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• **Supplementary File** •

Bipartite Consensus of Edge Dynamics on Coopetition Multi-Agent Systems

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Appendix A Theory of Graph

We denote a signed digraph by $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$, where $\mathcal{V} = \{1, 2, \cdots, n\}$ is the set of nodes and $\mathcal{E} = \{(i, j)|$ if i can receive information from $j\} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges. $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ is the adjacency matrix of $\mathcal{G}$, where $a_{ij} \neq 0 \Leftrightarrow (i, j) \in \mathcal{E}$. $a_{ij} = 1$, if the connection from j to i is cooperation; $a_{ij} = -1$, if the connection from j to i is competition. We assume that $(i, i) \notin \mathcal{E}$ and hence $a_{ii} = 0,\ \forall i = 1, 2, \cdots, n$. We call a signed digraph which is digon sign-symmetric if $a_{ij}a_{ji} \geqslant 0$, we only discuss the case of digon sign-symmetric in this letter. The set of neighbors of node i is defined by $\mathcal{N}(i) = \{j|a_{ij} \neq 0\}$, the in-degree and out-degree of node i are defined as $d_{in}(i) = \sum\limits_{j=1}^{n} |a_{ij}|$ and $d_{out}(i) = \sum\limits_{j=1}^{n} |a_{ji}|$, respectively. If the graph is undirected, the adjacency matrix $\mathcal{A}$ is regardless with the order of edges in $\mathcal{E}$, and the matrix $\mathcal{A}$ is a symmetric matrix. For a digraph, we assume that $\mathcal{A}_u = \frac{\mathcal{A}^T+\mathcal{A}}{2}$, and hence $\mathcal{G}(\mathcal{A}_u)$ can be seen as an undirected graph derived from the digraph $\mathcal{G}(\mathcal{A})$.

A directed path of $\mathcal{G}(\mathcal{A})$ is a series of interrelated edges in $\mathcal{E}$:

$$\mathcal{P} = \{(i_2, i_1), (i_3, i_2), \cdots, (i_p, i_{p-1})\} \subset \mathcal{E},$$

where nodes $i_1, i_2, \cdots$, and i_p are different, and the length of the directed path $\mathcal{P}$ is $p - 1$. A cycle $\mathcal{C}$ of $\mathcal{G}(\mathcal{A})$ means that the end point of $\mathcal{P}$ coincides with the starting point (that is, $i_1 = i_p$). A cycle is positive, which means that it contains an even number of negative edges, i.e. $a_{i_2,i_1} \cdots a_{i_1,i_{p-1}} > 0$. If $a_{i_2,i_1} \cdots a_{i_1,i_{p-1}} < 0$, the cycle is negative. A semi cycle of $\mathcal{G}(\mathcal{A})$ is a cycle in its derived undirected graph $\mathcal{G}(\mathcal{A}_u)$. A digraph is strongly connected if there is at least one path from i to j and another one from j to i, $\forall i, j \in \mathcal{V},\ i \neq j$. A digraph contains a spanning tree if there is a node i called root which has paths to all the other nodes of the graph.

Appendix A.1 Structural Balance

A digraph is structurally balanced if all of its semi cycles are positive. In other words, a digraph is structurally balanced if all of its nodes can be divided into $\mathcal{V}_1, \mathcal{V}_2$, where $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}, \mathcal{V}_1 \cap \mathcal{V}_2 = \phi$, and $a_{ij} \geqslant 0, \forall i, j \in \mathcal{V}_p, (p \in \{1, 2\}),\ a_{ij} \leqslant 0, \forall i \in \mathcal{V}_p, j \in \mathcal{V}_q, (p, q \in \{1, 2\},\ p \neq q)$. Otherwise, it is called structural unbalance. It is worth noting that the existence of a semi cycle is a necessary condition for the structural balance of a graph. Let $\mathcal{D} = \{D|D = diag\{d_1, \cdots, d_n\}, d_i = \pm 1, i = 1, 2, \cdots, n\}$ be the set of gauge transformations, we have the following Lemma.

Lemma 1. [1] A strongly connected, digon sign-symmetric signed digraph $\mathcal{G}(\mathcal{A})$ is structurally balanced if and only if any of the following equivalent conditions holds:
(1) $\mathcal{G}(\mathcal{A}_u)$ is structurally balanced;
(2) all directed cycles of $\mathcal{G}(\mathcal{A})$ are positive;
(3) $\exists D \in \mathcal{D}$ such that $D\mathcal{A}D$ has all nonnegative entries;
(4) 0 is an eigenvalue of L.

Moreover, we have the following corollary:

Corollary 1. A strongly connected, digon sign-symmetric signed digraph $\mathcal{G}(\mathcal{A})$ is structurally balanced if and only if 0 is a single eigenvalue of Laplacian matrix L, that is, $rank(L) = n - 1$.
Proof. From [2] and [3], this result is obviously.

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Appendix A.2 Line Graph

For a digraph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ with n nodes and m edges, its line graph $\mathcal{L}(\mathcal{G})$ is defined as follows [4]:

(1) A node (i, j) of $\mathcal{L}(\mathcal{G})$ corresponds to a directed edge (i, j) of $\mathcal{G}$;

(2) For node i of $\mathcal{G}$, its incoming edge (i, j) is adjacent to its outgoing edge (k, i) in $\mathcal{L}(\mathcal{G})$.

It is noteworthy that we have the following rules in a signed digraph and its line graph:

(1) edges in the line graph generated from negative weighted incoming edges of the original graph take negative weights;

(2) edges in the line graph generated from positive weighted incoming edges of the original graph take positive weights.

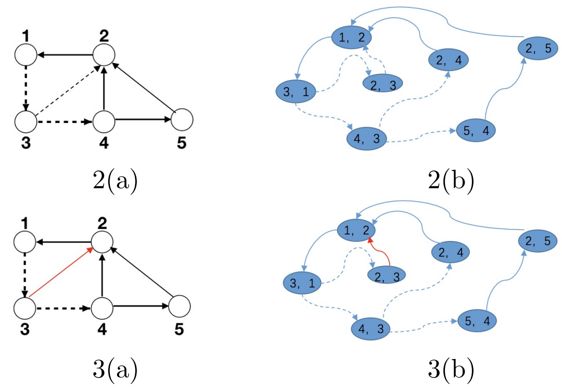

Figure A1 2(a) is a structurally balanced digraph and 2(b) is its line graph; 3(a) is a structurally unbalanced digraph and 3(b) is its line graph.

In Figure A1, 2(a) and 3(a) are all strongly connected, and their line graph 2(b) and 3(b) are also strongly connected. In addition, 2(a) is a structurally balanced digraph and its line graph 2(b) is also structurally balanced. 3(a) is to change negative weighted edge $(2, 3)$ in 2(a) to a positive weight (marked in red line), resulting in structural unbalanced. In its line graph 3(b), $(2, 3)$ is adjacent to $(1, 2)$, thus, the negative weighted edge $(2, 3) \rightarrow (1, 2)$ is turned into positive weight. Therefore, the structure of the line graph 3(b) is not balanced. Figure A1 shows that, in a coopetition networks, if graph $\mathcal{G}$ is structurally balanced, then its line graph $\mathcal{L}(\mathcal{G})$ is also structurally balanced; if graph $\mathcal{G}$ is structurally unbalanced, then its line graph $\mathcal{L}(\mathcal{G})$ is also structurally unbalanced, and vice versa.

With respect to the special properties of the original graph and its line graph, there are several lemmas in the following:

Lemma 2. [4]If a digraph $\mathcal{G}$ contains more than one node and is strongly connected, then its line graph $\mathcal{L}(\mathcal{G})$ is also strongly connected.

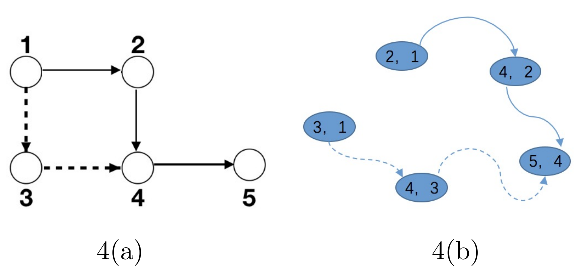

Figure A2 A line graph corresponding to a digraph with a spanning tree no longer contains a spanning tree.

Remark 1. Figure A2 shows that if the original graph contains a spanning tree, the corresponding line graph may not also contain a spanning tree. The graph condition for consensus of node dynamics system can be reduced from strong connectivity to a spanning tree, yet this result cannot be simply generalized to the edge dynamics.

Lemma 3. For a strongly connected, digon sign-symmetric signed digraph $\mathcal{G}$, its line graph $\mathcal{L}(\mathcal{G})$ is structurally balanced if and only if $\mathcal{G}$ is structurally balanced.

Proof. (Sufficiency.) From Lemma 1, a digraph $\mathcal{G}(\mathcal{A})$ is structurally balanced if and only if all directed cycles of $\mathcal{G}(\mathcal{A})$ are positive. By the definition of line graph, a directed cycle in the original digraph still corresponds to a directed cycle in the line graph, no matter whether the sign on the edge is positive or negative, as shown in Figure A3.

A directed path with length of $r (r \geqslant 2)$ in the original digraph generates a directed path with length of $r - 1$ in the line digraph, and does not generate a directed cycle, as shown in Figure A4.

If there are two or more than two directed cycles in the original graph sharing a common node, the *union* of these directed cycles will generate larger directed cycles in the line graph. For the sake of simplicity, a concrete example is employed below to illustrate the proof without loss of generality. The general case can be proved in the same way. As shown in Figure A5, there are two directed cycles in the original graph 5(a): "$1 \rightarrow 2 \rightarrow 3 \rightarrow 1$" and "$3 \rightarrow 4 \rightarrow 3$", and

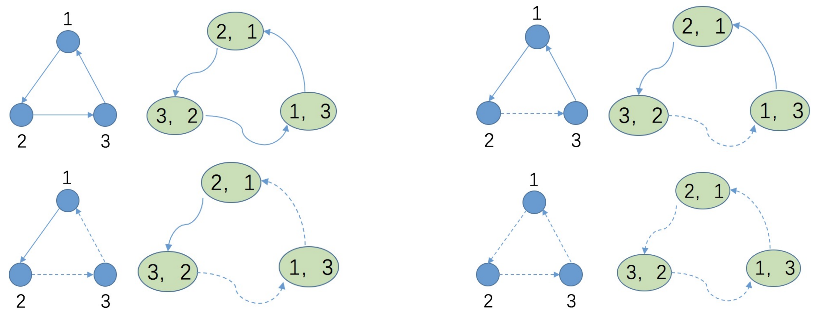

Figure A3 The directed cycle in the original digraph still corresponds to a directed cycle in the line graph.

Figure A4 A directed path of $r-1$ in the line graph is generated from the directed path of r in the original digraph.

these two directed cycles have a common node "3". 5(b) is the line graph generated from 5(a). Besides directed cycles "$(2,1) \to (3,2) \to (1,3) \to (2,1)$" generated from "$1 \to 2 \to 3 \to 1$" and "$(3,4) \to (4,3) \to (3,4)$" generated from "$3 \to 4 \to 3$", there is a larger directed cycle "$(2,1) \to (3,2) \to (4,3) \to (3,4) \to (1,3) \to (2,1)$" generated from the *union* of the two directed cycles.

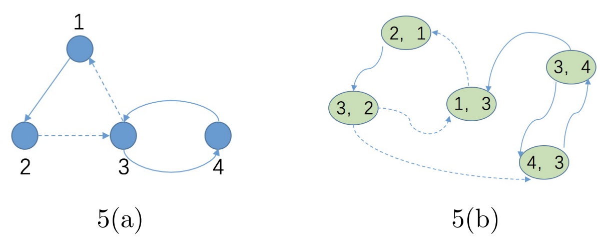

Figure A5 The union of some directed cycles with a common node will generate an additional larger directed cycle.

Therefore, the directed cycles in the line graph are all generated from the directed cycles, as well as from the *union* of directed cycles with a common node in the original graph. If the original digraph is structurally balanced, all the directed cycles in the original digraph are positive, that is, the number of the negative weighted edges in the directed cycle is even. Following the rules of the weight of edge in line graph, if a directed cycle of the corresponding line graph is generated from one directed cycle in the original graph, the number of the negative weighted edges is the same as the number of the negative weighted edges in the corresponding directed cycle of the original graph. And if the directed cycle of the corresponding line graph is generated from the *union* of directed cycles sharing a common node in the original graph, the number of the negative weighted edges is the sum of the numbers of negative weighted edges in such corresponding directed cycles of the original graph, which is even too. Therefore, if the original digraph is structurally balanced, then cycles in the line graph are all positive and the line graph is then structurally balanced.

(Necessity.) If the line graph corresponding to a signed digraph is structurally balanced, then the number of the negative weighted edges in each directed cycles of the line graph is even. It is known from the previous analysis that we only need to consider the directed cycles of line graphs corresponding to single directed cycles in the original digraph, without considering the directed cycles in the line graphs that are generated from the union of directed cycles which share a common node. For those single directed cycles in the original digraph, the number of negative weighted edges is equivalent to the number of negative weighted edges of the corresponding cycles in the line graph. It follows that if the numbers of negative weighted edges in the directed cycles of line graph are all even, then the numbers of negative weighted edges in the corresponding cycles of original digraph are even too. As a result, the original graph is also structurally balanced.

Appendix A.3 Laplacian Matrix of Line Graph

Take the digraph 2(a) in Figure A1 as an example, the number of nodes of its line graph 2(b) is equivalent to 7 and the number of edges of 2(b) is equivalent to 9. The adjacency matrix $\mathcal{A}_2$ of 2(a), the adjacency matrix $\mathcal{A}_2'$ and the Laplacian matrix L'_2 of 2(b) are as follows, respectively:

$$\mathcal{A}_2=\begin{bmatrix}0&1&0&0&0\\0&0&-1&1&1\\-1&0&0&0&0\\0&0&-1&0&0\\0&0&0&1&0\end{bmatrix},\mathcal{A}'_2=\begin{bmatrix}0&-1&1&1&0&0&0\\0&0&0&0&-1&0&0\\0&0&0&0&0&-1&0\\0&0&0&0&0&0&1\\1&0&0&0&0&0&0\\0&0&0&0&-1&0&0\\0&0&0&0&0&-1&0\end{bmatrix},L'_2=\begin{bmatrix}3&1&-1&-1&0&0&0\\0&1&0&0&1&0&0\\0&0&1&0&0&1&0\\0&0&0&1&0&0&-1\\-1&0&0&0&1&0&0\\0&0&0&0&1&1&0\\0&0&0&0&0&1&1\end{bmatrix},$$

where the label of nodes in matrix $\mathcal{A}'$ corresponds to the order of increments in the subscript.

Appendix B Our Results and Discussions of Edge Bipartite Consensus

Appendix B.1 First-Order Edge Dynamics Systems

For a nonnegative weighted digraph $\mathcal{G}$, there are some results as follows:

Lemma 4. [2] (Spectral Localization): Let $\mathcal{G}=\{\mathcal{V},\mathcal{E},\mathcal{A}\}$ be a weighted digraph with Laplacian L. Denote the maximum node out-degree of the digraph $\mathcal{G}$ by $d_{max}(\mathcal{G})=\max_i d_{out}(i)$. Then, all the eigenvalues of L are located in the following disk:

$$D(\mathcal{G})=\{z\in\mathbb{C}:|z-d_{max}(\mathcal{G})|\leqslant d_{max}(\mathcal{G})\},$$

which is centered at $z=d_{max}(\mathcal{G})+0j$ in the complex plane. In other words, except for the single eigenvalue 0, all of the other eigenvalues of L have positive real part.

Lemma 5. [2] Assume G is a strongly connected digraph with Laplacian matrix L satisfying $Lw_r=0$, $w_l^TL=0$, and $w_l^Tw_r=1$, then $\lim_{t\to\infty}e^{-Lt}=w_rw_l^T$.

Let $x_{ij}(t)$ represent the state of edge (i,j) at time t, the first-order continuous-time edge dynamics model is

$$\dot{x}_{ij}(t)=u_{ij}(t),\ \forall(i,j)\in\mathcal{E},\tag{B1}$$

For system (B1), we assume that

$$u_{ij}(t)=\sum_{r\in\mathcal{N}(j)}|a_{jr}|[sgn(a_{jr})x_{jr}(t)-x_{ij}(t)],\ \forall(i,j)\in\mathcal{E}.\tag{B2}$$

Let $X=(x_{ij})\in\mathbb{R}^{\mathcal{M}\times 1},\ i=1,2,\cdots n,\ j\in\mathcal{N}(i),\ \mathcal{M}=\sum_{i=1}^{n}d_{in}(i)$, the edge bipartite consensus model (B1) can be simply denoted as $\dot{X}(t)=-L'X(t)$, where L' is the Laplacian matrix of the line graph $\mathcal{L}(\mathcal{G})$. For a signed digraph $\mathcal{G}$, we have the following result on the dynamics of edge.

Theorem 1. Let signed digraph $\mathcal{G}(\mathcal{A})$ be digon sign-symmetric and strongly connected, for a continuous-time edge dynamics system (B1), under protocol (B2), all states of edges asymptotically reach the bipartite consensus if and only if $\mathcal{G}(\mathcal{A})$ is structurally balanced.

In this case, $\lim_{t\to\infty}X(t)=w_l^TDX(0)Dw_r$, where D is the gauge transformation such that $D\mathcal{A}'D$ is nonnegative, $\mathcal{A}'$ is the adjacency matrix of the corresponding line graph $\mathcal{L}(\mathcal{G})$, w_r and w_l are the right and left eigenvectors associated with the single eigenvalue $\mu=0$ of the matrix $DL'D$, respectively, and $w_l^Tw_r=1$, L' is the Laplacian matrix of the line graph $\mathcal{L}(\mathcal{G})(\mathcal{A}')$. If $\mathcal{G}(\mathcal{A})$ is structurally unbalanced, system (B1) is asymptotically stable, that is, $\lim_{t\to\infty}X(t)=0$.

Proof. From Lemmas 2 and 3, the signed digraph $\mathcal{G}(\mathcal{A})$ is strongly connected, which is equivalent to that the corresponding line graph $\mathcal{L}(\mathcal{G})(\mathcal{A}')$ is strongly connected. And $\mathcal{G}(\mathcal{A})$ is structurally balanced, which is equivalent to that $\mathcal{L}(\mathcal{G})(\mathcal{A}')$ is structurally balanced. From Lemma 1, that the line graph $\mathcal{L}(\mathcal{G})(\mathcal{A}')$ is structurally balanced is equivalent to that there exists a diagonal matrix $D\in\mathcal{D}$, such that all elements of $D\mathcal{A}'D$ are nonnegative, where $\mathcal{A}'$ is the adjacency matrix of the line graph $\mathcal{L}(\mathcal{G})$. By definition, we know that $D^{-1}=D$.

Let $Y=DX$, then $\dot{Y}=D\dot{X}=-DL'X=-DL'DY=-\tilde{L}'Y$, where $\tilde{L}'=DL'D$. Therefore, the Laplacian matrix L' of line graph $\mathcal{L}(\mathcal{G})$ is similar to matrix $\tilde{L}'$, and they are with the same eigenvalues. By the property of structural balance, from Lemma 3 and Corollary 1, $rank(L')=\mathcal{M}-1,\ \mathcal{M}=\sum_{i=1}^{n}d_{in}(i)$, and then $rank(\tilde{L}')=\mathcal{M}-1$. Moreover, for the new system $\dot{Y}=-\tilde{L}'Y$, all of the elements of the adjacency matrix $D\mathcal{A}'D$ of the corresponding digraph are nonnegative. And from Lemma 4, the other eigenvalues of $\tilde{L}'$ except the single eigenvalue 0 are all located in the right half complex plane. From the stability theorem, there exists a $c\in\mathbb{R}$ such that $\lim_{t\to\infty}Y_m(t)=c,\ m=1,2,\cdots,\mathcal{M}$, that is, $\lim_{t\to\infty}|x_{ij}(t)|=|c|$, or $\lim_{t\to\infty}(|x_{ij}(t)|-|x_{ks}(t)|)=0$. Hence, all states of edges asymptotically reach the bipartite consensus. Moreover, by Lemma 5, we have

$$\begin{aligned}\lim_{t\to\infty}X(t)&=\lim_{t\to\infty}e^{-L't}X(0)=\lim_{t\to\infty}e^{-D\tilde{L}'Dt}X(0)\\&=\lim_{t\to\infty}De^{-\tilde{L}'t}DX(0)=Dw_rw_l^TDX(0)=w_l^TDX(0)Dw_r,\end{aligned}$$

where w_l and w_r are the left and right eigenvector of $\tilde{L}' = DL'D$, respectively, and $w_l^T w_r = 1$.

If $\mathcal{G}(\mathcal{A})$ is structurally unbalanced, then so is the corresponding line graph $\mathcal{L}(\mathcal{G})(\mathcal{A}')$. From [1], we know that the real parts of eigenvalues of L' are all greater than 0. Similarly, the real parts of eigenvalues of $\tilde{L}'$ are all greater than 0, so $\lim_{t\to\infty} Y_m(t) = 0,\ m = 1, 2, \cdots, \mathcal{M}$, namely, $\lim_{t\to\infty} x_{ij}(t) = 0$.

Remark 2. Under the condition that the line graph is structurally balanced, the edge dynamics of the system can asymptotically achieve bipartite consensus. The line graph is structurally balanced, which means that all edges of the original graph can be divided into "$\mathcal{E}_1$" and "$\mathcal{E}_2$", $\mathcal{E}_1 \cup \mathcal{E}_2 = \mathcal{E},\ \mathcal{E}_1 \cap \mathcal{E}_2 = \phi$. "$\lim_{t\to\infty} |x_{ij}(t)| = |c|$" means that $\lim_{t\to\infty} x_{ij}(t) = -c,\ (i,j) \in \mathcal{E}_2$, if $\lim_{t\to\infty} x_{st}(t) = c,\ (s,t) \in \mathcal{E}_1$.

Appendix B.2 Second-Order Edge Dynamics Systems

Below is a second-order continuous-time edge dynamics model

$$\dot{x}_{ij} = v_{ij},\ \dot{v}_{ij} = u_{ij}, \tag{B3}$$

where $x_{ij},\ v_{ij}(k)$ denote the position and velocity of edge (i,j) at time t,respectively. Let

$$u_{ij}(t) = -k_1 v_{ij} + \sum_{r\in\mathcal{N}(j)} |a_{jr}|[(sgn(a_{jr})x_{jr}(t) - x_{ij}(t)) + k_2(sgn(a_{jr})v_{jr}(t) - v_{ij}(t))],\ \forall (i,j) \in \mathcal{E}, \tag{B4}$$

where $k_1, k_2 > 0$ are feedback gains. If $k_1 = 0$, protocol (B4) contains velocity information of neighbors. Information in the protocol are all relative information, and there is no velocity damping term. Therefore, when the system tends to asymptotic consensus, the velocities of edges are not necessarily 0, and the consensus is dynamic. If $k_1 \neq 0,\ k_2 = 0$, protocol (B4) does not contain the relative velocity information of neighbors, which is replaced by its own absolute velocity information term $-k_1 v_{ij}$. This term can be considered as a damping term. Under protocol (B4), the absolute values of positions of all edges asymptotically convergent to a common value over time, and the velocities of all edges tend to 0, that is, static consensus is asymptotically achieved. It is worth noting that, under certain conditions, protocol (B4) can guarantee that all the positions and velocities of all edges tend to be asymptotical consensus, and that the final states of all edges can be static or dynamic.

Let $X = (x_{ij}), V = (v_{ij}) \in \mathbb{R}^{\mathcal{M}\times 1},\ i = 1, 2, \cdots, n,\ j \in \mathcal{N}(i),\ \mathcal{M} = \sum_{i=1}^{n} d_{in}(i)$. Denote $W = [X^T\ \ V^T]^T$, system (B3) can be simplified as: $\dot{W}(t) = \Gamma W(t)$, where

$$\Gamma_{2\mathcal{M}\times 2\mathcal{M}} = \begin{bmatrix} \mathbf{0} & I_{\mathcal{M}} \\ -L' & -k_1 I_{\mathcal{M}} - k_2 L' \end{bmatrix},$$

matrix L' is the Laplacian of line graph $\mathcal{L}(\mathcal{G})(\mathcal{A}')$ defined as above. Furthermore, let $\tilde{X} = DX,\ \tilde{V} = DV,\ D \in \mathcal{D}$, and $\tilde{W} = [\tilde{X}^T\ \ \tilde{V}^T]^T$, we have

$$\dot{\tilde{W}}(t) = \tilde{\Gamma}\tilde{W}(t), \tag{B5}$$

where

$$\tilde{\Gamma}_{2\mathcal{M}\times 2\mathcal{M}} = \begin{bmatrix} \mathbf{0} & I_{\mathcal{M}} \\ -\tilde{L}' & -k_1 I_{\mathcal{M}} - k_2 \tilde{L}' \end{bmatrix},$$

with $\tilde{L}' = DL'D$. Notice that

$$\begin{aligned} det[\lambda I_{2\mathcal{M}} - \tilde{\Gamma}] &= det\left(\begin{bmatrix} \lambda I_{\mathcal{M}} & -I_{\mathcal{M}} \\ \tilde{L}' & (\lambda + k_1) I_{\mathcal{M}} + k_2 \tilde{L}' \end{bmatrix}\right) \\ &= det[(\lambda^2 + \lambda k_1) I_{\mathcal{M}} + (\lambda k_2 + 1)\tilde{L}'] \\ &= \prod_{i=1}^{\mathcal{M}} [\lambda^2 + (k_1 - k_2\mu_i)\lambda - \mu_i] \end{aligned}$$

where $\mu_i, i = 1, 2, \cdots, \mathcal{M}$ are the eigenvalues of $-L'$, so that the eigenvalues of matrix $\tilde{\Gamma}$ are given directly from the following:

$$\lambda_{i\pm} = \frac{(k_2\mu_i - k_1) \pm \sqrt{(k_2\mu_i - k_1)^2 + 4\mu_i}}{2}. \tag{B6}$$

Lemma 6. [5] Given a complex coefficient polynomial of order two, in the form of

$$p_2(s) = s^2 + (\xi_1 + i\eta_1)s + \xi_0 + i\eta_0,$$

then, $p_2(s)$ is stable if and only if

$$\xi_1 > 0,$$

and

$$\xi_1\eta_1\eta_0 + \xi_1^2\xi_0 - \eta_0^2 > 0,$$

where i is an imaginary number with $i^2 = -1$.

Thus, for the second-order edge dynamics system (B3), we have the following result.

Theorem 2. For a continuous-time edge dynamics system (B3), let the corresponding signed digraph $\mathcal{G}(\mathcal{A})$ be strongly connected, digon sign-symmetric and structurally balanced. Then system (B3) can asymptotically reach the edge bipartite consensus under protocol (B4), if

$$k_1 - k_2 Re(\mu_i) > 0 \tag{B7}$$

and

$$(k_1 - k_2 Re(\mu_i)) \cdot k_2 \cdot (Im(\mu_i))^2 - (k_1 - k_2 Re(\mu_i))^2 Re(\mu_i) - (Im(\mu_i))^2 > 0, \tag{B8}$$

where $\mu_i, i = 1, 2, \cdots, \mathcal{M}$ are the eigenvalues of $-L'$ of the line graph. Moreover, if $k_1 = 0$, the consensus values of edges are, respectively, as follows

$$\lim_{t\to\infty} X(t) = w_l^T DX(0)Dw_r + tw_l^T DV(0)Dw_r,$$
$$\lim_{t\to\infty} V(t) = w_l^T DV(0)Dw_r,$$

where w_r and w_l are the right and left eigenvectors associated with the single eigenvalue 0 of the Laplacian matrix $DL'D$, respectively, and $w_l^T w_r = 1$, $D \in \mathcal{D}$ is defined as above. If $k_1 \neq 0$, the consensus values of edges are as follows, respectively

$$\lim_{t\to\infty} X(t) = w_l^T DX(0)Dw_r + \frac{1}{k_1} w_l^T DV(0)Dw_r,$$
$$\lim_{t\to\infty} V(t) = 0.$$

If the signed digraph $\mathcal{G}(\mathcal{A})$ is structurally unbalanced, the positions and velocities of edges tend to be 0, under conditions (B7) and (B8).

Proof. (1) If $k_1 = 0$, from Lemma 6 and the conditions of Theorem 2, matrix $\tilde{\Gamma}$ has exactly two zero eigenvalues and all the other eigenvalues have negative real parts. By the results presented in [6], we have that system (B5) can asymptotically reach the edge consensus, and $\lim_{t\to\infty} \tilde{X}(t) = w_r w_l^T \tilde{X}(0) + tw_r w_l^T \tilde{V}(0)$, $\lim_{t\to\infty} \tilde{V}(t) = w_r w_l^T \tilde{V}(0)$, where w_r and w_l are the right and left eigenvectors associated with $\mu_i = 0$ of the Laplacian matrix $\tilde{L}' = DL'D$, respectively, and $w_l^T w_r = 1$. Thus, from the definition of D, system (B3) can asymptotically reach the edge bipartite consensus under protocol (B4), and

$$\lim_{t\to\infty} X(t) = D(w_r w_l^T DX(0) + tw_r w_l^T DV(0)) = w_l^T DX(0)Dw_r + tw_l^T DV(0)Dw_r,$$
$$\lim_{t\to\infty} V(t) = D(w_r w_l^T DV(0)) = w_l^T DV(0)Dw_r.$$

In this case, the conditions (B7) and (B8) can be further simplified to

$$k_2 > \max_{\mu_i} \sqrt{\frac{(Im(\mu_i))^2}{|\mu_i|^2(-Re(\mu_i))}}. \tag{B9}$$

(2) If $k_1 \neq 0$, similar to the analysis in (1), matrix $\tilde{\Gamma}$ has exactly one zero eigenvalue and all the other eigenvalues have negative real parts, under conditions (B7) and (B8). And then, system (B5) can asymptotically reach the edge consensus, and $\lim_{t\to\infty} \tilde{X}(t) = w_r w_l^T \tilde{X}(0) + \frac{1}{k_1} w_r w_l^T \tilde{V}(0)$, $\lim_{t\to\infty} \tilde{V}(t) = 0$, where w_l, w_r are defined as above. Thus, system (B3) can asymptotically reach the edge bipartite consensus under protocol (B4), and

$$\lim_{t\to\infty} X(t) = D(w_r w_l^T DX(0) + \frac{1}{k_1} w_r w_l^T DV(0)) = w_l^T DX(0)Dw_r + \frac{1}{k_1} w_l^T DV(0)Dw_r,$$
$$\lim_{t\to\infty} V(t) = 0.$$

(3) If the signed digraph $\mathcal{G}(\mathcal{A})$ is structurally unbalanced, then all of the eigenvalues of Laplacian matrix $-L'$ have negative real parts. Therefore, all of the eigenvalues of matrix $\tilde{\Gamma}$ have negative real parts under conditions (B7) and (B8). Thus system (B5) is asymptotically stable, that is $\lim_{t\to\infty} \tilde{X}(t) = 0$, $\lim_{t\to\infty} \tilde{V}(t) = 0$. Further, we have $\lim_{t\to\infty} X(t) = 0$, $\lim_{t\to\infty} V(t) = 0$.

Remark 3. In the first-order edge dynamics, the consensus problem depends only on the topology of the system. Therefore, under the given protocol, the main difficulties are to find the condition which makes the topology of the edge dynamics system structurally balanced. In the second-order edge dynamics, the consensus problem depends not only on the topological structure of the system, but also on the parameters in the protocol. We should not only find the conditions that the topological structure should satisfy, but also find the appropriate range of parameters, such that the eigenvalues corresponding to the Laplace matrix of the edge dynamics all have positive real parts except 0.

Appendix C Numerical Simulation

The digraph is 2(a) of Figure A1 with $n = 5$. The solid line between nodes indicates that there is a cooperative relationship between agents, and the corresponding weight is taken as 1. The dotted line between nodes indicates that there is a competitive relationship between agents, and the corresponding weight is taken as -1. Let $x_{12}(0) = -3, x_{23}(0) = 2, x_{24}(0) =$

$1.5, x_{25}(0) = -2, x_{31}(0) = -4, x_{43}(0) = 2.5, x_{54}(0) = 3.7$. Let $D = diag(1, -1, 1, 1, 1, -1, 1)$,

$$\tilde{L}' = DL'D = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix},$$

0 is a single eigemvalue of $\tilde{L}'$, the corresponding left eigenvector is

$$w_l = [-0.2357, -0.2357, -0.2357, -0.2357, -0.7071, -0.4714, -0.2357]^T,$$

and the corresponding right eigenvector is

$$w_r = [-0.4243, -0.4243, -0.4243, -0.4243, -0.4243, -0.4243, -0.4243]^T,$$

satisfying $w_l^T w_r = 1$. For the first-order edge dynamics system (B1), by calculation, we have

$$\lim_{t\to\infty} X(t) = [-1.0800, 1.0800, -1.0800, -1.0800, -1.0800, 1.0800, -1.0800]^T, \ (i,j) \in \mathcal{E}.$$

Figure C1 shows that the absolute value of states of all edges in the network do converge to the common value 1.0800. Figure C2 shows that if the digraph is structurally unbalanced (3(a) of Figure A1), the states of all edges converge to 0.

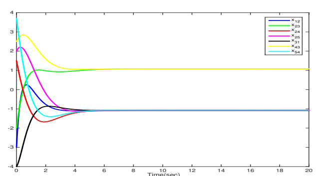

Figure C1 State trajectories of all edges under protocol (B2) with 2(a).

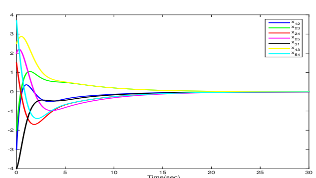

Figure C2 State trajectories of all edges under protocol (B2) with 3(a).

For second-order dynamics system (B3), Let $v_{12}(0) = 2.6, v_{23}(0) = -1, v_{24}(0) = -3, v_{25}(0) = 0.4, v_{31}(0) = 1.8, v_{43}(0) = 4.2, v_{54}(0) = -1.6$. if $k_1 = 0$, then $k_2 > 0.522$ by calculation. Thus, $\lim_{t\to\infty} |x_{ij}(t)| = 1.08 + 0.36t$, $\lim_{t\to\infty} |v_{ij}(t)| = 0.36$. Figure C3 shows that the position and velocity trajectories of all edges asymptotically reach the edge bipartite consensus under protocol (B4) with 2(a), $k_1 = 0$, $k_2 = 2$. Figure C4 and Figure C5 show that the case of $k_1 = 0$, $k_2 = 5$ and $k_1 = 0$, $k_2 = 0.6$, respectively.

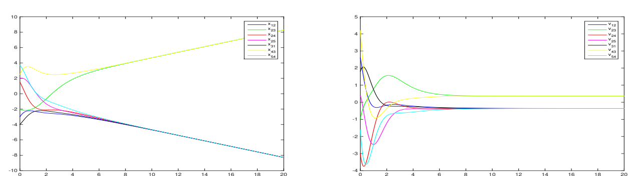

Figure C3 Position and velocity trajectories of all edges under protocol (B4) with 2(a), $k_1 = 0$, $k_2 = 2$.

Figures C3, C4 and C5 show that if $k_1 = 0$ and condition (B9) is satisfied, then the selection of k_2 only affects the speed of convergence and does not affect the final consensus value. k_2 is either too large or too small, which all lead to slower convergence rate.

Figure C6 shows that k_2 cannot be less than the right side of the condition (B9), otherwise the system does not converge.

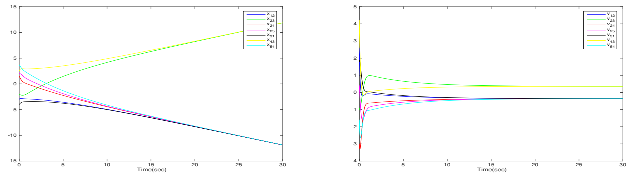

Figure C4 Position and velocity trajectories of all edges under protocol (B4) with 2(a), $k_1 = 0,\ k_2 = 5$.

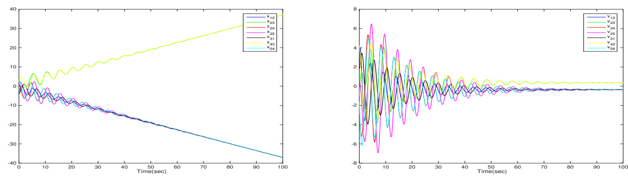

Figure C5 Position and velocity trajectories of all edges under protocol (B4) with 2(a), $k_1 = 0,\ k_2 = 0.6$.

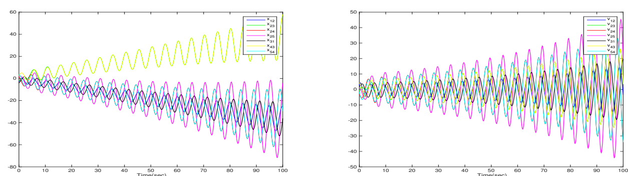

Figure C6 Position and velocity trajectories of all edges under protocol (B4) with 2(a), $k_1 = 0,\ k_2 = 0.5$.

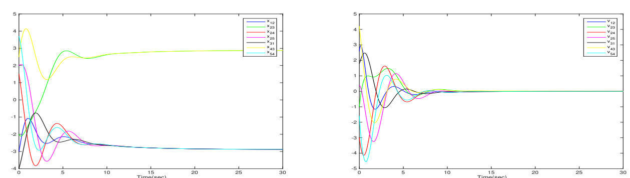

Figure C7 Position and velocity trajectories of all edges under protocol (B4) with 2(a), $k_1 = 0.2,\ k_2 = 1$.

If $k_1 = 0.2$, then $k_2 > 0.371$, and $\lim_{t\to\infty} |x_{ij}(t)| = 2.88,\ \lim_{t\to\infty} |v_{ij}(t)| = 0$. Figure C7 shows that the position and velocity trajectories of all edges asymptotically reach the edge bipartite consensus under protocol (B4) with 2(a),$k_1 = 0.2,\ k_2 = 1$.

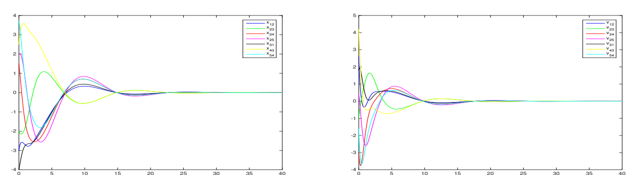

Figure C8 Position and velocity trajectories of all edges under protocol (B4) with 3(a), $k_1 = 0,\ k_2 = 2$.

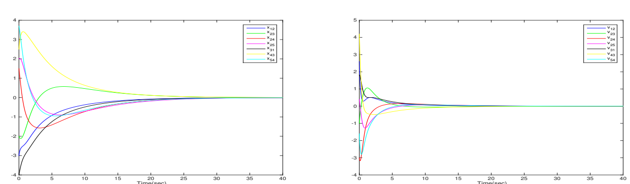

Figure C9 Position and velocity trajectories of all edges under protocol (B4) with 3(a), $k_1 = 1,\ k_2 = 2$.

For the case of structural unbalance, Figure C8 shows that the position and velocity trajectories of all edges asymptotically reach the edge consensus under protocol (B4) with 3(a), $k_1 = 0,\ k_2 = 2$, and Figure C9 shows that the case of $k_1 = 1,\ k_2 = 2$, respectively.

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