SCIENCE CHINA
Information Sciences

• RESEARCH PAPER •

January 2020, Vol. 63 112205:1–112205:10
https://doi.org/10.1007/s11432-019-9934-5

Mean square stability for Markov jump Boolean networks

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Received 7 March 2019/Accepted 30 May 2019/Published online 25 December 2019

Abstract In this paper, one of the stability definitions of Markov jump Boolean networks (MJBNs), called mean square stability (MSS), is investigated. Some necessary and sufficient conditions are presented to guarantee the MSS of such MJBNs. Moreover, one of the necessary and sufficient conditions for MSS is obtained in terms of linear programming, which implies that MSS is equivalent to global stability with probability 1 for MJBNs. Furthermore, the construction of Lyapunov function is given and also another theorem based on the Lyapunov function is derived to ensure the MSS of MJBNs. Finally, a numerical example is provided to illustrate the profits of our results.

Keywords mean square stability, MSS, Boolean network, BN, stochastic stability, Markovian jump parameters, semi-tensor product, STP, Lyapunov function

Citation Wang L Q, Fang M, Wu Z-G. Mean square stability for Markov jump Boolean networks. Sci China Inf Sci, 2020, 63(1): 112205, https://doi.org/10.1007/s11432-019-9934-5

1 Introduction

Boolean network (BN), introduced first in 1969 [1], is a typical formalism to model and describe biological regulatory networks dynamics. The three elements that comprise BNs are nodes, connected lines, and update rules. The dynamics of BNs can be used to express the genetic variables in a much simpler and effective manner. Here, 1 represents expression and 0 represents no expression. For every node in the network, its state variable follows the logical relationship between the nodes that were connected in the network, and the nonlinear update relationship is also given in terms of a Boolean function. A new mathematical method called semi-tensor product (STP) of matrices was proposed in [2] to give an algebraic structure for BNs. It was an effective and novel tool to analyze and synthesize this model, and many studies have investigated its dynamic properties. A multitude of modern control theories have been applied to BNs [3, 4] and PBNs (probabilistic Boolean networks), such as controllability [5, 6], reachability [7], observability [8, 9], stabilization [10], and other applications, including shifting register [11]. Besides, many modern control designs have been applied to achieve the dynamic goals, such as state feedback control [12, 13], sampled-date state feedback control [14], pinning control [15], and event-triggered control [16].

Subject to external disturbance at each discrete time, some genes in the gene regulatory networks would update their state variables by different update rules from the possible update rule set. Consider the example of bacteriophage. Lysis and lysogeny are two different behaviors of this example, where which

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one is active is uncertain. Moreover, asynchronous BNs describe a phenomenon in that among a group of BNs, and only one BN is active at each time [17]. These are both switching phenomena in BNs. When the switched signal randomly switches among a set, it generates a new model called PBN, by following a probability distribution. PBN was first proposed by Shmulevich [18, 19] to describe the stochasticity and uncertainties in gene regulation networks. Many studies show that PBNs can model rule-based properties better than determined BNs for gene regulatory networks. Therefore, with the development of STP method, it is back in the spotlight. Nowadays, some interesting studies on PBNs have performed, such as stability analysis [20], controllability [21], stabilization, and set stabilization [22,23]. Besides, some studies investigate a BN with external disturbances, which follows a conditional probability distribution according to the state value [24–26]. Totally speaking, Markov jump Boolean networks (MJBNs) is a kind of switched BNs following a jump rule. Recently, many studies focus on Markov chains; thus, it is being applied to many fields, especially to model gene networks [27, 28]. For example, Ref. [20] shows that in complex multicellular organisms systems, the update rules are different at each step even for the same gene. Moreover, in a normal gene regulatory network, the update directions among states have been observed to be random.

We observe that Ref. [20] shows the definition of global stability for BNs in stochastic sense. Mean square stability (MSS) is a key concept for Markov jump linear systems [29]. Therefore, it is pointed that for a normal Markov jump linear system, MSS implies stability with probability 1, not vice versa. In a BN with stochastic switched signals, global stability is equivalent to stability with probability 1. To the best of our knowledge, there are few studies considering MSS for BNs considered in this paper, not to mention the relationship between the stability with probability 1 and MSS. STP of matrices is the main method in this paper, based on which MSS of MJBNs is mainly investigated. The contributions of this paper are listed as follows:

- Some necessary and sufficient conditions are obtained for MSS of MJBNs.
- Lyapunov function is an important tool in stability analysis for BNs [30]. A kind of linear Lyapunov function has been applied to obtain a necessary and sufficient condition in terms of linear programming for MSS.
- Based on a one-to-one mapping from MSS to global stability with probability 1 [20], MSS for MJBNs has been proved to be equivalent to global stability with probability 1.

The organization of the paper is listed as follows. Section 2 prestents some necessary preliminaries of some notations and some propositions of STP. Section 3 presents the major results of this paper, including the definitions of stability, propositions, lemmas and theorems. Section 4 gives a numerical example to illustrate the effectiveness of our results. Finally, Section 5 is a brief conclusion.

2 Preliminaries

2.1 Notations

Let the delta set $\Delta_s := \{\delta_s^i | i = 1, 2, \ldots, s\}$, where δ_s^i is the i-th column of identity matrix I_s with degree s. An $m \times n$ logical matrix M is a matrix whose columns are elements of Δ_m. Let $\mathrm{Col}_i(M)$ be the i-th column of matrix M. The set of all $m \times n$ logical matrices is denoted by $\mathcal{L}_{m\times n}$. Assume $A = [\delta_m^{i_1}, \delta_m^{i_2}, \ldots, \delta_m^{i_n}] \in \mathcal{L}_{m\times n}$; then, we simplify it by $A = \delta_m[i_1, i_2, \ldots, i_n]$. Define a field $\mathcal{F} = \{1, 0\}$; then a Boolean matrix B is an $m \times n$ matrix with all entries in $\mathcal{F}$. The set of such Boolean matrices is denoted by $\mathcal{B}_{m\times n}$. Identify each element in $\mathcal{F}$ by vectors as 1 (True) $\sim \delta_2^1$ and 0 (False) $\sim \delta_2^2$, respectively; then $\mathcal{F} \sim \Delta_2$. Set $\mathbb{R}$ represents the set of real number and $\mathbb{R}^n$ denotes the n-dimensional real vector space. $\mathbb{R}_{m\times n}$ is a set that contains all $m \times n$-dimensional real matrices. For a stochastic variable x, $\mathrm{E}\{x\}$ represents the expectation of x. A matrix $M > 0$ means that all entries of M, $(M)_{ij} > 0$ hold. If there is another matrix B such that $A > B$, it implies $A - B > 0$. For an n dimensional nonnegative real vector $x = (x_1 \ \cdots \ x_n)^{\mathrm{T}}$, $\sqrt{x} := (\sqrt{x}_1 \ \cdots \ \sqrt{x}_n)^{\mathrm{T}}$.

2.2 STP of matrices

Definition 1 ([2]). For two given matrices $P \in \mathbb{R}_{m\times n}$ and $Q \in \mathbb{R}_{r\times l}$, the semi-tensor product of them is defined as

$$P \ltimes Q = (P \otimes I_{\alpha/n})(Q \otimes I_{\alpha/r}),$$

where $\alpha = \mathrm{lcm}(n, r)$, $\mathrm{lcm}(,)$ is the least common multiple operator, and $\otimes$ is the tensor (or Kronecker) product.

Lemma 1 ([2]). Any logical function $f(x_1, x_2, \ldots, x_n)$ with logical arguments $x_1, x_2, \ldots, x_n \in \Delta_2$ can be expressed in a multi-linear form as

$$f(x_1, \ldots, x_n) = M_f x_1 \ltimes x_2 \ltimes \cdots \ltimes x_n,$$

where the unique structure matrix of f $M_f \in \mathcal{L}_{2\times 2^n}$.

3 Main results

3.1 Mean square stability

In this subsection, we consider a Markov jump Boolean network (MJBN) with n nodes $x_1, x_2, \ldots, x_n$ and N update rules. $x_i(t) \in \Delta_2$ is the state value of node x_i at time t. Let $x(t) = \ltimes_{i=1}^n x_i(t)$, and let the algebraic form of the BN be

$$x(t+1) = L_{\theta(t)} x(t), \tag{1}$$

where $x(t) \in \Delta_{2^n}$. $\theta(t)$ represents the discrete time homogeneous Markov chain, which takes its values in a finite set $\mathbb{N} = \{1, \ldots, N\}$ with an initial distribution $\boldsymbol{p} = [p_1 \cdots p_N]^{\mathrm{T}}$. Thus, the transition probability matrix $P = (p_{ij})_{N\times N}$ is

$$p_s = \boldsymbol{P}\{\theta(0) = s\}, \tag{2}$$

$$p_{ij} = \boldsymbol{P}\{\theta(t+1) = j | \theta(t) = i\}, \tag{3}$$

where $\sum_{s=1}^N p_s = 1$ and $\sum_{j=1}^N p_{ij} = 1$.

Definition 2. MJBN (1) is said to be globally stable at $x_e \in \Delta_{2^n}$ with probability 1, if for any initial state value and switched signal distribution $\lim_{t\to\infty} \mathrm{E}\{x(t)|x(0), \theta(0)\} = x_e$.

Definition 3. MJBN (1) is said to be MSS at $X_e \in \Delta_{2^{2n}}$, if for any initial state value and switched signal distribution $\lim_{t\to\infty} \mathrm{E}\{x(t) \ltimes x(t)|x(0), \theta(0)\} = X_e$.

Without loss of generality, we assume x_e to be $\delta_{2^n}^{2^n}$ and $X_e = \delta_{2^{2n}}^{2^{2n}}$. Now, we consider a new system induced by (1). Let $X(t) = x(t+1) \ltimes x(t+1)$, and then we have

$$X(t+1) = L_{\theta(t)} \otimes L_{\theta(t)} X(t), \tag{4}$$

where $X(t) \in \mathcal{S} = \{\delta_{2^{2n}}^{(k-1)2^n+k}, k = 1, 2, \ldots, 2^n\}$ and $L_i \otimes L_i \in \mathcal{L}_{2^{2n}\times 2^{2n}}$ for all $i \in \mathbb{N}$.

Let $Z_j(t) = X(t) 1_{\theta(t)=j}$, where

$$1_{\theta(t)=j} = \begin{cases} 1, & \theta(t) = j, \\ 0, & \theta(t) \in \mathbb{N} \setminus \{j\}. \end{cases}$$

Then $\mathrm{E}\{X(t)\} = \sum_{j=1}^N Z_j(t)$. For $j \in \mathbb{N}$, we have

$$\begin{aligned} Z_j(t+1) &= \mathrm{E}\{X(t+1) 1_{\theta(t+1)=j}\} \\ &= \sum_{i=1}^N \mathrm{E}\{X(t+1) 1_{\theta(t+1)=j} 1_{\theta(t)=i}\} \end{aligned}$$

$$= \sum_{i=1}^{N} p_{ij} L_i \otimes L_i Z_i(t).$$

Let $Z_j(t) = (W_j(t)^{\mathrm{T}}\ \pi_j(t))^{\mathrm{T}}$, where $W_j(t) \in \mathbb{R}^{2^{2n}-1}$ and $\pi_j(t) \in \mathbb{R}$ for all $j \in \mathbb{N}$. Then we have the following result.

Proposition 1. MJBN (1) is MSS at $X_e = \delta_{2^{2n}}^{2^{2n}}$ if and only if for any initial state value and any initial distribution of $\theta(t)$, the following equation holds:

$$\lim_{t\to\infty} W_j(t) = 0_{2^{2n}-1}, \quad \forall j \in \mathbb{N}. \tag{5}$$

Proof. (Sufficiency) Assume for any initial state value and any initial distribution of $\theta(t)$, Eq. (5) holds. It implies $\lim_{t\to\infty} \sum_{j=1}^{N} \pi_j(t) = 1$. Because $\lim_{t\to\infty} \mathrm{E}\{X(t)\} = \lim_{t\to\infty} \sum_{j=1}^{N} Z_j(t)$, it can be concluded that $\lim_{t\to\infty} \mathrm{E}\{X(t)\} = \delta_{2^{2n}}^{2^{2n}}$.

(Necessity) If system (1) is MSS at $X_e = \delta_{2^{2n}}^{2^{2n}}$, we have $\lim_{t\to\infty} \mathrm{E}\{X(t)\} = \delta_{2^{2n}}^{2^{2n}}$, which implies $\lim_{t\to\infty} \sum_{j=1}^{N} W_j(t) = 0_{2^{2n}-1}$. Because $W_j(t) \geqslant 0$, Eq. (5) holds.

We recall a result from [29] that if the Markov chain $\{\theta(t), t = 0, 1, \ldots\}$ is an ergodic Markov chain, then it implies that for any initial distribution of $\{p_i, i \in \mathbb{N}\}$, there exists a limit probability distribution $\{\pi_i; \pi_i > 0, i \in \mathbb{N}\}$ such that

$$\sum_{i=1}^{n} p_{ij}\pi_i = \pi_j, \quad \sum_{i=1}^{N} \pi_i = 1.$$

Then, we have the following proposition.

Proposition 2. MJBN (1) is MSS at $X_e = \delta_{2^{2n}}^{2^{2n}}$ if and only if for any initial state value and any initial distribution of $\theta(k)$, the following equation holds:

$$\lim_{t\to\infty} \pi_j(t) = \pi_j, \quad \forall j \in \mathbb{N}. \tag{6}$$

Let $Z(t) = (Z_1(t)^{\mathrm{T}} \cdots Z_N(t)^{\mathrm{T}})^{\mathrm{T}}$ and define a matrix $\mathbb{B} = (P' \otimes I_{2^{2n}})\mathrm{diag}[L_i \otimes L_i]$. Then, we obtain

$$Z(t+1) = \mathbb{B}Z(t). \tag{7}$$

The following theorem then is obvious.

Theorem 1. MJBN (1) is MSS at $X_e = \delta_{2^{2n}}^{2^{2n}}$ if and only if for any initial state value and any initial distribution of $\theta(0)$, the following equation holds:

$$\lim_{t\to\infty} Z(t) = (0 \cdots \pi_1\ 0 \cdots \pi_2\ 0 \cdots \pi_N)^{\mathrm{T}}. \tag{8}$$

Remark 1. The above theorem shows that, the fixed point of (7) is related to the limit probability distribution for the ergodic Markov process [29]. Thus, it is easy to verify the MSS of the considered system via checking the fixed point if the limit probability distribution is known.

Lemma 2. If MJBN (1) is MSS at $X_e = \delta_{2^{2n}}^{2^{2n}}$, then $\delta_{2^n}^{2^n}$ is a fixed point of all subsystems, that is

$$L_i\delta_{2^n}^{2^n} = \delta_{2^n}^{2^n}. \tag{9}$$

Proof. BN (1) is MSS at $X_e = \delta_{2^{2n}}^{2^{2n}}$, then the limit of $Z(t)$ exists. Hence, take limit of both sides in (7), and then we have

$$\pi_j\delta_{2^{2n}}^{2^{2n}} = \sum_{i=1}^{N} p_{ij}(L_i \otimes L_i)\pi_i\delta_{2^{2n}}^{2^{2n}}$$
$$= \sum_{i=1}^{N} \pi_i p_{ij}(L_i \otimes L_i)\delta_{2^{2n}}^{2^{2n}}$$

$$= \pi_j (L_i \otimes L_i)\delta_{2^{2n}}^{2^{2n}}$$
$$= \pi_j (L_i \delta_{2^n}^{2^n}) \ltimes (L_i \delta_{2^n}^{2^n}),$$

which implies $L_i\delta_{2^n}^{2^n} = \delta_{2^n}^{2^n}$.

Theorem 2. System (1) is MSS at $X_e = \delta_{2^{2n}}^{2^{2n}}$ if and only if there exist nonnegative vectors $\lambda_i \in \mathbb{R}^{2^n}, i \in \mathbb{N}$ such that the following conditions hold:

$$(\lambda_i \ltimes \lambda_i)^{\mathrm{T}} \delta_{2^{2n}}^{2^{2n}} = 0, \tag{10a}$$

$$(\lambda_i \ltimes \lambda_i)^{\mathrm{T}} \delta_{2^{2n}}^{(t-1)2^n+t} > 0, \tag{10b}$$

$$\sum_{j=1}^{N} p_{ij} (\lambda_j \ltimes \lambda_j)^{\mathrm{T}} (L_i \otimes L_i) \delta_{2^{2n}}^{2^{2n}} = 0, \tag{10c}$$

$$\left[\sum_{j=1}^{N} p_{ij} (\lambda_j \ltimes \lambda_j)^{\mathrm{T}} (L_i \otimes L_i) - (\lambda_i \ltimes \lambda_i)^{\mathrm{T}} \right] \delta_{2^{2n}}^{(t-1)2^n+t} < 0, \tag{10d}$$

for $i = 1, \ldots, N$ and $t = 1, 2, \ldots, 2^n - 1$.

Proof. (Sufficiency) We first claim that all subsystems have a common fixed point $\delta_{2^n}^{2^n}$ under conditions (10a)–(10d). Assume that there exists $i_0 \in \mathbb{N}$, such that $L_{i_0}\delta_{2^n}^{2^n} = \delta_{2^n}^{t}$, where $t \neq 2^n$. From (10b), we know that for all $j \in \mathbb{N}$,

$$(\lambda_j \ltimes \lambda_j)^{\mathrm{T}} (L_{i_0}\delta_{2^n}^{2^n}) \ltimes (L_{i_0}\delta_{2^n}^{2^n}) > 0.$$

On the other hand, $\sum_{j=1}^{N} p_{i_0 j} = 1, p_{i_0 j} \geqslant 0$ and there exists at least one $\overline{j}$, such that $p_{i_0\overline{j}} > 0$. Thus

$$\begin{aligned} 0 &= \sum_{j=1}^{N} p_{i_0 j} (\lambda_j \ltimes \lambda_j)^{\mathrm{T}} (L_{i_0} \otimes L_{i_0}) \delta_{2^{2n}}^{2^{2n}} \\ &= \sum_{j=1}^{N} p_{i_0 j} (\lambda_j \ltimes \lambda_j)^{\mathrm{T}} (L_{i_0}\delta_{2^n}^{2^n}) \ltimes (L_{i_0}\delta_{2^n}^{2^n}) \\ &= \sum_{j=1}^{N} p_{i_0 j} (\lambda_j \ltimes \lambda_j)^{\mathrm{T}} \delta_{2^{2n}}^{(t-1)2^n+t} \\ &> p_{i_0\overline{j}} (\lambda_{\overline{j}} \ltimes \lambda_{\overline{j}})^{\mathrm{T}} \delta_{2^{2n}}^{(t-1)2^n+t} > 0, \end{aligned}$$

which contradicts (10c). Therefore $\delta_{2^n}^{2^n}$ is a common fixed point of all subsystems in (1). Then we have

$$L_i \otimes L_i = \begin{bmatrix} L_{11}^i & 0_{2^{2n}-1} \\ \alpha_i^{\mathrm{T}} & 1 \end{bmatrix},$$

which implies that $W_j(t+1) = \sum_{i=1}^{N} p_{ij} L_{11}^i W_i(t)$. Let

$$W(t)^{\mathrm{T}} = (W_1(t)^{\mathrm{T}} \ W_2(t)^{\mathrm{T}} \cdots W_N(t)^{\mathrm{T}})^{\mathrm{T}},$$

and

$$Q = P' \otimes I_{2^{2n}-1} \mathrm{diag}\{L_{11}^1 \cdots L_{11}^N\}.$$

Then we have

$$W(t+1) = QW(t). \tag{11}$$

Let $\lambda_i = [\hat{\lambda}_i^{\mathrm{T}} \ 0]^{\mathrm{T}}$, where positive vector $\hat{\lambda}_i \in \mathbb{R}^{2^n-1}$; then

$$\lambda_i \ltimes \lambda_i = [(\hat{\lambda}_i \ltimes \hat{\lambda}_i)^{\mathrm{T}} \ (\hat{\lambda}_i \ltimes 0)^{\mathrm{T}} \ (0 \ltimes \hat{\lambda}_i)^{\mathrm{T}} \ 0]^{\mathrm{T}}.$$

Define

$$\widetilde{\lambda}_i = [(\hat{\lambda}_i \ltimes \hat{\lambda}_i)^{\mathrm{T}} \ (\hat{\lambda}_i \ltimes 0)^{\mathrm{T}} \ (0 \ltimes \hat{\lambda}_i)^{\mathrm{T}}]^{\mathrm{T}}.$$

According to (10d), it can be obtained that for all $i \in \mathbb{N}$, and $t = 1, 2, \dots, 2^n - 1$,

$$\left[\sum_{j=1}^{N} p_{ij}(\widetilde{\lambda}_i^{\mathrm{T}} L_{11}^i \ 0) - (\widetilde{\lambda}_i^{\mathrm{T}} \ 0)\right] \delta_{2^{2n}}^{(t-1)2^n+t} < 0.$$

Let $\widetilde{\lambda} = (\widetilde{\lambda}_1^{\mathrm{T}} \cdots \widetilde{\lambda}_N^{\mathrm{T}}) > 0$; then for positive system (11), we have

$$\widetilde{\lambda}^{\mathrm{T}}(Q - I) < 0,$$

which implies $W(k)$ convergences to $0_{2^{2n}-1}$. As a result, system (4) is MSS at $\delta_{2^{2n}}^{2^{2n}}$.

(Necessity) Because square stability implies stochastic stability of (1) and if system (1) is stochastic stable, $\delta_{2^n}^{2^n}$ is a fixed point of $L_i, i \in \mathbb{N}$. Then, we assume

$$L_i = \begin{bmatrix} \overline{L}_i & 0_{2^n-1} \\ l_i^{\mathrm{T}} & 1 \end{bmatrix},$$

$z_i(t) = \mathrm{E}\{x(t)1_{\theta(t)=i}\} \triangleq (w_i(t)^{\mathrm{T}} \ \mu_i(t))^{\mathrm{T}}, w_i(t) \in \mathbb{R}^{2^n-1}$. Then

$$\begin{aligned} z_j(t+1) &= \sum_{i=1}^{N} \mathrm{E}\{x(t+1)1_{\theta(t+1)=j}1_{\theta(t)=i}\} \\ &= \sum_{i=1}^{N} p_{ij} E\{L_i x(t) 1_{\theta(t)=i}\} \\ &= \sum_{i=1}^{N} p_{ij} L_i z_i(t), \end{aligned}$$

and

$$w_j(t+1) = \sum_{i=1}^{N} p_{ij} \overline{L}_i w_i(t).$$

Let $w(t) = (w_1(t)^{\mathrm{T}} \ w_2(t)^{\mathrm{T}} \cdots w_N(t)^{\mathrm{T}})^{\mathrm{T}}$, and $\mathbb{M} = P^{\mathrm{T}} \otimes I_{2^n-1}\mathrm{diag}\{\overline{L}_1 \cdots \overline{L}_N\}$; then

$$w(t+1) = \mathbb{M}w(t).$$

Because $\mathbb{M}$ is positive and $w(k)$ convergences to 0_{2^n-1}, thus there exist positive vectors, $\lambda_i \in \mathbb{R}^{2^n-1}$ such that

$$\lambda^{\mathrm{T}}(\mathbb{M} - I_{N(2^n-1)}) < 0,$$

which implies $\sum_{i=1}^{N} p_{ji}\lambda_i\overline{L}_j - \lambda_j < 0$. Let $\overline{\lambda}_i = (\lambda_i^{\mathrm{T}} \ 0)^{\mathrm{T}}$, and as a result of this, for $t = 1, \dots, 2^n - 1$, we have

$$\left(\sum_{i=1}^{N} p_{ji}\overline{\lambda}_i L_j - \overline{\lambda}_j\right) \delta_{2^n}^t < 0. \tag{12}$$

Letting $\widetilde{\lambda}_i = (\sqrt{\lambda_i}^{\mathrm{T}} \ 0)^{\mathrm{T}}$, then we claim that for $i = 1, \dots, N$,

$$\left(\sum_{i=1}^{N} p_{ji}(\widetilde{\lambda}_i \ltimes \widetilde{\lambda}_i)^{\mathrm{T}}(L_j \otimes L_j) - \widetilde{\lambda}_j \ltimes \widetilde{\lambda}_j\right) \delta_{2^{2n}}^{(t-1)2^n+t} < 0. \tag{13}$$

Without loss of generality, we prove it by case $j = 1$ and $L_1\delta_{2^n}^t = \delta_{2^n}^s$. According to (12), we have

$$\sum_{i=1}^{N} p_{1i}\overline{\lambda}_i\delta_{2^n}^s - \overline{\lambda}_1\delta_{2^n}^t < 0,$$

and

$$\begin{aligned}
&\left(\sum_{i=1}^{N} p_{1i}(\widetilde{\lambda}_i \ltimes \widetilde{\lambda}_i)^{\mathrm{T}}(L_1 \otimes L_1) - \widetilde{\lambda}_1 \ltimes \widetilde{\lambda_1}\right)\delta_{2^{2n}}^{(t-1)2^n+t}\\
&= \sum_{i=1}^{N} p_{1i}(\widetilde{\lambda}_i \ltimes \widetilde{\lambda}_i)^{\mathrm{T}}(L_1\delta_{2^n}^t) \ltimes (L_1\delta_{2^n}^t) - (\widetilde{\lambda}_1\delta_{2^n}^t) \ltimes (\widetilde{\lambda_1\delta_{2^n}^t})\\
&= \sum_{i=1}^{N} p_{1i}(\widetilde{\lambda}_i \ltimes \widetilde{\lambda}_i)^{\mathrm{T}}\delta_{2^{2n}}^{(s-1)2^n+s} - (\widetilde{\lambda}_1\delta_{2^n}^t) \ltimes (\widetilde{\lambda_1\delta_{2^n}^t})\\
&= \sum_{i=1}^{N} p_{1i}\overline{\lambda}_i\delta_{2^n}^s - \overline{\lambda}_1\delta_{2^n}^t < 0.
\end{aligned}$$

Hence, Eq. (13) holds, which is equal to (10d).

$(\widetilde{\lambda}_i \ltimes \widetilde{\lambda}_i)^{\mathrm{T}}\delta_{2^{2n}}^{2^{2n}} = (\widetilde{\lambda}_i\delta_{2^n}^{2^n})^2 = 0$ implies Eq. (10a) holds. As $\lambda_i > 0$, $(\widetilde{\lambda}_i \ltimes \widetilde{\lambda}_i)^{\mathrm{T}}\delta_{2^{2n}}^{(t-1)2^n+t} = (\widetilde{\lambda}_i\delta_{2^n}^t)^2 > 0$. Eq. (10c) also holds because $\delta_{2^n}^{2^n}$ is a fixed point of L_i. Thus, $\widetilde{\lambda}_i$ satisfies (10a)–(10d).

Remark 2. When $\mathbb{N} \equiv \{1\}$, system (1) considered becomes a BN without switching and thus the results degenerate the case of MSS of deterministic BNs. Because the MSS can be regarded as the synchronization of two same systems with the initial state value, according to the necessity part of the proof to Theorem 2, we claim that in an MJBN, MSS is equivalent to global stability with probability 1. Observe that MSS is equivalent to the existence of λ_i in Theorem 2 and global stability with probability 1 is also equivalent to the existence of λ'_i in [20]. Also, there is a one-to-one mapping of Lyapunov parameters from λ_i and λ'_i. Thus we have the following result.

Theorem 3. For MJBN (1) considered in this paper, global stability with probability 1 implies MSS.

3.2 Lyapunov function

Definition 4. For an MJBN (1), a stochastic function V defined on $\mathcal{S} \times \mathbb{N} \to \mathbb{R}$ is called a Lyapunow function for MSS, if

(a) $V(X, \theta) > 0, \forall X \in \mathcal{S} \setminus \{X_e\}, \forall \theta \in \mathbb{N}$ and $V(X_e, \theta) = 0, \forall \theta \in \mathbb{N}$, and

(b) Along with the Markov process and the expectation of the trajectories to (4), $\Delta(X(t), \theta(t)) := \mathrm{E}\{V(X(t+1), \theta(t+1))|X(t), \theta(t)\} - V(X(t), \theta(t)) < 0, \forall X(t) \neq X_e, \forall X(t) \in \mathbb{N}$, and $\Delta(X_e, \theta(t)) = 0, \forall \theta(t) \in \mathbb{N}$.

Observe that if the considered system (4) has a Lyapunov function as defined in Definition 4, then system (1) is MSS. Based on this idea, our next aim is to find a way of constructing a Lyapunov function for system (4). Consider the following function assuming $\theta(t) = i$ and

$$V(X(t), i) = (\lambda_i \ltimes \lambda_i)^{\mathrm{T}}X(t), \tag{14}$$

where $\lambda_i \in \mathbb{R}^{2^n}$ is a positive vector.

We claim that if $\lambda_i, i \in \mathbb{N}$ satisfies conditions (10a)–(10d), then we would prove $V(X(t), \theta(t))$ is a Lyapunov function. According to (10a)–(10b), we claim that condition (a) of Definition 4 is satisfied. Without loss of generality, we assume that $\theta(t) = i$; then based on the Markov jump transition matrix, we know that $\mathrm{E}(\theta(t+1) = j) = p_{ij}$. Hence, along with the trajectories of switched BN (1), one has

$$\begin{aligned}
\Delta V(X(t), i) &= \mathrm{E}\{(\lambda_{\theta(t+1)} \ltimes \lambda_{\theta(t+1)})^{\mathrm{T}}X(t+1)|X(t), \theta(t) = i\} - (\lambda_i \ltimes \lambda_i)^{\mathrm{T}}X(t)\\
&= \left(\sum_{j=1}^{N} p_{ij}(\lambda_j \ltimes \lambda_j)^{\mathrm{T}}(L_i \otimes L_i) - (\lambda_i \ltimes \lambda_i)^{\mathrm{T}}\right)X(t).
\end{aligned}$$

Condition (b) of Definition 4 is satisfied by (10a), (10b) and (10d). Therefore, Eq. (14) can be regarded as one of the Lyaponov functions for the MSS of system (1). Moreover, because Theorem 2 presents a necessary and sufficient condition, the MSS of (1) is equivalent to the existence of the Lyapunov function for (4) in the form of (14). Consequently, we have the following theorem.

Theorem 4. System (1) is MSS at $X_e = \delta_{2^{2n}}^{2^{2n}}$ if and only if it has a Lyapunov function defined in Definition 4.

The proof of Theorem 2 also presents an approach to finding the Lyapunov function for (4) in the form of (14). In Section 4, we use an example to show how to find it.

Remark 3. Observe that Markov jump Boolean network is a kind of positive system and linear co-positive Lyapunov function approach plays an important role in tackling positive systems because it captures the very nature of positive systems, namely, that their states are always nonnegative [31]. The Lyapunov function considered in this paper is so called linear co-positive Lyapunov function. Besides, other type Lyapunov functions can be used, such as $V(X(t),i) = x(t)^{\mathrm{T}}\lambda_i\lambda_i^{\mathrm{T}}x(t), x(t)\ltimes x(t) = X(t)$ for $\theta(t) = i$ and $\Delta V(X(t),i) = \mathrm{E}\{x(t+1)^{\mathrm{T}}(\lambda_{\theta(t+1)}\lambda_{\theta(t+1)}^{\mathrm{T}})x(t+1)|X(t),\theta(t)=i\} - x(t)^{\mathrm{T}}\lambda_i\lambda_i^{\mathrm{T}}x(t) = x(t)^{\mathrm{T}}(\sum_{j=1}^{N}p_{ij}L_i^{\mathrm{T}}\lambda_j^{\mathrm{T}}\lambda_jL_i - \lambda_i^{\mathrm{T}}\lambda_i)x(t)$.

4 Example

Consider a BN motivated from [32] as follows:

$$x_i(t+1) = f_i^{\theta(t)}(x_1(t),x_2(t),x_3(t)), \quad i=1,2,3, \tag{15}$$

where $x_1(t),x_2(t),x_3(t)$ represent the concentration level of the inhibitor of apoptosis proteins, active caspase 3, and cative caspase 8, respectively. And $\theta(t)\in\mathcal{N}=\{1,2\}$ follows $P=\left[\begin{smallmatrix}1/3 & 2/3\\ 1/2 & 1/2\end{smallmatrix}\right]$. The subnetworks are $f^1=(x_1,\neg x_1\wedge x_3,x_2)$ and $f^2=(\neg x_1\wedge x_2, x_1\wedge x_2, x_1\wedge x_2)$. Using STP method, the corresponding transition matrices are $L_1=\delta_8[3\ 3\ 4\ 4\ 5\ 7\ 6\ 8]$ and $L_2=\delta_8[5\ 5\ 8\ 8\ 4\ 4\ 8\ 8]$, respectively. The states' update rule of (15) switches between f^1 and f^2 stochastically according to the Markov chain $\theta(t)$. Then it can be calculated that

$$\mathbb{M} = P^{\mathrm{T}}\otimes I_7\begin{bmatrix}\overline{L}_1 & \\ & \overline{L}_2\end{bmatrix}.$$

Also,

$$\mathbb{M}-I_{14}=\begin{bmatrix}
-1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
\frac13&\frac13&-1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\frac13&-\frac23&0&0&0&0&0&0&0&\frac12&\frac12&0\\
0&0&0&0&-\frac23&0&0&\frac12&\frac12&0&0&0&0&0\\
0&0&0&0&0&-1&\frac13&0&0&0&0&0&0&0\\
0&0&0&0&0&\frac13&-1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-1&0&0&0&0&0\\
\frac23&\frac23&0&0&0&0&0&0&0&-1&0&0&0&0\\
0&0&\frac23&\frac23&0&0&0&0&0&0&-1&\frac12&\frac12&0\\
0&0&0&0&\frac23&0&0&\frac12&\frac12&0&0&-1&0&0\\
0&0&0&0&0&0&\frac23&0&0&0&0&0&-1&0\\
0&0&0&0&0&\frac23&0&0&0&0&0&0&0&-1
\end{bmatrix}.$$

There exists

$$\lambda_1 = (20.6667\ 21.6667\ 21.0000\ 22.0000\ 28.7500\ 24.0625\ 28.1875)^{\mathrm{T}},$$
$$\lambda_2 = (39.1250\ 38.1250\ 11.0000\ 11.0000\ 27.5000\ 27.5000\ 11.0000)^{\mathrm{T}},$$

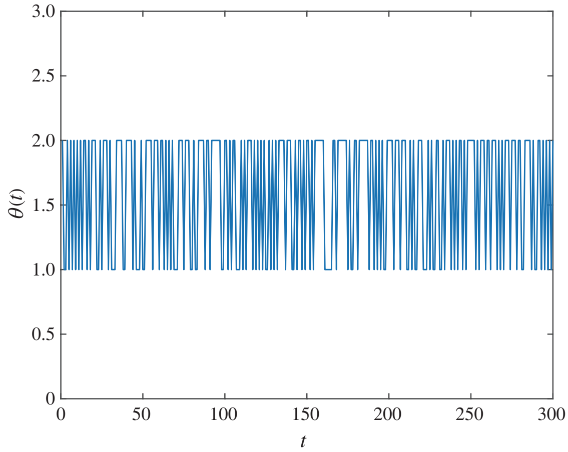

Figure 1 (Color online) A random sequence of the Markov jump switching signals based on the transition matrix.

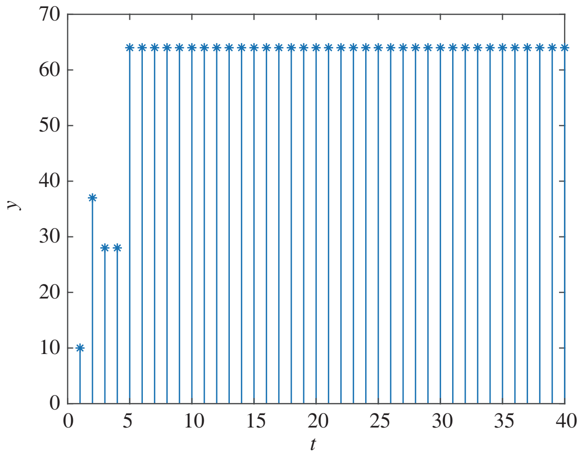

Figure 2 (Color online) $X(t) := \delta_{64}^{y}$ is the state trajectory under the Markov jump switching signals in Figure 1 with initial state δ_{64}^{10}.

such that $(\lambda_1^{\rm T}\ \lambda_2^{\rm T})^{\rm T}(\mathbb{M} - I_9) < 0$. Let $\overline{\lambda}_i = (\lambda_i\ 0)^{\rm T}, i = 1, 2$, and $\widetilde{\lambda}_i = \sqrt{\overline{\lambda}_i}$; precisely,

$$\widetilde{\lambda}_1 = (4.5461\ 4.6548\ 4.5826\ 4.6904\ 5.3619\ 4.9054\ 5.3092)^{\rm T},$$
$$\widetilde{\lambda}_2 = (6.2550\ 6.1745\ 3.3166\ 3.3166\ 5.2440\ 5.2440\ 3.3166)^{\rm T}.$$

Then $V(X(t), i) = (\widetilde{\lambda}_i \ltimes \widetilde{\lambda}_i)^{\rm T} X(t), i = 1, 2$ are the Lyapunov functions that we need. On the other hand, we show the MSS using a simulation case. Under the stochastic Markov jump with Figure 1, the trajectory of initial state δ_{64}^{10} converges to δ_{64}^{64} in finite steps as shown in Figure 2.

5 Conclusion

To the best of our knowledge, the definition of mean square stability of Markov jump Boolean networks, was first given in this paper. Then, some necessary and sufficient conditions were presented to guarantee the MSS based on the propositions of switched signals and the STP method. Furthermore, one of the most important necessary and sufficient conditions for MSS was obtained in terms of linear programming, based on which, we claimed that MSS was equivalent to global stability with probability 1 in MJBNs. It was quite different from normal Markov jump linear systems. Moreover, the construction of Lyapunov function was given and Lyapunov theorem was derived. Finally, a numerical example to illustrate the effectiveness of the results was provided.

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