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Bipartite consensus of edge dynamics on coopetition multi-agent systems

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Dear editor,
The cooperative control of multi-agent systems has attracted wide attention from scholars in various fields because of its wide applications. The controllability and consensus problem of multi-agent systems are two basic issues, which have been widely studied in recent years [1–4].

Most of the existing studies on consensus are concerned with the states of agents (i.e., nodes dynamics). But in many physical problems, we should consider not only the states of agents in the system, but also the states of links between agents (i.e., edges dynamics), such as transport network system, electric power system, friendship, trust, and communication channel. Zelazo et al. [5] considered the consensus problems of the node system after adding the edge information. By studying the relationship between edges and nodes, incidence matrices were introduced. The Laplacian matrix was defined by using the incidence matrices. In [6–8], Wang et al. investigated the dynamics of edges and established discrete-time and continuous-time edge consensus protocols for directed and undirected multi-agent systems, respectively. By mapping the original node digraph to its edge topology (i.e., line graph), the edge consensus was analyzed under the condition that the original digraph/undirected graph is strongly connected/connected.

Despite the above results, further research is needed to characterize the relationship between agents rather than agents themselves. This requires line graphs to convey the relationship between agents. It is worth pointing out that the study on edge states and edge interactions was well developed in social balance theory, which states that the relationship can be positive or negative. In this study, we discuss the bipartite consensus of the edges under a directed graph with antagonistic interactions in the first-order and second-order edge dynamic system. We consider coopetition systems rather than cooperative systems. The cooperative relationship among agents is characterized by positive weight, while competitive relationship is represented by negative weight. That is, if there are antagonistic relationships among agents, negative weights will be produced in the corresponding topologies. If we still use the consensus protocols under the nonnegative weights, the corresponding Laplacian matrix will have negative eigenvalues, which leads to the instability of the system, and the system would not asymptotically reach consensus. In this study, in order to achieve a certain consensus, new distributed protocols are designed and necessary and sufficient conditions are derived. Under the designed protocols, the first-order and second-order edge dynamics systems asymptotically reach the edge bipartite consensus, respectively.

Model and methodology. We define a signed di-

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graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$, where $\mathcal{E} = \{(i,j)|$ if i can receive information from $j\}$ and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ is the adjacency matrix of $\mathcal{G}$, where $a_{ij} \neq 0 \Leftrightarrow (i,j) \in \mathcal{E}$. $a_{ij} = 1$ and $a_{ij} = -1$ represent the cooperation and competition relationships from j to i, respectively. (More theories of graph are in Appendix A.) For a digraph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ with n nodes and m edges, its line graph $\mathcal{L}(\mathcal{G})$ is defined as follows [9]:

(1) A node (i,j) of $\mathcal{L}(\mathcal{G})$ corresponds to a directed edge (i,j) of $\mathcal{G}$;

(2) For node i of $\mathcal{G}$, its incoming edge (i,j) is adjacent to its outgoing edge (k,i) in $\mathcal{L}(\mathcal{G})$.

It can be seen that the line graph of a digraph is still a directed graph, and the number of nodes in the line graph is equivalent to $\sum_{i=1}^{n} d_{\text{in}}(i)$, which is the number of edges in the original graph. For a digraph, it is obvious that $\sum_{i=1}^{n} d_{\text{in}}(i) = \sum_{i=1}^{n} d_{\text{out}}(i)$. In the original digraph $\mathcal{G}$, each node i has $d_{\text{in}}(i)$ incoming edges and $d_{\text{out}}(i)$ outgoing edges, so the node i can derive $d_{\text{in}}(i) \cdot d_{\text{out}}(i)$ edges in the line graph $\mathcal{L}(\mathcal{G})$, and the total number of edges of $\mathcal{L}(\mathcal{G})$ is $\sum_{i=1}^{n} d_{\text{in}}(i) \cdot d_{\text{out}}(i)$.

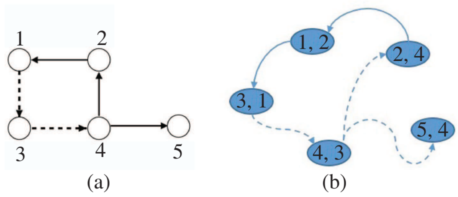

Figure 1 (Color online) (a) A digraph; (b) its line graph.

For example, Figure 1(b) is a line graph corresponding to the original graph in Figure 1(a), where the real lines represent the cooperative relationship among agents, and the dotted lines represent the competitive relationship between agents. It is noteworthy that we have the following rules in a signed digraph and its line graph:

(1) Edges in the line graph derived from negative weighted edges of the original graph take negative weights;

(2) Edges in the line graph derived from positive weighted edges of the original graph take positive weights.

If a digraph $\mathcal{G}$ contains more than one node and is strongly connected, then its line graph $\mathcal{L}(\mathcal{G})$ is also strongly connected (Lemma 2 in Appendix A.2). And for a strongly connected, digon sign-symmetric signed digraph $\mathcal{G}$, its line graph $\mathcal{L}(\mathcal{G})$ is structurally balanced if and only if $\mathcal{G}$ is structurally balanced (Lemma 3 in Appendix A.2).

Let $x_{ij}(t)$, $v_{ij}(t)$ represent the position and velocity of edge (i,j) at time t, respectively. Consider the following first-order continuous time edge dynamics model:

$$\dot{x}_{ij}(t) = u_{ij}(t), \quad \forall (i,j) \in \mathcal{E}, \tag{1}$$

and second-order continuous time edge dynamics model

$$\dot{x}_{ij}(t) = v_{ij}(t), \quad \dot{v}_{ij}(t) = u_{ij}(t), \quad \forall (i,j) \in \mathcal{E}. \tag{2}$$

We want to design distributed edge bipartite consensus protocols such that all edges can asymptotically reach certain consensus. We consider (1) to solve the bipartite consensus of edges if and only if $|x_{ij}| - |x_{ks}| \to 0$, $\forall i,k = 1,2,\ldots,n$, $j \in \mathcal{N}(i)$, $s \in \mathcal{N}(k)$ as $t \to \infty$, and (2) to solve the bipartite consensus of edges if and only if $|x_{ij}| - |x_{ks}| \to 0$ and $|v_{ij}| - |v_{ks}| \to 0$ as $t \to \infty$. Notice that the state of edge (i,j) in the original graph corresponds to the state of node (i,j) in the line graph, where the labels of x_{ij} are arranged in ascending order (first i, and then j). Therefore, we can transform the characterization of the edge dynamics in the original digraph to the characterization of the node dynamics in the corresponding line graph.

Results and discussion. For system (1), we assume that

$$u_{ij}(t) = \sum_{r\in\mathcal{N}(j)} |a_{jr}|[\text{sgn}(a_{jr})x_{jr}(t) - x_{ij}(t)], \tag{3}$$

where $i = 1,2,\ldots,n$, $j \in \mathcal{N}(i)$. Let $X = (x_{ij}) \in \mathbb{R}^{\mathcal{M}\times 1}$, $\mathcal{M} = \sum_{i=1}^{n} d_{\text{in}}(i)$. System (1) can be denoted by $\dot{X}(t) = -L'X(t)$, where L' is the Laplacian matrix of the line graph $\mathcal{L}(\mathcal{G})$. Let $\mathcal{A}' = [a'_{st}]$ be the adjacency matrix of $\mathcal{L}(\mathcal{G})$, C' be the in-degree matrix of $\mathcal{L}(\mathcal{G})$, which is a diagonal matrix and the diagonal element is $\sum_{t\in\mathcal{N}(s)} |a'_{st}|$, $s = 1,2,\ldots,n$. L' is defined as follows: $L' = [l'_{st}]_{\mathcal{M}\times\mathcal{M}} = C' - \mathcal{A}'$, where

$$l'_{st} = \begin{cases} -a'_{st}, & s \neq t, \\ \displaystyle\sum_{t\in\mathcal{N}(s)} |a'_{st}|, & s = t. \end{cases}$$

Let signed digraph $\mathcal{G}(\mathcal{A})$ be digon sign-symmetric and strongly connected. For a continuous time edge dynamics system (1), under protocol (3), all states of edges asymptotically reach the bipartite consensus if and only if $\mathcal{G}(\mathcal{A})$ is structurally balanced (Theorem 1 in Appendix B.1). In this case, $\lim_{t\to\infty} X(t) = w_l^{\mathrm{T}} DX(0)Dw_r$, where $D = \text{diag}\{d_1,\ldots,d_n\}$, $d_i = \pm 1$, $i \in \mathcal{M}$ is the gauge transformation such that $D\mathcal{A}'D$ is nonnegative, w_r and w_l are the right and left eigenvectors associated with $\mu = 0$ of the Laplacian matrix $DL'D$, respectively, and $w_l^{\mathrm{T}} w_r = 1$. If $\mathcal{G}(\mathcal{A})$ is structurally unbalanced, system (1) is asymptotically stable, that is, $\lim_{t\to\infty} X(t) = 0$.

For second-order edge dynamics system (2), the bipartite consensus protocol is as follows:

$$u_{ij}(t) = -k_1 v_{ij} + \sum_{r\in\mathcal{N}(j)} |a_{jr}|[(\mathrm{sgn}(a_{jr})x_{jr}(t) - x_{ij}(t)) + k_2(\mathrm{sgn}(a_{jr})v_{jr}(t) - v_{ij}(t))], \quad (4)$$

where $k_1, k_2 > 0$ are feedback gains. Let the corresponding signed digraph $\mathcal{G}(\mathcal{A})$ be strongly connected, digon sign-symmetric and structurally balanced. System (2) can asymptotically reach the edge bipartite consensus under protocol (4), if

$$k_1 - k_2\mathrm{Re}(\mu_i) > 0 \quad (5)$$

and

$$(k_1 - k_2\mathrm{Re}(\mu_i)) \cdot k_2 \cdot (\mathrm{Im}(\mu_i))^2 - (k_1 - k_2\mathrm{Re}(\mu_i))^2\mathrm{Re}(\mu_i) - (\mathrm{Im}(\mu_i))^2 > 0, \quad (6)$$

where μ_i $(i = 1, 2, \ldots, \mathcal{M})$ are the eigenvalues of Laplacian matrix $-L'$ of the line graph. Moreover, if $k_1 = 0$, the consensus values of edges are as follows:

$$\lim_{t\to\infty} X(t) = w_l^{\mathrm{T}} DX(0)Dw_r + tw_l^{\mathrm{T}} DV(0)Dw_r,$$
$$\lim_{t\to\infty} V(t) = w_l^{\mathrm{T}} DV(0)Dw_r,$$

where w_r and w_l are the right and left eigenvectors associated with $\mu_i = 0$ of the Laplacian matrix $-DL'D$, respectively, $w_l^{\mathrm{T}} w_r = 1$, and D is defined as above. If $k_1 \neq 0$, the consensus values of edges are as follows:

$$\lim_{t\to\infty} X(t) = w_l^{\mathrm{T}} DX(0)Dw_r + \frac{1}{k_1} w_l^{\mathrm{T}} DV(0)Dw_r,$$
$$\lim_{t\to\infty} V(t) = 0.$$

If the signed digraph $\mathcal{G}(\mathcal{A})$ is structurally unbalanced, the positions and velocities of edges tend to 0, under conditions (5) and (6) (Theorem 2 in Appendix B.2).

We give numerical simulation in Appendix C.

Conclusion. We discuss the bipartite consensus problems of the edge dynamics for multi-agent systems with cooperation and competition interactions, where signed digraphs are used to represent such systems. The strongly connected digraph is mapped to its line graph. With the help of line graph, it is found and proved that the line graph corresponding to a structurally balanced topology is still structurally balanced, and vice versa. Under the first-order and second-order edge dynamics, the necessary or sufficient conditions for the edge states to asymptotically achieve bipartite consensus are obtained by using some analysis methods of matrix theory and graph theory, and the decision values are obtained. We also study the structural unbalanced case. It is found that the states of edges can still achieve consensus under the given protocol, but this consensus is trivial, that is, the state of each edge tends to 0.

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Supporting information Appendixes A–C. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

References

1 Chen F, Ren W, Lin Z L. Multi-leader multi-follower coordination with cohesion, dispersion, and containment control via proximity graphs. Sci China Inf Sci, 2017, 60: 110204
2 Guan Y Q, Wang L. Structural controllability of multi-agent systems with absolute protocol under fixed and switching topologies. Sci China Inf Sci, 2017, 60: 092203
3 Fu Q, Gu P P, Li X D, et al. Iterative learning control approach for consensus of multi-agent systems with regular linear dynamics. Sci China Inf Sci, 2017, 60: 079202
4 Tian L, Ji Z J, Hou T, et al. Bipartite consensus on coopetition networks with time-varying delays. IEEE Access, 2018, 6: 10169–10178
5 Zelazo D, Mesbahi M. Edge agreement: graph-theoretic performance bounds and passivity analysis. IEEE Trans Autom Control, 2011, 56: 544–555
6 Wang X L, Wang X F, Su H S. Consensus of edge dynamics on directed multi-agent systems. In: Proceedings of the 33rd Chinese Control Conference, Nanjing, 2014. 1372–1376
7 Wang X L, Su H S, Wang X F, et al. Nonnegative edge quasi-consensus of networked dynamical systems. IEEE Trans Circ Syst II, 2017, 64: 304–308
8 Wang X L, Su H S, Chen M Z Q, et al. Reaching non-negative edge consensus of networked dynamical systems. IEEE Trans Cybern, 2018, 48: 2712–2722
9 Harary F, Norman R Z. Some properties of line digraphs. Rend Circ Mat Palermo, 1960, 9: 161–168