k_2+k_3+\Delta k_3)}{(m_1+\Delta m_1)(m_2+\Delta m_2)}\tilde{x}_3 \\ &+ \left(\frac{k_2+\Delta k_2}{(m_1+\Delta m_1)(m_2+\Delta m_2)} - \frac{k_2}{m_1m_2}\right)u + \frac{(k_2+\Delta k_2)}{m_1(m_2+\Delta m_2)}w_1 + \frac{1}{m_2+\Delta m_2}w_2^{(2)} \\ &- \frac{(k_1+\Delta k_1+k_2+\Delta k_2)(m_2+\Delta m_2)+2(k_2+\Delta k_2+k_3+\Delta k_3)(m_1+\Delta m_1)}{(m_1+\Delta m_1)(m_2+\Delta m_2)^2}w_2. \end{aligned} \tag{F16}$$

It can be seen from (F16) that neither f_1 nor f_2 is total disturbance. The total disturbance (F16) appears in the integrator chain (F15), which reveals the essential dynamics from the control input u to the controlled variable y. The total disturbance (F16) may not be a certain concrete disturbance, but an equivalent total effect of f_1 and f_2 and their higher derivatives. Moreover, it's the key factor influencing the controlled variable. Therefore, if the total disturbance can be well handled, the control objective can be achieved in spite of multiple disturbances.

Next, an ADRC design for overcoming the effects of the multiple disturbances f for the uncertain system (1), via total disturbance estimation and rejection, is proposed.

Appendix F.2 ADRC design

From the integrator chain form (F13), it can be seen that, to realize a satisfied tracking process of the controlled variable despite multiple disturbances f, it only needs to deal with the total disturbance f_{total}, which it is not necessarily a certain concrete disturbance.

To estimate the total disturbance f_{total}, a commonly designed ESO, which corresponds to the integrator chain form (F13), is presented as follows [26]:

$$\begin{bmatrix}\dot{\hat{\tilde{x}}}(t)\\ \dot{\hat{f}}_{total}(t)\end{bmatrix} = \begin{bmatrix}\tilde{A} & \tilde{b}_{f_t}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\begin{bmatrix}\hat{\tilde{x}}(t)\\ \hat{f}_{total}(t)\end{bmatrix} + \begin{bmatrix}\tilde{b}_u\\ 0\end{bmatrix}u + l_{ESO}\tilde{c}^T(\tilde{x}-\hat{\tilde{x}}), \tag{F17}$$

where $\hat{\tilde{x}} = [\hat{\tilde{x}}_1\ \hat{\tilde{x}}_2\ \cdots\ \hat{\tilde{x}}_n]^T$ and $\hat{f}_{total}$ are expected to be the online estimations of the up to $(n-1)$-th order derivatives of the controlled variable $[y\ y^{(1)}\ \cdots\ y^{(n-1)}]^T$ and the total disturbance f_{total}, respectively. Since the total disturbance f_{total} is definitely observable, the eigenvalues of the matrix $A_{ESO} \triangleq \left[\begin{smallmatrix}\tilde{A} & \tilde{b}_{f_t}\\ \boldsymbol{0} & \boldsymbol{0}\end{smallmatrix}\right] - l_{ESO}[\tilde{c}^T\ 0]$ can be assigned in $\mathcal{C}^-$ via the designed parameter $l_{ESO} = [l_1\ l_2\ \cdots\ l_{n+1}]^T$. In many literatures, to simplify the tuning of the ESO's parameter, l_{ESO} is designed as

$$l_{ESO} = \left[\phi_1\omega_o\ \phi_2\omega_o^2\ \cdots\ \phi_{n+1}\omega_o^{n+1}\right]^T, \quad \phi_i = \frac{(n+1-i)!i!}{(n+1)!}, \quad \omega_o \geqslant 0, \tag{F18}$$

such that all the eigenvalues of A_{ESO} are set at $-\omega_o$ and ω_o is defined as the bandwidth of the ESO (F17) [44].

Furthermore, since $\tilde{x}$ and f_{total} are definitely observable, Proposition 2 shows that the estimation error of the ESO (F17), which is developed based on the integrator chain form (F13), can be tuned by the ESO parameter l_{ESO}, if the total disturbance f_{total} has bounded derivative. However, from (F12), the total disturbance is not just caused by the external disturbance, but also caused by the model uncertainties, which are related to the state $\tilde{x}$ and the control input u. Hence, its hard to pre-assume that the total disturbance f_{total} has bounded derivative. Next, an ADRC law is given and then the analysis on ESO is intricately connected with that on the ADRC based closed-loop system in which it is used.

By utilizing the estimation from the ESO (F17), a possible ADRC law can be designed as follows.

$$u(t) = \begin{cases} 0, & t_0 \leqslant t < \tilde{t}_0, \\ -\dfrac{\hat{f}_{total}(t)}{\bar{b}} + \dfrac{r^{(n)}(t) - \sum_{i=1}^n k_{c,i}(\hat{\tilde{x}}_i(t) - r^{(i-1)}(t))}{\bar{b}}, & t \geqslant \tilde{t}_0, \end{cases} \tag{F19}$$

where the feedback gain $k_c = [k_{c,1}\ k_{c,2}\ \cdots\ k_{c,n}]^T$ is chosen such that $A_{k_c} \triangleq \tilde{A} - \tilde{b}_{f_t} k_c^T$ is Hurwitz. Additionally, $\tilde{t}_0$ is the time after which the peaking of the ESO (F17) ends. Since it is hard to get minimal $\tilde{t}_0$ [45], the following design for $\tilde{t}_0$ is used [46]: $\tilde{t}_0 = t_0 + 2(n-1)\|\bar{P}_1\| \frac{\max\left\{\ln\left(\omega_o \max\limits_{2\leqslant i\leqslant n} |\tilde{x}_i(t_0) - \hat{\tilde{x}}_i(t_0)|\right),\ 0\right\}}{\omega_o}$, where the positive definite matrix $\bar{P}_1$ satisfies

$$\bar{A}_1^T \bar{P}_1 + \bar{P}_1 \bar{A}_1 = -\boldsymbol{I}, \quad \bar{A}_1 = \begin{bmatrix} -\phi_1 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -\phi_n & 0 & \cdots & 0 & 1 \\ -\phi_{n+1} & 0 & \cdots & \cdots & 0 \end{bmatrix}. \tag{F20}$$

It is remarkable that $\tilde{t}_0 = t_0$ if the initial condition satisfies $\max\limits_{2\leqslant i\leqslant n} |\tilde{x}_i(t_0) - \hat{\tilde{x}}_i(t_0)| \leqslant \frac{1}{\omega_o}$.

The capability of the ADRC design (F17)–(F19) for the uncertain system (1) will be studied under the following rational assumption.

Assumption 2.

(i). $f(x,u,t)$ is smooth for $(x,u) \in R^{n+1}$ and piecewise smooth for $t \in (t_0, \tilde{t}_1) \bigcup \bigcup_{i\geqslant 1} (\tilde{t}_i, \tilde{t}_{i+1})$. There exists a positive φ_d such that $\min\limits_{i\geqslant 1}\{|\tilde{t}_{i+1} - \tilde{t}_i|, |\tilde{t}_1 - t_0|\} \geqslant \varphi_d$.

(ii). $g_{f,x_0}(\tilde{x}, t)$ and its partial derivatives with respect to $\tilde{x}$ and t are bounded by a continuous function $\psi_1(\tilde{x})$ for $(f, x_0, t) \in \Omega_f \times R^n \times ((t_0, \tilde{t}_1) \bigcup \bigcup_{i\geqslant 1} (\tilde{t}_i, \tilde{t}_{i+1}))$.

(iii). $Q_k(x,u,t)$ $(0 \leqslant k \leqslant n-1)$ and its partial derivatives with respect to x and t are bounded by a continuous function $\psi_{2,k}(x,u)$ for $t \in (t_0, \tilde{t}_1) \bigcup \bigcup_{i\geqslant 1} (\tilde{t}_i, \tilde{t}_{i+1})$.

(iv). There exist positives φ_1 and φ_2 such that $(\frac{\partial f_{total}}{\partial u} + \bar{b})/\bar{b} \in [\varphi_1, \varphi_2] \subset \left(0, 2 + \frac{2}{n}\right)$ for $t \geqslant t_0$.

In the assumption (i), the disturbances f contain the disturbances from interior which are smooth with respect to (x,u,t), and the disturbances from exterior which are piecewise smooth with respect to t. Additionally, the length of each smooth region of external disturbance is assumed to be larger than a constant, which is natural in practice. The assumption (ii) illuminates that the state x and its changing rates with respect to $\tilde{x}$ and t are bounded by a continuous function of the controlled variables $\tilde{x}$ in each time interval. The assumption (iii) illustrates that Q_k, which only appears in the total disturbance and is composed of the derivatives of multiple disturbances, and its partial derivatives with respect to x and t, are bounded by a continuous function in each time interval. The assumption (iv) indicates that the nominal part of control input $\bar{b}u$ is a dominant term and the margin for the uncertainty $\frac{\partial f_{total}}{\partial u}$ is not narrow. Hence, Assumption 2 permits $f(x,u,t)$ to describe a wide class of disturbances in engineering practice, such as square wave, sinusoidal signal and internal disturbances in quadratic form and even exponential form.

Remark 6. If the multiple disturbances only depend on the state and control input, i.e., $f = f(x,u)$, Assumption 2 can be simplified as: (i). $f(x,u)$ is smooth for $(x,u) \in R^{n+1}$; (ii). There exist positives φ_1 and φ_2 such that $(\frac{\partial f_{total}}{\partial u} + \bar{b})/\bar{b} \in [\varphi_1, \varphi_2] \subset \left(0, 2 + \frac{2}{n}\right)$ for $t \geqslant t_0$.

Check Assumption 2 for the TMS system (1) and (D1). A simplified constraint on the parametric uncertainties $(\Delta m_i, \Delta k_i)$ $(i = 1,2)$ and the external disturbances w_i $(i = 1,2)$ can be obtained as follows:

(i). The conditions in (i)-(iii) of Assumption 2 can be simplified as that the external disturbances $w_i(t)$ $(i = 1,2)$ are piecewise smooth, the smallest time length between the discontinuous point of $w_i^{(j)}$ $(i = 1,2,\ j \geqslant 0)$ is greater than a constant, and there exists a positive η_w^* such that

$$|w_1^{(j)}(t)| \leqslant \eta_w^*, \quad |w_2^{(i)}(t)| \leqslant \eta_w^*, \quad j = 0,1, \quad i = 0,1,2,3, \tag{F21}$$

in the corresponding continuous region.

(ii). The condition (iv) of Assumption 2 can be simplified as that there exist positives φ_1^* and φ_2^* such that

$$\frac{1 + \frac{\Delta k_2}{k_2}}{\left(1 + \frac{\Delta m_1}{m_1}\right)\left(1 + \frac{\Delta m_2}{m_2}\right)} \in [\varphi_1^*, \varphi_2^*] \subset \left(0,\ \frac{5}{2}\right). \tag{F22}$$

Let $\tilde{r} \triangleq \left[r\ r^{(1)}\ \cdots\ r^{(n-1)}\right]^T$. Then, the following theorem illuminates the capability of the ADRC design (F17)–(F19).

Theorem 3. Consider the system (1) with the ADRC design (F17)–(F19). Assume that Assumption 1 and Assumption 2 are satisfied. There exist positives ω^* and $\eta_i^* (1 \leqslant i \leqslant 3)$ which depend on $(\tilde{x}(t_0), \psi_\cdot, \varphi_\cdot, k_c)$, such that, for all $\omega_o \geqslant \omega^*$, the closed-loop system has the following properties:

$$\sup_{t\in[t_0,\infty)} |y(t) - (r(t) + \tilde{c}e^{A_{k_c}(t-t_0)}(\tilde{x}(t_0) - \tilde{r}(t_0)))| \leqslant \eta_1^* \max\left\{\frac{\ln\omega_o}{\omega_o}, \frac{1}{\omega_o}\right\}, \tag{F23}$$

$$\left\|\begin{bmatrix} \tilde{x}(t) \\ f_{total} \end{bmatrix} - \begin{bmatrix} \hat{\tilde{x}}(t) \\ \hat{f}_{total}(t) \end{bmatrix}\right\| \leqslant \eta_2^* \left(\frac{1}{\omega_o} + e^{-\eta_3^*\omega_o(t-\tilde{t}_i)}\right), \quad t \in [\tilde{t}_i, \tilde{t}_{i+1}), \quad i \geqslant 0. \tag{F24}$$

Proof of Theorem 3: Based on Assumption 2 (i) and the proof of Theorem 1, the transformation $x = g_{f,x_0}(\tilde{x}, t)$ exists for $x_0 \in R^n$ and $x \in U(x_0)$, and the system (1) can be reformulated as the integrator chain form (F13) in each time interval $t \in (t_0, \tilde{t}_1) \bigcup \bigcup_{i\geqslant 1} (\tilde{t}_i, \tilde{t}_{i+1})$. The reference signal $r(t)$ and its derivatives are bounded by some positive η_r, i.e., $\sup_{t\geqslant t_0, i\geqslant 0} |r^{(i)}| \leqslant \eta_r$.

Define

$$e = \tilde{x} - \tilde{r}, \quad \tilde{r} = \begin{bmatrix} r & r^{(1)} & \cdots & r^{(n-1)} \end{bmatrix}^T, \tag{F25}$$

and

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{n+1} \end{bmatrix} = T_1^{-1} \begin{bmatrix} \tilde{x} - \hat{\tilde{x}} \\ f_{total} - \hat{f}_{total} \end{bmatrix}, \quad T_1 = \begin{bmatrix} \omega_o^{-n} & 0 & \cdots & 0 \\ 0 & \omega_o^{1-n} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{F26}$$

Then, the control input (6) can be reformulated as follows.

$$u(t) = \begin{cases} 0, & t_0 \leqslant t < \tilde{t}_0, \\ \dfrac{-\hat{f}_{total}(t) - k_c^T e(t) + k_c^T(\tilde{x}(t) - \hat{\tilde{x}}(t)) + r^{(n)}(t)}{\bar{b}}, & t \geqslant \tilde{t}_0. \end{cases} \tag{F27}$$

Therefore, for all $x_0 \in R^n$, the closed-loop system (1) and (4)–(6) can be written as follows.

$$\dot{e} = \tilde{A}e + \tilde{b}_{f_t}\Gamma_0(e, 0, t), \quad \dot{\xi} = \omega_o \bar{A}_1 \xi + b_2 \Gamma_1(e, 0, t), \quad t_0 \leqslant t < \tilde{t}_0, \tag{F28}$$

$$\dot{e} = A_{k_c} e + \tilde{b}_{f_t} k_{ce}^T T_1 \xi, \quad \dot{\xi} = \omega_o \bar{A}_2 \xi + b_2 \Gamma_2(e, u, \xi, t), \quad t \geqslant \tilde{t}_0, \tag{F29}$$

where $\bar{A}_1$ is defined in (F20) and

$$\begin{cases} b_2 = \begin{bmatrix} 0 \\ \tilde{b}_{f_t} \end{bmatrix}, \quad k_{ce} = \begin{bmatrix} k_c \\ 1 \end{bmatrix}, \quad \bar{A}_2 = \bar{A}_1 - b_2 \begin{bmatrix} \tilde{c}^T & 0 \end{bmatrix} \cdot (\dfrac{\partial f_{total}}{\partial u}/\bar{b}), \\ \Gamma_0(e, u, t) = f_{total}(e + \tilde{r}, u, t) - r^{(n)}, \\ \Gamma_1(e, u, t) = \dfrac{\partial f_{total}(e + \tilde{r}, u, t)}{\partial(e + \tilde{r})}(\tilde{A}(e + \tilde{r}) + \tilde{b}_{f_t} f_{total}(e + \tilde{r}, u, t)) + \dfrac{\partial f_{total}(e + \tilde{r}, u, t)}{\partial t}, \\ \Gamma_2(e, u, \xi, t) = \dfrac{\partial f_{total}(e + \tilde{r}, u, t)}{\partial(e + \tilde{r})}(A_{k_c} e + \tilde{A}\tilde{r} + \tilde{b}_{f_t} r^{(n)}) \\ \qquad + \dfrac{\partial f_{total}(e + \tilde{r}, u, t)}{\partial t} + (\dfrac{\partial f_{total}}{\partial u}/\bar{b})(-k_c^T A_{k_c} e + r^{(n+1)}) \\ \qquad + \left(\dfrac{\partial f_{total}(e + \tilde{r}, u, t)}{\partial(e + \tilde{r})}\tilde{b}_{f_t} k_{ce}^T T_1 + (\dfrac{\partial f_{total}}{\partial u}/\bar{b})(k_c^T T_2 - k_c^T \tilde{b}_{f_t} \begin{bmatrix} k_c^T & 0 \end{bmatrix} T_1)\right)\xi, \\ T_2 = \begin{bmatrix} -\phi_1 \omega_o^{1-n} & \omega_o^{1-n} & 0 & \cdots & 0 \\ -\phi_2 \omega_o^{2-n} & 0 & \omega_o^{2-n} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -\phi_{n-1}\omega_o^{-1} & \vdots & \ddots & \ddots & \omega_o^{-1} \\ -\phi_n & 0 & \cdots & \cdots & 0 \end{bmatrix}. \end{cases} \tag{F30}$$

Next, the properties of A_{k_c}, $\bar{A}_1$, $\bar{A}_2$ and Γ_i $(0 \leqslant i \leqslant 2)$ in the closed-loop system (F28)-(F29) are analyzed, respectively.

The analysis for A_{k_c}, $\bar{A}_1$ *and* $\bar{A}_2$.

Since A_{k_c} and $\bar{A}_1$ are Hurwitz, there exist positive definite matrices P_{k_c} and $\bar{P}_1$ such that

$$A_{k_c}^T P_{k_c} + P_{k_c} A_{k_c} = -\boldsymbol{I}, \quad \bar{A}_1^T \bar{P}_1 + \bar{P}_1 \bar{A}_1 = -\boldsymbol{I}. \tag{F31}$$

Moreover, according to Assumption 2 (iv) and Lemma 1 in [48], there exist a positive definite matrix $\bar{P}_2$ and a positive c_0 such that

$$\bar{A}_2^T \bar{P}_2 + \bar{P}_2 \bar{A}_2 \leqslant -c_0 \bar{P}_2, \quad \forall \left(\left(\frac{\partial f_{total}}{\partial u} + \bar{b}\right)/\bar{b}\right) \in [\varphi_1, \varphi_2]. \tag{F32}$$

The analysis for Γ_i $(0 \leqslant i \leqslant 2)$.

Define

$$\Psi_1(a) = \max_{\|x\| \leqslant a} \psi_1(x), \quad \Psi_{2,i}(a, b) = \max_{\|x\| \leqslant a, |u| \leqslant b} \psi_{2,i}(x, u), \quad 0 \leqslant i \leqslant n-1, \tag{F33}$$

which are non-decreasing with respect to their independent variables.

Let ρ_e and ρ_u be any positives. From Assumption 2, for any $e \in \{e|\ \|e\| \leqslant \rho_e\}$ and $u \in \{u|\ |u| \leqslant \rho_u\}$, the total disturbance and its partial derivatives have the following bound for $t \in (t_0, \tilde{t}_1) \bigcup \bigcup_{i \geqslant 1} (\tilde{t}_i, \tilde{t}_{i+1})$:

$$|f_{total}(e + \tilde{r}, u, t)| \leqslant \Psi_{f,1}(\rho_e, \rho_u), \quad \left\| \frac{\partial f_{total}(e + \tilde{r}, u, t)}{\partial(e + \tilde{r})} \right\| \leqslant \Psi_{f,2}(\rho_e, \rho_u), \quad \left| \frac{\partial f_{total}(e + \tilde{r}, u, t)}{\partial t} \right| \leqslant \Psi_{f,3}(\rho_e, \rho_u), \tag{F34}$$

where $\Psi_{f,1}(\rho_e, \rho_u) \triangleq \|c^T A^n\| \Psi_1(\rho_e + \sqrt{n}\eta_r) + \sum_{i=0}^{n-1} \Psi_{2,i}(\Psi_1(\rho_e + \sqrt{n}\eta_r), \rho_u)$, $\Psi_{f,2}(\rho_e, \rho_u) \triangleq \|c^T A^n\| \Psi_1(\rho_e + \sqrt{n}\eta_r) + \sum_{i=0}^{n-1} \Psi_{2,i}(\Psi_1(\rho_e + \sqrt{n}\eta_r), \rho_u) \Psi_1(\rho_e + \sqrt{n}\eta_r)$ and $\Psi_{f,3}(\rho_e, \rho_u) \triangleq \Psi_{f,2}(\rho_e, \rho_u) + \sum_{i=0}^{n-1} \Psi_{2,i}(\Psi_1(\rho_e + \sqrt{n}\eta_r), \rho_u)$. Therefore, Γ_i $(i = 0, 1)$ has the following bound:

$$|\Gamma_0| \leqslant \pi_0(\rho_e, \rho_u), \quad |\Gamma_1| \leqslant \pi_1(\rho_e, \rho_u), \quad t \in (t_0, \tilde{t}_1) \bigcup \bigcup_{i \geqslant 1} (\tilde{t}_i, \tilde{t}_{i+1}), \tag{F35}$$

where the positives

$$\pi_0(\rho_e,\rho_u)\triangleq\Psi_{f,1}(\rho_e,\rho_u)+\eta_r,\quad \pi_1(\rho_e,\rho_u)\triangleq\Psi_{f,2}(\rho_e,\rho_u)(\|\tilde{A}\|(\rho_e+\sqrt{n}\eta_r)+\Psi_{f,1}(\rho_e,\rho_u))+\Psi_{f,3}(\rho_e,\rho_u), \tag{F36}$$

are non-decreasing with respect to (ρ_e,ρ_u).

Assumption 2 (iv) implies that $\frac{\partial(f_{total}+\bar{b}u)}{\partial u}\neq 0$. According to implicit function theorem, it can be deduced that for any given constant c, the equation $(f_{total}(\tilde{x},u,t)+\bar{b}u=c)$ has a unique solution $u=u_f(\tilde{x},c,t)$. Next, we will proof that there exists a non-decreasing continuous function $\Psi_u(\cdot)$ such that

$$|u_f(\tilde{x},c,t)|\leqslant\Psi_u(\rho_c),\quad \text{if}\ \ |c|\leqslant\rho_c. \tag{F37}$$

Firstly, by contradiction, we proof that $u_f(\tilde{x},c,t)$ is bounded for any $(\tilde{x},t)$ if c is bounded. Without loss of generality, assume that $\bar{b}>0$ and there exist $(\tilde{x}^*,t^*)$ and a constant c^* such that $u^*\triangleq u_f(\tilde{x}^*,c^*,t^*)=+\infty$. Hence,

$$f_{total}(\tilde{x}^*,u^*,t^*)+\bar{b}u^*=c^*. \tag{F38}$$

Let $\Delta c^*>0$ be a constant. Define $u^{**}\triangleq u_f(\tilde{x}^*,c^*+\Delta c^*,t^*)$, which means that

$$f_{total}(\tilde{x}^*,u^{**},t^*)+\bar{b}u^{**}=c^*+\Delta c^*. \tag{F39}$$

Since $\bar{b}>0$, Assumption 2 (iv) implies that $\frac{\partial(f_{total}+\bar{b}u)}{\partial u}>0$. Therefore, if $u^{**}<+\infty$, there is

$$f_{total}(\tilde{x}^*,u^{**},t^*)+\bar{b}u^{**}\leqslant f_{total}(\tilde{x}^*,u^*,t^*)+\bar{b}u^*=c^*<c^*+\Delta c^*. \tag{F40}$$

In the other hand, if $u^{**}=+\infty$, from (F38), there is

$$f_{total}(\tilde{x}^*,u^{**},t^*)+\bar{b}u^{**}=c^*<c^*+\Delta c^*. \tag{F41}$$

From (F39)–(F41), it contradicts the assumption that $\bar{b}>0$ and there exist $(\tilde{x}^*,t^*)$ and a constant c^* such that $u_f(\tilde{x}^*,c^*,t^*)=+\infty$. The similar results and proofs can be made for the cases that $\bar{b}<0$ or $u_f(\tilde{x}^*,c^*,t^*)=-\infty$. In conclusion, the proposition that $u_f(\tilde{x},c,t)$ is bounded for any $(\tilde{x},t)$ if c is bounded is proved. Hence, there exists a function $\psi_u(c)$ such that $|u_f(\tilde{x},c,t)|\leqslant\psi_u(c)$. Define the non-decreasing continuous function $\Psi_u(a)\triangleq\max_{|c|\leqslant a}\psi_u(c)$, which satisfies (F37).

From (F27) and the definition of u_f, the control input can also be expressed as

$$u(t)=u_f(e+\tilde{r},r^{(n)}-k_c^Te+k_{ce}^TT_1\xi,t),\quad t\geqslant\tilde{t}_0. \tag{F42}$$

Let ρ_ξ be any positive and the positive ω_0^* is chosen to ensure that $\tilde{t}_0<\tilde{t}_1$ for any $\omega_o\in[\omega_0^*,\infty)$. Owing to (F37). it yields that, for any $e\in\{e|\ \|e\|\leqslant\rho_e\}$, $\xi\in\{\xi|\ \|\xi\|\leqslant\rho_\xi\}$ and $\omega_o\in\{\omega_o|\ \omega_o\geqslant\omega_0^*\}$,

$$|u(t)|\leqslant\eta_u(\rho_e,\rho_\xi,\omega_0^*),\quad t\in\bigcup_{i\geqslant 0}(\tilde{t}_i,\tilde{t}_{i+1}), \tag{F43}$$

where $\eta_u(\rho_e,\rho_\xi,\omega_0^*)\triangleq\Psi_u(\eta_r+\|k_c\|\rho_e+\|k_{ce}^TT_1(\omega_0^*)\|\rho_\xi))$ is non-decreasing with respect to (ρ_e,ρ_ξ) and non-increasing with respect to ω_0^*. Hence, for any $e\in\{e|\ \|e\|\leqslant\rho_e\}$, $\xi\in\{\xi|\ \|\xi\|\leqslant\rho_\xi\}$ and $\omega_o\in\{\omega_o|\ \omega_o\geqslant\omega_0^*\}$, the bound of Γ_2 can be obtained as follows.

$$|\Gamma_2|\leqslant\pi_2(\rho_e,\rho_\xi,\omega_0^*),\quad t\in\bigcup_{i\geqslant 0}(\tilde{t}_i,\tilde{t}_{i+1}), \tag{F44}$$

where the positive

$$\begin{aligned}\pi_2(\rho_e,\rho_\xi,\omega_0^*)\triangleq{}&\max\{|\varphi_1-1|,|\varphi_2-1|\}\cdot(\|k_c^TA_{k_c}\|\rho_e+\eta_r+(\|k_c^TT_2(\omega_0^*)\|+\|k_c^T\tilde{b}_{ft}[k_c^T\ 0]T_1(\omega_0^*)\|)\rho_\xi)\\&+\Psi_{f,1}(\rho_e,\eta_u(\rho_e,\rho_\xi,\omega_0^*))(\|A_{k_c}\|\rho_e+(1+\sqrt{n}\|\tilde{A}\|)\eta_r+\|k_{ce}^TT_1(\omega_0^*)\|\rho_\xi)+\Psi_{f,3}(\rho_e,\eta_u(\rho_e,\rho_\xi,\omega_0^*))\end{aligned} \tag{F45}$$

is non-decreasing with respect to (ρ_e,ρ_ξ) and non-increasing with respect to ω_0^*.

Then the trajectories of the tracking error and the estimation error in the time sequences $(t_0,\tilde{t}_0)$ and $(\tilde{t}_i,\tilde{t}_{i+1})$ $(i\geqslant 0)$ will be analyzed.

The analysis for the trajectories of the closed-loop system in $t\in[t_0,\tilde{t}_0)$.

Similar to Step 5 in the proof of Theorem 1 in [46], there exist ω_1^*, η_{e1} and $\eta_{\xi1}$ such that for any $\omega_o\in[\omega_1^*,\infty)$, $(\tilde{t}_0,e(t),\xi(t))$ satisfies

$$t_0<\tilde{t}_0\leqslant t_0+\varphi_d/2,\quad \|e(t)\|\leqslant\eta_{e1},\quad \|\xi(t)\|\leqslant\eta_{\xi1},\quad \forall t\in[t_0,\tilde{t}_0). \tag{F46}$$

Denote $e^*(t)$ by the following differential equation.

$$\dot{e}^*(t)=A_{k_c}e^*(t),\quad t\geqslant t_0,\quad e^*(t_0)=e(t_0), \tag{F47}$$

where $e^*(t)$ can be explicitly expressed as follows.

$$e^*(t)=e^{A_{k_c}(t-t_0)}e(t_0). \tag{F48}$$

Due to the definition of $\tilde{t}_0$, there exists a positive η_{e^*1} such that

$$|\|e^*(t)\|-\|e^*(t_0)\||\leqslant 2(n-1)\|\bar{P}_1\|\bar{\mu}_1(\omega_o)\eta_{e^*1},\quad t\in[t_0,\tilde{t}_0), \tag{F49}$$

where $\bar{\mu}_1(\omega_o)\triangleq\frac{\max\left\{\ln\left(\omega_o\max\limits_{2\leqslant i\leqslant n}\{|\tilde{x}_i(t_0)-\hat{\tilde{x}}_i(t_0)|\}\right),0\right\}}{\omega_o}$. Owing to (F28), (F35) and (F46), there is

$$|\|e(t)\|-\|e(t_0)\||\leqslant 2(n-1)\|\bar{P}_1\|\bar{\mu}_1(\omega_o)(\|\tilde{A}\|\eta_{e1}+\pi_0(\eta_{e1},0)),\quad t\in[t_0,\tilde{t}_0). \tag{F50}$$

Define

$$e_z(t) = e(t) - e^*(t). \tag{F51}$$

With the combination of (F49) and (F50), the bound of $\|e_z\|$ is shown as follows.

$$\|e_z(t)\| \leqslant 2(n-1)\|\bar{P}_1\|(\eta_{e^*1} + \|\tilde{A}\|\eta_{e1} + \pi_0(\eta_{e1}, 0))\bar{\mu}_1(\omega_o),\ t \in [t_0, \tilde{t}_0), \omega_o \geqslant \omega_1^*. \tag{F52}$$

Then, the rest of the proof is based on $\omega_o \in [\omega_1^*, \infty)$.

The analysis for the trajectories of the closed-loop system in $t \in [\tilde{t}_0, \tilde{t}_1)$.

Firstly, the initial condition $(e(\tilde{t}_0), \xi(\tilde{t}_0))$ is considered. Since $\tilde{x}(t)$ and $\tilde{r}(t)$ are continuous at $\tilde{t}_0$, from (F46), there is

$$\|e(\tilde{t}_0)\| \leqslant \eta_{e1}. \tag{F53}$$

Moveover, $\xi_i(t)$ $(1 \leqslant i \leqslant n)$ is continuous at $\tilde{t}_0$, which implies

$$\sqrt{\sum_{i=1}^{n} \xi_i(\tilde{t}_0)^2} \leqslant \eta_{\xi 1}. \tag{F54}$$

Due to the continuity of $(\hat{\tilde{x}}, \tilde{r}, \hat{f}_{total})$, the following inequalities are hold.

$$|k_c^T(\hat{\tilde{x}}(\tilde{t}_0) - \tilde{r}(\tilde{t}_0))| \leqslant \lim_{t \to \tilde{t}_0} \left| k_c^T e - \sum_{i=1}^{n} k_{c,i}\xi_i/\omega_o^{n+1-i} \right| \leqslant \|k_c\|(\eta_{e1} + \bar{\mu}_2(\omega_1^*)\eta_{\xi 1}), \tag{F55}$$

$$|\hat{f}_{total}(\tilde{t}_0)| = \left| \lim_{t \to \tilde{t}_0^-} (f_{total} - \xi_{n+1}) \right| = \left| \lim_{t \to \tilde{t}_0^-} (\Gamma_0 + r^{(n)} - \xi_{n+1}) \right| \leqslant \pi_0(\eta_{e1}, 0) + \eta_r + \eta_{\xi 1}, \tag{F56}$$

where $\bar{\mu}_2(\omega_o) \triangleq \max\{1, 1/\omega_o^n\}$. Due to (6), (F55) and (F56), the bound of $u(\tilde{t}_0)$ is presented as follows.

$$|u(\tilde{t}_0)| \leqslant \eta_{u1} \triangleq \frac{\pi_0(\eta_{e1}, 0) + 2\eta_r + \eta_{\xi 1} + \|k_c\|(\eta_{e1} + \bar{\mu}_2(\omega_1^*)\eta_{\xi 1})}{|\bar{b}|}. \tag{F57}$$

The definitions of $\xi_{n+1}(\tilde{t}_0)$ and $u(\tilde{t}_0)$ mean that

$$\xi_{n+1}(\tilde{t}_0) = f_{total} - \hat{f}_{total}(\tilde{t}_0) = \Gamma_0(e(\tilde{t}_0), u(\tilde{t}_0), t) + \bar{b}u(\tilde{t}_0) + k_c^T e(\tilde{t}_0) - k_c^T(\hat{\tilde{x}}(\tilde{t}_0) - \tilde{r}(\tilde{t}_0)). \tag{F58}$$

Then, combined with (F54)–(F58), $\xi(\tilde{t}_0)$ has the following bound.

$$\|\xi(\tilde{t}_0)\| \leqslant \eta_{\xi 2} \triangleq \eta_{\xi 1} + \pi_0(\eta_{e1}, \eta_{u1}) + |\bar{b}|\eta_{u1} + \|k_c\|\eta_{e1} + \|k_c\|(\eta_{e1} + \bar{\mu}_2(\omega_1^*)\eta_{\xi 1}). \tag{F59}$$

Define $V_{k_c}(e) = e^T P_{k_c} e$ and $\bar{V}_2(\xi) = \xi^T \bar{P}_2 \xi$. In addition, denote

$$c_{k1} = \lambda_{\min}(P_{k_c}), \quad c_{k2} = \lambda_{\max}(P_{k_c}), \quad c_{21} = \lambda_{\min}(\bar{P}_2), \quad c_{22} = \lambda_{\max}(\bar{P}_2). \tag{F60}$$

Then, (F53) and (F59) indicate that $(e(\tilde{t}_0), \xi(\tilde{t}_0))$ belongs to the set

$$\Omega_1 = \left\{ (e, \xi) \mid \sqrt{V_{k_c}(e)} \leqslant \rho_1, \quad \sqrt{\bar{V}_2(\xi)} \leqslant \sqrt{c_{22}}\eta_{\xi 2} \right\}, \tag{F61}$$

where $\rho_1 \triangleq \sqrt{c_{k2}} \max\{\eta_{e1}, 2\|P_{k_c}\tilde{b}_{ft}k_{ce}^T\|\bar{\mu}_2(\omega_2^*)\sqrt{c_{22}}\eta_{\xi 2}/\sqrt{c_{21}}\}$. Next, we proceed to prove that there exists $\omega_2^* \geqslant \omega_1^*$ such that $(e(t), \xi(t))$ lies in Ω_1 for any $\omega_o \in [\omega_2^*, \infty)$ and $t \in [\tilde{t}_0, \tilde{t}_1)$ by the following two steps:

(1): Assume that there exists $t^* \in (\tilde{t}_0, \tilde{t}_1)$ such that

$$\sqrt{\bar{V}_2(\xi(t^*))} = \sqrt{c_{22}}\eta_{\xi 2},\ \sqrt{V_{k_c}(e(t))} \leqslant \rho_1,\ \forall t \in [\tilde{t}_0, t^*]. \tag{F62}$$

Thus, there is

$$\|\xi(t^*)\| \leqslant \frac{\sqrt{c_{22}}\eta_{\xi 2}}{\sqrt{c_{21}}}, \quad \|e(t)\| \leqslant \frac{\rho_1}{c_{k2}}, \quad t \in [\tilde{t}_0, t^*]. \tag{F63}$$

Owing to (F32) and (F44), the definition of $\bar{V}_2$ indicates that

$$\frac{d\sqrt{\bar{V}_2(\xi(t^*))}}{dt} \leqslant -\frac{\sqrt{c_{22}}c_0\eta_{\xi 2}\omega_o}{2} + \frac{\|\bar{P}_2 b_2\|}{\sqrt{c_{21}}}\pi_2\left(\frac{\rho_1}{c_{k2}}, \frac{\sqrt{c_{22}}\eta_{\xi 2}}{\sqrt{c_{21}}}, \omega_1^*\right). \tag{F64}$$

Therefore, there exists $\omega_2^* \geqslant \omega_1^*$ such that $\frac{d\sqrt{\bar{V}_2(\xi(t^*))}}{dt} \leqslant 0$ for any $\omega_o \in [\omega_2^*, \infty)$.

(2): Assume that there exists $t^* \in (\tilde{t}_0, \tilde{t}_1)$ such that

$$\sqrt{V_{k_c}(e(t^*))} = \rho_1,\ \sqrt{\bar{V}_2(\xi(t))} \leqslant \sqrt{c_{22}}\eta_{\xi 2},\ \forall t \in [\tilde{t}_0, t^*]. \tag{F65}$$

Due to (F31), the definition of V_{k_c} implies that

$$\frac{d\sqrt{V_{k_c}(e(t^*))}}{dt} \leqslant -\frac{\|e\|}{2\sqrt{V_{k_c}(e)}}\left(\frac{\rho_1}{\sqrt{c_{k2}}} - 2\bar{\mu}_2(\omega_2^*)\|P_{k_c}\tilde{b}_{ft}k_{ce}^T\|\sqrt{c_{22}}\eta_{\xi 2}/\sqrt{c_{21}}\right) \leqslant 0 \tag{F66}$$

for any ω_o and ω_2^* satisfying $\omega_o \geqslant \omega_2^* \geqslant \omega_1^*$.

From (F62) to (F66), we conclude that $(e(t),\xi(t))$ will stay in Ω_1 for any $\omega_o\in[\omega_2^*,\infty)$ and $t\in[\tilde{t}_0,\tilde{t}_1)$. Hence, there is

$$\frac{d\sqrt{\bar{V}_2(\xi(t))}}{dt}\leqslant-\frac{c_0\omega_o}{2}\sqrt{\bar{V}_2(\xi(t))}+\frac{\tilde{\eta}_1}{2},\quad t\in[\tilde{t}_0,\tilde{t}_1),\tag{F67}$$

where $\tilde{\eta}_1\triangleq\frac{2\|\bar{P}_2b_2\|}{\sqrt{c_{21}}}\pi_2\left(\frac{\rho_1}{c_{k2}},\frac{\sqrt{c_{22}}\eta_{\xi2}}{\sqrt{c_{21}}},\omega_2^*\right)$. Moreover, with the help of Gronwall Lemma [35], the bound of $\sqrt{\bar{V}_2(\xi(t))}$ can be obtained as follows.

$$\sqrt{\bar{V}_2(\xi(t))}\leqslant\frac{\tilde{\eta}_1}{c_0\omega_o}+\sqrt{c_{22}}\eta_{\xi2}e^{-\frac{c_0\omega_o}{2}(t-\tilde{t}_0)},\quad t\in[\tilde{t}_0,\tilde{t}_1).\tag{F68}$$

Define $t_{p,1}(\omega_o)=\tilde{t}_0+\frac{2\bar{\mu}_3(\omega_o)}{c_0}$ and $\bar{\mu}_3(\omega_o)=\frac{\max\{\ln(\omega_o),0\}}{\omega_o}$, then there exists $\omega_3^*\geqslant\omega_2^*$ such that

$$\frac{2\bar{\mu}_3(\omega_o)}{c_0}\leqslant\frac{\varphi_d}{2},\quad\forall\omega_o\in[\omega_3^*,\infty),\tag{F69}$$

which implies

$$t_{p,1}\leqslant\tilde{t}_0+\varphi_d/2<t_0+\varphi_d\leqslant\tilde{t}_1,\quad\forall\omega_o\in[\omega_3^*,\infty).\tag{F70}$$

Then, from (F68), we have

$$\sqrt{\bar{V}_2(\xi(t))}\leqslant\frac{\tilde{\eta}_2}{\omega_o},\quad t\in[t_{p,1},\tilde{t}_1),\tag{F71}$$

where $\tilde{\eta}_2\triangleq\frac{\tilde{\eta}_1}{c_0}+\sqrt{c_{22}}\eta_{\xi2}$.

According to the definition of e_z (F51), there is

$$\dot{e}_z(t)=A_{k_c}e_z(t)+\tilde{b}_{ft}k_{ce}^TT_1\xi(t),\quad t\in[\tilde{t}_0,\infty).\tag{F72}$$

Moreover, it can be deduced from (F72) that for $t\in[\tilde{t}_0,\infty)$,

$$\sqrt{V_{k_c}(e_z(t))}\leqslant\sqrt{V_{k_c}(e_z(\tilde{t}_0))}e^{-\frac{t-\tilde{t}_0}{2c_{k2}}}+\frac{\|\tilde{b}_{ft}k_{ce}^T\|\bar{\mu}_2(\omega_o)}{\sqrt{c_{k1}}}\int_{\tilde{t}_0}^{t}e^{-\frac{t-\tau}{2c_{k2}}}\|\xi(\tau)\|d\tau.\tag{F73}$$

Therefore, the bound of e_z for $t\in[\tilde{t}_0,\tilde{t}_1)$ can be expressed as follows.

$$\|e_z(t)\|\leqslant\frac{\sqrt{V_{k_c}(e_z(\tilde{t}_0))}}{\sqrt{c_{k1}}}+\frac{\|\tilde{b}_{ft}k_{ce}^T\|\bar{\mu}_2(\omega_o)}{c_{k1}}\left(\frac{2c_{k2}\tilde{\eta}_2}{\omega_o}+\frac{2\bar{\mu}_3(\omega_o)\sqrt{c_{22}}\eta_{\xi2}}{c_0\sqrt{c_{21}}}\right).\tag{F74}$$

Since for any given positive η_ε, there exists $\omega_4^*\geqslant\omega_3^*$ such that

$$\frac{\|\tilde{b}_{ft}k_{ce}^T\|\bar{\mu}_2(\omega_o)}{c_{k1}}\left(\frac{2c_{k2}\tilde{\eta}_2}{\omega_o}+\frac{2\bar{\mu}_3(\omega_o)\sqrt{c_{22}}\eta_{\xi2}}{c_0\sqrt{c_{21}}}\right)\leqslant\eta_\varepsilon,\tag{F75}$$

the bound of $e(\tilde{t}_1)$ for $\omega_o\in[\omega_4^*,\infty)$ is given as follows.

$$\|e(\tilde{t}_1)\|\leqslant\eta_{e2}\triangleq\|e^*(\tilde{t}_1)\|+\frac{\sqrt{V_{k_c}(e_z(\tilde{t}_0))}}{\sqrt{c_{k1}}}+\eta_\varepsilon.\tag{F76}$$

Additionally, due to the continuity of $(\hat{\tilde{x}}(t),\hat{f}_{total}(t),\tilde{r}(t))$, (F43) implies that

$$\|u(\tilde{t}_1)\|\leqslant\eta_{u2}\triangleq\eta_u(\eta_{e2},\frac{\tilde{\eta}_2}{\sqrt{c_{21}}\omega_4^*},\omega_4^*).\tag{F77}$$

The rest of the proof is based on $\omega_o\in[\omega_4^*,\infty)$.

The analysis for the trajectories of the closed-loop system in $t\in[\tilde{t}_1,\tilde{t}_2)$.

First, the continuity of $e(t)$ implies that

$$\sqrt{V_{k_c}(e(\tilde{t}_1))}\leqslant\min\{\rho_1,\sqrt{c_{k2}}\eta_{e2}\}.\tag{F78}$$

Notice that $\xi_{n+1}(t)$ contains f_{total} which might be discontinuous at $\tilde{t}_1$. Based on (F35), the bound of $\xi(\tilde{t}_1)$ is obtained.

$$\|\xi(\tilde{t}_1)\|\leqslant\|\lim_{t\to\tilde{t}_1^-}\xi(t)\|+\|b_2(\Gamma_0(e(\tilde{t}_1),u(\tilde{t}_1),\tilde{t}_1^+)-\Gamma_0(e(\tilde{t}_1),u(\tilde{t}_1),\tilde{t}_1^-))\|\leqslant\frac{\tilde{\eta}_2}{\sqrt{c_{21}}\omega_o}+2\pi_0(\eta_{e2},\eta_{u2}).\tag{F79}$$

Thus, there exists $\omega_5^*\geqslant\omega_4^*$ such that for any $\omega_o\in[\omega_5^*,\infty)$,

$$\|\xi(\tilde{t}_1)\|\leqslant\eta_{\xi3}\triangleq3\pi_0(\eta_{e2},\eta_{u2}).\tag{F80}$$

Define

$$\Omega_2=\left\{(e,\xi)\mid\sqrt{V_{k_c}(e)}\leqslant\rho_2,\quad\sqrt{\bar{V}_2(\xi)}\leqslant\sqrt{c_{22}}\eta_{\xi3}\right\},\tag{F81}$$

where $\rho_2\triangleq\max\{\min\{\rho_1,\sqrt{c_{k2}}\eta_{e2}\},2\sqrt{c_{k2}}\|P_{k_c}\tilde{b}_{ft}k_{ce}^T\|\bar{\mu}(\omega_6^*)\sqrt{c_{22}}\eta_{\xi3}/\sqrt{c_{21}}\}$. Similar with the deduction (F62)-(F66), there exists $\omega_6^*\geqslant\omega_5^*$ such that $(e(t),\xi(t))$ will stay in Ω_2 for any $\omega_o\in[\omega_6^*,\infty)$ and $t\in[\tilde{t}_1,\tilde{t}_2)$. Hence, similar with (F67)-(F68), there exists a positive $\tilde{\eta}_3$ such that the following inequality is satisfied.

$$\sqrt{\bar{V}_2(\xi(t))}\leqslant\frac{\tilde{\eta}_3}{c_0\omega_o}+\sqrt{c_{22}}\eta_{\xi3}e^{-\frac{c_0\omega_o}{2}(t-\tilde{t}_1)},\quad t\in[\tilde{t}_1,\tilde{t}_2).\tag{F82}$$

Define $t_{p,2} = \tilde{t}_1 + (t_{p,1} - \tilde{t}_0)$. According to the same deduction with (F69)-(F71), there exists a positive $\tilde{\eta}_4$ such that $\sqrt{\bar{V}_2(\xi(t))} \leqslant \tilde{\eta}_4/\omega_o$ for any $\omega_o \in [\omega_6^*, \infty)$ and $t \in [t_{p,2}, \tilde{t}_2)$. Moreover, by defining $\omega_7^* = \max\{\omega_6^*, \tilde{\eta}_4\omega_4^*/\tilde{\eta}_2\}$, for any $\omega_o \in [\omega_7^*, \infty)$, there is

$$\sqrt{\bar{V}_2(\xi(t))} \leqslant \frac{\tilde{\eta}_4}{\omega_o} \leqslant \frac{\tilde{\eta}_2}{\omega_4^*}, \quad t \in [t_{p,2}, \tilde{t}_2). \tag{F83}$$

Notice that the following inequality is satisfied.

$$\sum_{i=1}^{j} \int_{\tilde{t}_i}^{t_{p,i+1}} e^{-\frac{t-\tau}{2c_{k2}}} d\tau \leqslant \frac{2\bar{\mu}_3(\omega_o)}{c_0(1-\eta_c)}, \quad t \geqslant t_{p,j+1} + \frac{\varphi_d}{2}, \quad j \geqslant 1 \tag{F84}$$

where $\eta_c = e^{-\frac{\varphi_d}{4c_{k2}}}$ and $t_{p,i+1} = \tilde{t}_i + t_{p,1} - \tilde{t}_0$ $(i \geqslant 1)$. It can be deduced from (F73) that for $t \in [\tilde{t}_1, \tilde{t}_2)$,

$$\|e_z(t)\| \leqslant \frac{\sqrt{V_{k_c}(e_z(\tilde{t}_0))}}{\sqrt{c_{k1}}} + \frac{\|\tilde{b}_{f_t} k_{ce}^T\| \bar{\mu}_2(\omega_o)}{c_{k1}} \left(\frac{2c_{k2}\max\{\tilde{\eta}_2, \tilde{\eta}_4\}}{\omega_o} + \frac{2(2-\eta_c)\bar{\mu}_3(\omega_o)\sqrt{c_{22}}\eta_{\xi 3}}{c_0(1-\eta_c)\sqrt{c_{21}}} \right). \tag{F85}$$

Moreover, since $\|e^*(t)\| \leqslant \|e^*(\tilde{t}_1)\|$ for $t \in [\tilde{t}_1, \infty)$, there exists $\omega_8^* \geqslant \omega_7^*$ such that $\|e(\tilde{t}_2)\| \leqslant \eta_{e2}$. Additionally, owing to (F43) and the continuity of $(\hat{\tilde{x}}(t), \hat{f}_{total}(t), \tilde{r}(t))$, the control input satisfies that $\|u(\tilde{t}_2)\| \leqslant \eta_{u2}$. Then, the rest of the proof is based on $\omega_o \in [\omega_8^*, \infty)$.

The analysis for the trajectories of the closed-loop system in $t \in [\tilde{t}_i, \tilde{t}_{i+1})$ $(i \geqslant 2)$.

First, consider $t \in [\tilde{t}_2, \tilde{t}_3)$ for which the proof is almost the same with the case of $t \in [\tilde{t}_1, \tilde{t}_2)$. The continuity of $e(t)$ implies

$$\sqrt{V_{k_c}(e(\tilde{t}_2))} \leqslant \min\{\rho_2, \sqrt{c_{k2}}\eta_{e2}\}. \tag{F86}$$

Similar with (F79)-(F80), there exists $\omega^* \geqslant \omega_8^*$ such that $\|\xi(\tilde{t}_2)\| \leqslant \eta_{\xi 3}$. Notice that

$$\rho_3 \triangleq \max\{\min\{\rho_2, \sqrt{c_{k2}}\eta_{e2}\}, 2\sqrt{c_{k2}}\|P_{k_c}\tilde{b}_{f_t} k_{ce}^T\| \bar{\mu}(\omega_6^*)\sqrt{c_{22}}\eta_{\xi 3}/\sqrt{c_{21}}\} = \rho_2. \tag{F87}$$

Then, it can be determined that $(e(t), \xi(t))$ lies in Ω_2 due to the similar process of $t \in [\tilde{t}_1, \tilde{t}_2)$. According to the similar deduction as (F82)-(F85), it yields that for $t \in [\tilde{t}_2, \tilde{t}_3)$, (F85) is satisfied,

$$\sqrt{\bar{V}_2(\xi(t))} \leqslant \frac{\tilde{\eta}_3}{c_0\omega_o} + \sqrt{c_{22}}\eta_{\xi 3} e^{-\frac{c_0\omega_o}{2}(t-\tilde{t}_2)}, \tag{F88}$$

and

$$e(\tilde{t}_3) \leqslant \eta_{e2}, \quad \|u(\tilde{t}_3)\| \leqslant \eta_{u2}. \tag{F89}$$

We proceed to consider the closed-loop system for $t \in [\tilde{t}_i, \tilde{t}_{i+1})$ $(i > 2)$. Similar with the deduction (F86)-(F89), it can be concluded that for any $\omega_o \in [\omega^*, \infty)$, there is

$$\begin{cases} \sqrt{\bar{V}_2(\xi(t))} \leqslant \dfrac{\tilde{\eta}_3}{c_0\omega_o} + \sqrt{c_{22}}\eta_{\xi 3} e^{-\frac{c_0\omega_o}{2}(t-\tilde{t}_i)}, \quad t \in [\tilde{t}_i, \tilde{t}_{i+1}), \\ \|e_z(t)\| \leqslant \dfrac{\sqrt{V_{k_c}(e_z(\tilde{t}_0))}}{\sqrt{c_{k1}}} + \dfrac{\|\tilde{b}_{f_t} k_{ce}^T\| \bar{\mu}_2(\omega_o)}{c_{k1}} \left(\dfrac{2c_{k2}\max\{\tilde{\eta}_2, \tilde{\eta}_4\}}{\omega_o} + \dfrac{2(2-\eta_c)\bar{\mu}_3(\omega_o)\sqrt{c_{22}}\eta_{\xi 3}}{c_0(1-\eta_c)\sqrt{c_{21}}} \right). \end{cases} \tag{F90}$$

Finally, from (F52), (F74), (F85) and (F90), (30) is proved. With the combination of (F68), (F82), (F88) and (F90), (31) is hold. ■

The result (F23) indicates that the error $|y(t) - (r(t) + \tilde{c}e^{A_{k_c}(t-t_0)}(\tilde{x}(t_0) - \tilde{r}(t_0)))|$ is bounded and, more importantly, tunable by the bandwidth ω_o of the ESO (F17), where $(r(t) + \tilde{c}e^{A_{k_c}(t-t_0)}(\tilde{x}(t_0) - \tilde{r}(t_0)))$ is the desired tracking trajectory exponentially converging to the reference signal $r(t)$. The eigenvalues of A_{k_c} determine the speed of the exponential decay of the desired tracking error. In addition, the larger the bandwidth of the ESO ω_o is, the smaller the difference between output and the desired tracking trajectory will be. Moreover, the result (F24) illustrates that the estimation errors for the derivatives of the controlled variable and the total disturbance are bounded in each smooth region, and tunable by the ESO's bandwidth ω_o. Additionally, the transient performance of the estimation errors is also adjustable by ω_o.

From Appendix F, the most important step in designing ADRC for the systems with multiple disturbances is to transform the unobservable and/or mismatched disturbances into observable and matched total disturbance, that is, from the system (1) to the system (F13). The transformation process and the principle behind are summarized in the following three steps:

Step 1: Investigating the relative degree of the uncertain system (1) and getting the integrator chain form (F2).

For a specific physical plant, the relative degree of the system depends on the physical mechanism whatever the mathematical description is. Then, Theorem 1 illustrates that the transformation from the new state $\tilde{x}$, which is composed by the controlled variable and its up to $(n-1)$-th derivatives, to the state x exists so that the integrator chain form (F2) can be obtained. The proof of Theorem 1 shows the details of transformation function $x = g_{f,x_0}(\tilde{x}, t)$.

Step 2: Conceptualizing the total disturbance due to the integrator chain form (F2).

Based on the integrator chain form (F2), the equivalent total effect of the multiple disturbances on the controlled variable is conceptualized as the total disturbance (F12).

Step 3: Reformulating the uncertain system (1) into the system (F13).

Then, for systems with multiple disturbances, the key is to handle the total disturbance, which can be estimated by ESO (F17) designed due to the system (F13). With the help of the online estimations from ESO, the ADRC controller, which combines the compensation for the total disturbance, rather than some specific disturbances, with the feedback of the controlled variable and the estimations of its derivatives, can be designed for the uncertain system (1).

Appendix G Extended discussions on ADRC designs

Appendix G.1 ADRC design with model information

Since the system matrices (A, b_u, B_f, c) are known model information, the total disturbance may contain some known information, which can be utilized to release the workload of the ESO for estimating the total disturbance. And a corresponding ADRC with known model information can be designed. For example, if the disturbances $f(t)$ only depend on t, then the transformation function $g_{f,x_0}(\tilde{x}, t)$ can be further expressed as $x = g_f(\tilde{x}, t) = T^{-1}(\tilde{x}+m_f)$, where $m_f = [0\ m_{f,2}\ \cdots\ m_{f,n}]^T$ and $m_{f,i} = \sum_{j=2}^{i} c^T A^{j-2} B_f f^{(i-j)}$ $(i \geqslant 2)$. Then the total disturbance f_{total} (F12) can be further expressed by the following known and unknown parts:

$$f_{total}(\tilde{x}, t) = f_{known}(\tilde{x}) + f_{unknown}(t), \tag{G1}$$

where $f_{known}(\tilde{x}) = c^T A^n T^{-1}\tilde{x}$ and $f_{unknown}(t) = T^{-1}m_f + m_{f,n+1}$. Hence, an ESO with known model information can be designed as follows.

$$\begin{bmatrix} \dot{\hat{\tilde{x}}}(t) \\ \dot{\hat{f}}_{unknown}(t) \end{bmatrix} = \begin{bmatrix} \tilde{A} + \tilde{b}_{f_t} c^T A^n T^{-1} & \tilde{b}_{f_t} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \hat{\tilde{x}}(t) \\ \hat{f}_{unknown}(t) \end{bmatrix} + \begin{bmatrix} \tilde{b}_u \\ 0 \end{bmatrix} u + l_{ESO}\tilde{c}^T(\tilde{x} - \hat{\tilde{x}}), \tag{G2}$$

where $\hat{\tilde{x}}$ and $\hat{f}_{unknown}$ are the estimations for the state $\tilde{x}$ (up to $(i-1)$-th order derivative of the controlled variable y) and the unknown part of the total disturbance $f_{unknown}$, respectively. Similar with the idea of the tuning law (F18), under the model information f_{known}, the ESO's parameter l_{ESO} can be designed such that all the eigenvalues of $\tilde{A}_{ESO} \triangleq \begin{bmatrix} \tilde{A}+\tilde{b}_{f_t} c^T A^n T^{-1} & \tilde{b}_{f_t} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} - l_{ESO}[\tilde{c}^T\ 0]$ are set at $-\omega_o$ $(\omega_o > 0)$. A corresponding ADRC law with known model information can be designed as

$$u(t) = \begin{cases} 0, \quad t_0 \leqslant t < \tilde{t}_0, \\ -\dfrac{\hat{f}_{unknown}(t) + f_{known}(\hat{\tilde{x}})}{c^T A^{n-1} b_u} + \dfrac{r^{(n)}(t) - \sum_{i=1}^{n} k_{c,i}(\hat{\tilde{x}}_i(t) - r^{(i-1)}(t))}{c^T A^{n-1} b_u}, \quad t \geqslant \tilde{t}_0. \end{cases} \tag{G3}$$

The theoretical analysis for the ADRC with known model information (G2) and (G3), which is similar to Theorem 3, can be obtained.

Appendix G.2 ADRC for some more general cases

Via the conceptualization of integrator chain and total disturbance discussed in this study, ADRC can be further applied for some more extended situations:

1). If the uncertain system (1) is minimum-phase and its relative degree is k $(0 < k < n)$, the corresponding integrator chain form (F2) will be k-th order. With the ESO (F17) being $(k+1)$-th order, ADRC similar to (F17)–(F19) can be designed and the theoretical results similar to Theorem 3 can be obtained.

2). For the following general nonlinear uncertain system:

$$\Sigma(f, g, h): \quad \dot{x}(t) = f(x, t) + g(x, t)u(t), \quad y(t) = h(x), \tag{G4}$$

ADRC similar to (F17)–(F19) can be designed and the theoretical results similar to Theorem 3 can be obtained under the assumptions similar to Assumption 1 and Assumption 2.

Appendix H Experimental verification

In this appendix, the proposed ADRC design for TMS system is verified in the hardware experiment, which is conducted on the rectilinear plant Model 210 from Educational Control Products as shown in Fig. D2. The optical encoder mounted at the body 2 (right) provides position measurement. The left mass is driven by a brush-less DC servo motor via rack and pinion transmission. Springs are installed from the left base to the body 1 (left), the body 1 (left) to the body 2 (right) and the body 2 (right) to the right base.

The control objective is to force the body 2 to track the desired reference signal $r(t)$, which is obtained by utilizing the tracking differentiator proposed in [47] to smooth the following piecewise function

$$r_0(t) = \begin{cases} 0.01\ (m), \quad 0 \leqslant t < 6, \\ -0.01\ (m), \quad 6 \leqslant t \leqslant 12. \end{cases} \tag{H1}$$

Since the relative degree from the control input u to the controlled variable $y(= x_2)$ is 4 despite of f_1 and f_2, an ADRC with a fifth order ESO (G2) and feedback control law (G3) is designed, where the known model information is $f_{known}(\tilde{x}) = -\frac{k_1k_2+k_1k_3+k_2k_3}{m_1m_2}\tilde{x}_1 - \frac{m_2(k_1+k_2)+m_1(k_2+k_3)}{m_1m_2}\tilde{x}_3$. What the ESO (G2) provides is the estimations of the

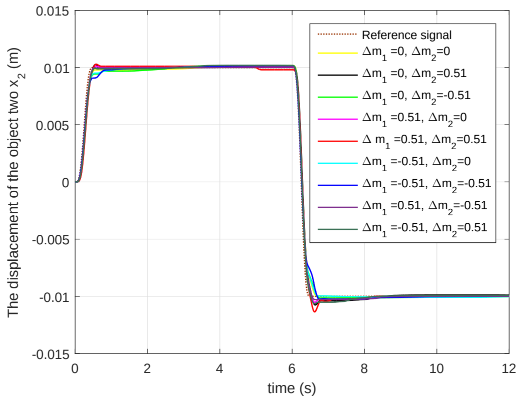

Figure H1 The response curves of the displacement of the body 2 x_2 for nine cases of parametric perturbations.

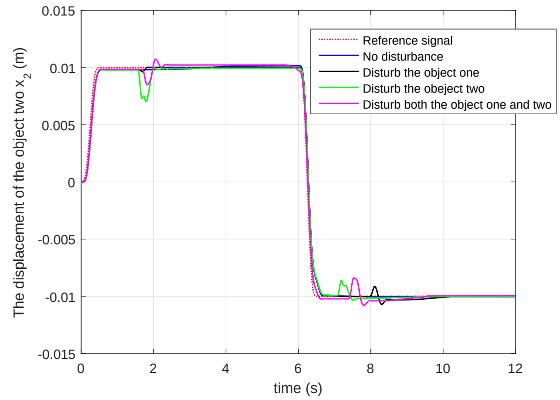

Figure H2 The response curves of the displacement of the body 2 x_2 for four cases of disturbances.

derivatives of the controlled variable $[y^{(1)}\ y^{(2)}\ y^{(3)}]^T$ and the unknown part of total disturbance

$$\begin{aligned} f_{unknown} = &\left(-\frac{(k_1+\Delta k_1)(k_2+\Delta k_2)+(k_1+\Delta k_1)(k_3+\Delta k_3)+(k_2+\Delta k_2)(k_3+\Delta k_3)}{(m_1+\Delta m_1)(m_2+\Delta m_2)}\right)\tilde{x}_1 \\ &-\frac{(m_2+\Delta m_2)(k_1+\Delta k_1+k_2+\Delta k_2)+(m_1+\Delta m_1)(k_2+\Delta k_2+k_3+\Delta k_3)}{(m_1+\Delta m_1)(m_2+\Delta m_2)}\tilde{x}_3 \\ &+\frac{k_1k_2+k_1k_3+k_2k_3}{m_1m_2}\tilde{x}_1+\frac{m_2(k_1+k_2)+m_1(k_2+k_3)}{m_1m_2}\tilde{x}_3 \\ &+\left(\frac{k_2+\Delta k_2}{(m_1+\Delta m_1)(m_2+\Delta m_2)}-\frac{k_2}{m_1m_2}\right)u+\frac{(k_2+\Delta k_2)}{m_1(m_2+\Delta m_2)}w_1+\frac{1}{m_2+\Delta m_2}w_2^{(2)} \\ &-\frac{(k_1+\Delta k_1+k_2+\Delta k_2)(m_2+\Delta m_2)+2(k_2+\Delta k_2+k_3+\Delta k_3)(m_1+\Delta m_1)}{(m_1+\Delta m_1)(m_2+\Delta m_2)^2}w_2. \end{aligned} \tag{H2}$$

What the ADRC (G2)–(G3) compensates for is the total disturbance (F16), rather than a specific disturbance f_1 or f_2.

The system parameters are $m_1 = 2.37\ (kg)$, $m_2 = 1.52\ (kg)$, $k_1 = 450\ (N/m)$, $k_2 = 405\ (N/m)$, $k_3 = 210\ (N/m)$. The initial values of the TMS system are $[x_1(0)\ x_2(0)\ x_3(0)\ x_4(0)\ w_1(0)\ w_2(0)] = \boldsymbol{0}$. Selecting the initial values of the ESO (G2) as zero, it can be deduced from the definition of $\tilde{t}_0$ that $\tilde{t}_0 = 0$. The control parameters (k_c, l_{ESO}) are designed via pole placement such that the eigenvalues of A_{k_c} and $\tilde{A}_{ESO}$ are assigned at $-17\ (rad/s)$ and $-60\ (rad/s)$, respectively.

In the first experiment, the following nine cases of parametric perturbations are considered.

$$\{(\Delta m_1, \Delta m_2) \mid \Delta m_1 \in \Delta_m,\ \Delta m_2 \in \Delta_m\}, \quad \Delta_m = \{0\ (kg), -0.51\ (kg), 0.51\ (kg)\}. \tag{H3}$$

The hardware experiment results of the ADRC for handling the nine cases of parametric perturbations are given in Fig. H1, which shows that the response curves of the displacement of the body 2 are highly consistent despite various parametric perturbations. This result demonstrates the strong capability of ADRC for handling multiple uncertainties.

In the second experiment, four cases of external disturbances are conducted to verify the robustness of the ADRC design, which are no external disturbances (Case 1), disturbing the body 1 by hand at about 2s and 8s (Case 2), disturbing the body 2 by hand at about 2s and 8s (Case 3) and disturbing both the body 1 and body 2 by hands at about 2s and 8s (Case 4). The experimental results for these four cases are shown in Fig. H2. From Fig. H2, no matter what channel the disturbances are added in, the tracking performance of the displacement of the body 2 is fairly consistent, which illustrates the strong capability of ADRC for various external disturbances.

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