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Stability analysis of switched positive nonlinear systems: an invariant ray approach

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Abstract This paper addresses the stability problem associated with a class of switched positive nonlinear systems in which each vector field is homogeneous, cooperative, and irreducible. Instead of using the Lyapunov function approach, we fully establish the invariant ray analysis method to establish several stability conditions that depend on the states, rays, and/or times. We illustrate the efficiency of our proposed approach using the example of a chemical reaction.

Keywords positive systems, switched nonlinear systems, stability

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1 Introduction

Positive systems appear in many application areas, such as ecology, biology, economics, chemical engineering, and network communications, where the states of interest take only nonnegative values. Studies on positive linear systems have reported fruitful results [1].

However, only a few results are devoted to positive nonlinear systems. In [2], researchers found a relationship between the sign and stability of a class of quasi-monotone positive nonlinear systems by using the comparison principle. In [3], a sufficient condition was established to guarantee the non-vanishing basin of attraction stability with respect to the positive orthant, which is meaningful when positive systems undergo bifurcations. The classical Lyapunov method was extended to nonnegative nonlinear systems in [4] wherein a Lyapunov-based unified framework was developed to analyze the stability and dissipativity. In [5], the researchers analyzed a class of monotone systems defined on the positive orthant by using the max-separable/sum-separable Lyapunov functions and the comparison principle.

The classical Perron-Frobenius theorem plays a key role in the analysis and design of positive linear systems [1]; this theorem was subsequently extended to nonlinear systems [6, 7]. In [8, 9], a class of nonlinear systems that satisfied homogeneous, cooperative, and irreducible properties were considered by using an extension of the Perron-Frobenius theorem; the asymptotical behavior of the system was also analyzed by using an "invariant ray", which resulted from homogeneity. These results were further enhanced in [10].

However, switched systems arise naturally in many engineering applications because of the existence of various jumping parameters [11]. Stability and stabilization problems of switched nonlinear systems have been investigated for many years [12–20]. Switched positive systems have also attracted attention lately;

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they have important applications, such as investigating epidemic spread over time-varying networks [21] and the mitigation of HIV mutations [22, 23]. Certain studies have made in-depth investigations of stability analysis and control design of switched positive linear systems [21–32], in which the co-positive Lyapunov function method is particularly emphasized [24, 26, 27, 30]. Indeed the nonnegativity of the states facilitates the construction of such types of Lyapunov functions. An interesting result reported in [18] shows that the stability problem of a switched nonlinear system can be transformed into the stability problem of a switched positive linear system by using a trajectory-based comparison approach.

To the best of our knowledge, Refs. [33, 34] are the only studies that have reported on the stability problem of switched nonlinear systems. In [33], the researchers considered impulsive nonnegative systems whose stability condition was derived in terms of the Lyapunov function method. However, it is well known that the Lyapunov function is not easy to construct for nonlinear systems. In [34], the researchers extended the results in [10] and considered a class of switched positive nonlinear systems (SPNSs) with homogeneous and cooperative vector fields under average dwell-time switching.

In this paper, we consider a class of SPNSs with each mode satisfying homogeneous, cooperative, and irreducible properties (these properties will be formally defined in Section 2). Inspired by [9], we analyze stability by adopting an approach that is different from the Lyapunov method. Our main idea is to map the true state onto the invariant ray of each mode and generate the virtual state on the ray; then, we analyze the dynamic behaviors of the virtual state. The behavior of the true state can thus be obtained by using the monotonicity of the system solutions. Several stability conditions have been established that depend on the states, rays, and/or times. The invariant ray of each mode can be constructed more easily than the Lyapunov function as the key tool for stability analysis by using the fixed point iteration methods. The obtained results provide a new clue for the stability analysis of SPNSs.

Section 2 provides some preliminaries and the problem formulation. Sections 3 analyzes the stability. An illustrative example is given in Section 4; this is followed by the conclusion in Section 5.

2 Preliminaries

Notations. Let $\mathbb{R}$ (respectively, $\mathbb{R}^+$, $\mathbb{R}_0^+$) be the set of (respectively, nonnegative, positive) real numbers and $\mathbb{R}^n$ (respectively, $\mathbb{R}_+^n$, $\mathrm{int}(\mathbb{R}_+^n)$) the set of n-tuples with all components belonging to $\mathbb{R}$ (respectively, $\mathbb{R}^+$, $\mathbb{R}_0^+$); $\mathrm{bd}(\mathbb{R}_+^n) \triangleq \mathbb{R}_+^n \backslash \mathrm{int}(\mathbb{R}_+^n)$. For a vector $a \in \mathbb{R}^n$, a_i, $i \in \mathcal{N} \triangleq \{1, 2, \dots, n\}$ denotes its ith element. For any $x, y \in \mathbb{R}_+^n$, $x \leqslant y$ means $x_i \leqslant y_i$, $\forall i \in \mathcal{N}$; $x < y$ means $x \leqslant y$ and $x \neq y$; $x \ll y$ means $x_i < y_i$, $\forall i \in \mathcal{N}$.

Next, we introduce some concepts that form the basis of the paper.

Definition 1 ([35])**.** Let $\varepsilon > 0$. For any set of positive scalars $r_i > 0, i = 1, 2, \dots, n$, we define the dilation operator δ_ε^r as $\delta_\varepsilon^r(x_1, x_2, \dots, x_n) \triangleq (\varepsilon^{r_1} x_1 \ \varepsilon^{r_2} x_2 \ \cdots \ \varepsilon^{r_n} x_n)^{\mathrm{T}}$. The scalars r_i are called the weights of the dilation. A vector field $g : \mathbb{R}^n \to \mathbb{R}^n$ is said to be homogeneous of degree p w.r.t. δ_ε^r if $g_i(\delta_\varepsilon^r(x_1, x_2, \dots, x_n)) = \varepsilon^{r_i + p} g_i(x_1, x_2, \dots, x_n)$, $\forall i \in \mathcal{N}$.

Definition 2 ([36])**.** A matrix is said to be Metzler if and only if its off-diagonal entries are nonnegative. A vector field $g(x), x \in \mathbb{R}^n$, is cooperative in $W \subset \mathbb{R}^n$ if the Jacobian matrix $\frac{\partial g}{\partial x}$ is Metzler, $\forall x \in W$.

Definition 3 ([1])**.** A matrix A is reducible if and only if there exists a permutation matrix P such that

$$PAP^{\mathrm{T}} = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix}, \tag{1}$$

where B and D are square matrices. When A is not reducible, it is said to be irreducible.

Consider the switched nonlinear system

$$\dot{x}(t) = f_{\sigma(t)}(x(t)), \tag{2}$$

where $x \in \mathbb{R}^n$ is the state that is continuous everywhere. We define $\mathcal{M} \triangleq \{1, 2, \dots, m\}$, where m is the number of modes. The switching function is denoted by $\sigma(t) : [0, \infty) \to \mathcal{M}$; it is a piecewise constant

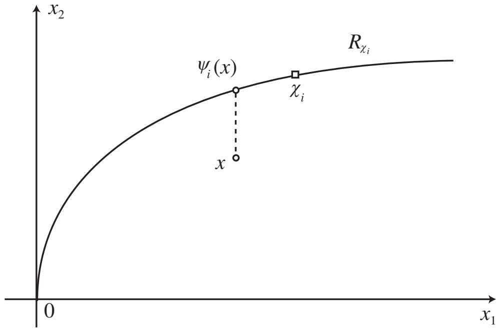

Figure 1 Invariant ray and the mapping ψ_i.

function continuous from the right. In addition, $\forall i \in \mathcal{M}$, f_i is continuous, and $f_i(0) = 0$. Moreover, f_i satisfies the following assumption.

Assumption 1. Let us assume that $\forall i \in \mathcal{M}$, f_i satisfies the following three conditions:

(1) It is homogeneous of degree $p \in \mathbb{R}^+$ with respect to δ_ε^r;

(2) It is cooperative in $\mathbb{R}_+^n \backslash \{0\}$;

(3) $\frac{\partial f_i}{\partial x}$ is irreducible for $x \in \mathrm{int}(\mathbb{R}_+^n)$. For $x \in \mathrm{bd}(\mathbb{R}_+^n) \backslash \{0\}$, either $\frac{\partial f_i}{\partial x}$ is irreducible or $f_{ij} > 0$ for $j \in \mathcal{N}$ such that $x_j = 0$. Here, f_{ij} denotes the jth element of f_i.

Suppose that there exist $\chi_i \in \mathbb{R}_+^n \backslash \{0\}$, $\forall i \in \mathcal{M}$, which satisfies

$$f_i(\chi_i) = \gamma_{\chi_i} \mathrm{diag}(r) \chi_i, \tag{3}$$

where $\gamma_{\chi_i} \in \mathbb{R}$, and $\mathrm{diag}(r)$ is a diagonal matrix where the ith diagonal entry is r_i. We define $R_{\chi_i} \triangleq \{\delta_\varepsilon^r(\chi_i) \,|\, \varepsilon \in \mathbb{R}_0^+\}$, $\forall i \in \mathcal{M}$ as a ray through χ_i. Such a ray depends on the value of r_i and may be a curved line or a straight line, as shown in Figure 1, where a system has two states and $r_1 > r_2$. R_{χ_i} is said to be invariant if $f_i(x)$ is tangent to R_{χ_i} at each point of R_{χ_i}. This means that the forward solution of the mode i starting from the arbitrary point R_{χ_i} stays on the ray for all future times [9]. It can be seen that the invariant ray generalizes the concept of an eigenvector of linear systems for application in nonlinear homogeneous systems.

Lemma 1 ([9])**.** Consider the mode i, $i \in \mathcal{M}$, of the switched system (2) $\dot{x} = f_i(x)$ with f_i satisfying Assumption 1 and the initial states belonging to $\mathbb{R}_+^n$. Then, the following hold:

- $\mathbb{R}_+^n$ is a forward invariant set; the flow of the system is strongly monotone in $\mathbb{R}_+^n$. Therefore, for any two flows $\bar{x}(t)$ and $\hat{x}(t)$ with $\bar{x}(0), \hat{x}(0) \in \mathbb{R}_+^n$, if $\bar{x}(0) < \hat{x}(0)$, then $\bar{x}(t) \ll \hat{x}(t)$, $\forall t > 0$.
- There exists at least one invariant ray $R_{\chi_i} \subset \mathrm{int}(\mathbb{R}_+^n)$ such that χ_i satisfies (3). Moreover, R_{χ_i} is unique in $\mathrm{int}(\mathbb{R}_+^n)$ if ($p = 0$) or ($p > 0$, $\gamma_{\chi_i} \leqslant 0$). The origin is asymptotically stable (respectively, stable) if and only if $\gamma_{\chi_i} < 0$ (respectively, $\gamma_{\chi_i} \leqslant 0$).

From Lemma 1, it follows that the existence and uniqueness of the solution as well as the invariant rays of the switched system (2) can be guaranteed under Assumption 1. The R_{χ_i} can be constructed by using various techniques; a typical way is the fixed point iteration method [37]. Because R_{χ_i} is invariant, we have $f_i(\delta_\varepsilon^r(\chi_i)) = \gamma_{\chi_i} \varepsilon^p \mathrm{diag}(r) \delta_\varepsilon^r(\chi_i)$. We can always pick a region $\mathcal{D} \in \mathbb{R}^n$ small enough such that $x \in \mathcal{D} \Rightarrow \rho(\frac{\partial f_i}{\partial x}) < 1$, where $\rho(\cdot)$ denotes the spectral radius. This can help us find a point $x^\star \in \mathcal{D} \cap R_{\chi_i}$ using fixed point iteration. Further, we obtain R_{χ_i}.

Let t_ι, $\iota = 0, 1, \ldots$ be the ιth switching instant, and $t_0 = 0$. It follows that the mode $\sigma(t_\iota)$ is activated in $[t_\iota, t_{\iota+1})$. Also, let t_{ik}, $i \in \mathcal{M}$, $k = 1, 2, \ldots$ be the kth time when the mode i is switched on.

Assumption 2. For $\iota = 0, 1, \ldots$, $\inf_\iota t_{\iota+1} - t_\iota \geqslant \tau$ where $\tau > 0$.

Assumption 2 imposes a minimal interval between the two switchings; this avoids the Zeno phenomenon.

Definition 4 ([11])**.** The origin of the switched system (2) is stable under σ if for any $\epsilon > 0$, there exists a $\delta > 0$ such that $|x(t)| \leqslant \epsilon$, $t \geqslant 0$ whenever $|x(0)| \leqslant \delta$. The origin of the switched system (2) is asymptotically stable if it is stable, and $\lim_{t \to \infty} |x(t)| = 0$.

In this study, we solve the problem of establishing switching conditions for the switched systems (2) with each mode satisfying Assumption 1 and $\gamma_{\chi_i} \leqslant 0$, $\forall i \in \mathcal{M}$ such that the origin is stable or asymptotically stable. We will show that several general stability criteria are provided without restricting to any specific switching law. Under some conditions, the arbitrary switching law is also allowed.

3 Stability analysis

3.1 General condition

Stability analysis is mainly based on the invariant ray in each mode. We shall first establish a relation between the true state and the invariant rays. For any given point $x \in \mathbb{R}_+^n$, we define the set

$$S_{x,i} \triangleq \{y \in R_{\chi_i} \mid x \leqslant y\}, \quad \forall i \in \mathcal{M}.$$

The set $S_{x,i}$ contains all points in the ray R_{χ_i}, which are not less than x. Also, we define the mapping $\psi_i : \mathbb{R}_+^n \to \mathbb{R}_+^n$ such that

$$\psi_i(x) \triangleq \min S_{x,i}, \tag{4}$$

where "min" means that if $a = \min S_{x,i}$, then $a < b$, $\forall b \in S_{x,i} \backslash a$.

It can be seen that $\psi_i(x)$ maps x onto a point of the ray, which is minimal among all the points not less than x in the ray, as shown in Figure 1. Note that $\psi_i(x)$ always exists and is unique. Moreover, ψ_i is monotone in $\mathbb{R}_+^n$ in the sense that $\psi_i(x) \leqslant \psi_i(y)$ for $x \leqslant y$.

We define $z_i(t) : [t_{ik}, t_{ik+1}) \to \mathbb{R}^n$, $\forall i \in \mathcal{M}$, for $k = 1, 2, \ldots$, as a virtual state of the mode i starting from the point in R_{χ_i} at each t_{ik} and running in each activating interval of the mode i. The result is given in the following theorem.

Theorem 1. Consider the switched system (2) with each mode satisfying Assumptions 1 and 2, and $\gamma_{\chi_i} \leqslant 0$, $\forall i \in \mathcal{M}$; the initial states $x(0) \in \mathbb{R}_+^n$. It holds that $x(t) \in \mathbb{R}_+^n$, $\forall t \geqslant 0$. At each switching instant t_{ik}, let

$$z_i(t_{ik}) = \psi_i(x(t_{ik})), \quad \forall i \in \mathcal{M}, \ k = 1, 2, \ldots \tag{5}$$

The origin is stable if $\gamma_{\chi_i} \leqslant 0$ and

$$z_i(t_{i(k+1)}) \leqslant z_i(t_{ik}). \tag{6}$$

The origin is asymptotically stable if $\gamma_{\chi_i} < 0$ and

$$z_i(t_{i(k+1)}) < z_i(t_{ik}). \tag{7}$$

Proof. As f_i satisfies Assumption 1, $\mathbb{R}_+^n$ is a forward invariant set for each mode, that is, $x(t_\iota) \in \mathbb{R}_+^n \Rightarrow x(t) \in \mathbb{R}_+^n$, $\forall t \in [t_\iota, t_{\iota+1})$. Therefore, $x(t) \in \mathbb{R}_+^n$, $\forall t \geqslant 0$ under any switching law.

As $z_i(t_{ik}) = \psi_i(x(t_{ik}))$, we consider the following two cases.

Case 1: $z_i(t_{ik}) = x(t_{ik})$. In this case, because R_{χ_i} is an invariant ray, $x(t)$ will always be on this ray. Therefore, $z_i(t) = x(t)$, $\forall t \in [t_\iota, t_{\iota+1})$ with $t_\iota = t_{ik}$.

Case 2: $z_i(t_{ik}) > x(t_{ik})$. Due to the strong monotonicity property, $z_i(t) \gg x(t)$, $\forall t \in [t_\iota, t_{\iota+1})$ with $t_\iota = t_{ik}$.

We can conclude that $x(t) \leqslant z_{\sigma(t_\iota)}(t)$, $\forall t \in [t_\iota, t_{\iota+1})$. It also follows from Lemma 1 that the flow of each mode in its activating period is strongly monotone in $\mathbb{R}_+^n$. Therefore, the boundedness of $z_i(t)$, $\forall i \in \mathcal{M}$, implies the boundedness of $x(t)$, and $\lim_{t\to\infty} |z_{\sigma(t)}(t)| = 0$ leads to $\lim_{t\to\infty} |x(t)| = 0$. Next, we analyze the behavior of z_i.

Stability. Because $z_i(t)$ starts from the point in R_{χ_i}, the solution $z_i(t)$ is always on R_{χ_i}, that is, $z_{is}(t)$ is always on $\varepsilon^{r_s}\chi_{is}$, for $s \in \mathcal{N}$. It follows from (3) that

$$\begin{aligned} \dot{z}_{is} &= \gamma_{\chi_i} r_s z_{is} \\ &= \gamma_{\chi_i} r_s \varepsilon^{r_s+p} \chi_{is} \end{aligned}$$

$$=\underbrace{\left(\gamma_{\chi_i} r_s \chi_{is}{}^{-\frac{p}{r_s}}\right)}_{\lambda_{is}} (z_{is})^{1+\frac{p}{r_s}}. \tag{8}$$

From (8), we can see that

$$\frac{\dot{z}_{is}}{(z_{is})^{1+\frac{p}{r_s}}} = \lambda_{is}. \tag{9}$$

By integrating the time derivative of z_{is} along the trajectory of (9), we obtain the following equations:

$$-\frac{p}{r_s}\left(\frac{1}{z_{is}(t)^{\frac{p}{r_s}}} - \frac{1}{z_{is}(t_{ik})^{\frac{p}{r_s}}}\right) = \lambda_{is}(t - t_{ik}),\ \text{for } p > 0, \tag{10}$$

which is equivalent to

$$\frac{1}{z_{is}(t)^{\frac{p}{r_s}}} - \frac{1}{z_{is}(t_{ik})^{\frac{p}{r_s}}} = -\frac{\lambda_{is} p}{r_s}(t - t_{ik}),\ \text{for } p > 0, \tag{11}$$

and

$$z_{is}(t) = \mathrm{e}^{\lambda_{is}(t-t_{ik})} z_{is}(t_{ik}),\ \text{for } p = 0, \tag{12}$$

where $t \in [t_\iota, t_{\iota+1})$ with $t_\iota = t_{ik}$.

We can see that all the elements of z_i are independent of one another and do not increase in the period in which the mode i is activated, if $\gamma_{\chi_i} \leqslant 0$.

For any given $\epsilon > 0$, we can always pick $\delta_i > 0$ such that if $|x(0)| \leqslant \delta_i$, then $|z_i(t_{i1})| \leqslant \epsilon$. Therefore, we can pick $\delta = \min_{i\in\mathcal{M}} \delta_i$ such that if $|x(0)| \leqslant \delta$, then $|z_i(t_{i1})| \leqslant \epsilon$, $\forall i \in \mathcal{M}$. Based on the condition (6) and the decreasing property of z_i in each activating period of the mode i, we have $|z_{\sigma(t_\iota)}(t)| \leqslant \epsilon\ \forall t \in [t_\iota, t_{\iota+1})$. This implies that $|x(t)| \leqslant \epsilon$, $\forall t \geqslant 0$, and the stability of the switched system (2) is achieved.

Asymptotical stability. For $\gamma_{\chi_i} < 0$, $\lambda_{is} < 0$, $\forall s \in \mathcal{N}$, it follows from (11) and (12) that $z_i(t)$ is always decreasing in the period in which the mode i is activated. Note that due to the finiteness of $\mathcal{M}$, there exists an index $i \in \mathcal{M}$ that is associated with an infinite sequence of switching instants $(t_{i1}, t_{i2}, \ldots)$. The condition (7) ensures that the sequence $z_i(t_{i1})$, $z_i(t_{i2})$, $\ldots$ is decreasing and positive and therefore has a limit vector $c \in \mathrm{int}(\mathbb{R}^n_+)$.

There also exists a family of positive definite continuous vector functions ϕ_i, $i \in \mathcal{M}$ such that Eq. (7) can be rewritten as

$$z_i(t_{i(k+1)}) - z_i(t_{ik}) \leqslant -\phi_i(z_i(t_{ik})). \tag{13}$$

It follows that

$$\begin{aligned} 0 = c - c &= \lim_{\tilde{k}\to\infty} z_i(t_{i\tilde{k}}) - \lim_{k\to\infty} z_i(t_{ik}) \\ &= \lim_{k\to\infty} \left(z_i(t_{i(k+1)}) - z_i(t_{ik})\right) \\ &\leqslant \lim_{k\to\infty} \left(-\phi_i(z_i(t_{ik}))\right) \leqslant 0. \end{aligned} \tag{14}$$

Therefore, $\lim_{k\to\infty} |z_i(t_{ik})| = 0$, which leads to $\lim_{k\to\infty} |x(t_{ik})| = 0$. It follows from the Lyapunov stability property that $\lim_{t\to\infty} |x(t)| = 0$, and the asymptotical stability of the switched system (2) at the origin follows. This completes the proof.

Remark 1. Rather than analyzing multiple Lyapunov functions, which are often difficult to construct for switched nonlinear systems, Theorem 1 fully relies on the invariant ray and the monotonicity of the solution of each mode. Conditions (6) and (7) mean that the switching should be slow enough to compensate for the differences between the various z_i values, such that the state of any mode when it is just switched on does not increase or even decrease as compared with the state when it was switched on the last time.

Remark 2. In fact, $\psi_i(x)$ can be regarded as the vector Lyapunov function of the mode i [38]; the existence of the invariant rays provides us a comparison system of $z_i(t)$ such that the behavior of $\psi_i(x)$ can be conveniently analyzed. The max-separable/sum-separable Lyapunov functions in [5] can also be constructed in this case, that is, by defining $V_i(x) \triangleq \max_{i\in\mathcal{N}} \psi_{is}(x_s)$ or $V_i(x) \triangleq \sum_{s=1}^{n} \psi_{is}(x_s)$, we can conclude that $\lim_{t\to\infty} V_{\sigma(t)}(t) = 0$.

Remark 3. The conditions (5)–(7) are analogous to the conditions for the classical multiple Lyapunov functions, which provide a general stability criterion without any restrictions for any specific switching laws, such as time-dependent switching or state-dependent switching. We find from (5) that at each switching time, the value of $z_i(t_{ik})$ depends on the value of $x(t_{ik})$; therefore, the conditions (6) and (7) can be checked by using real-time states information. If all modes share a common invariant ray, these conditions can be guaranteed under arbitrary switching. For an extreme case, such as when $\gamma_{\chi_i} = 0$ for $i \in \mathcal{M}$, $z_i(t)$ is fixed, that is, $z_i(t) = z_i(t_{ik})$, $\forall t \in [t_{ik}, t_{ik+1})$.

Theorem 1 is also available for the linear case. Consider a switched linear system $\dot{x} = A_i x$, $\forall i \in \mathcal{M}$, where A_i is Metzler, Hurwitz, and irreducible. Then, each mode is obviously homogeneous of degree 0 with $r = (1, \ldots, 1)$. We define $\vartheta(A_i) \triangleq \max\{\mathbb{R}(\lambda) : \lambda \in \sigma(A_i)\}$, where $\sigma(A_i) \triangleq \{z \in \mathcal{C} : \det(zI - A_i) = 0\}$ is the set of all eigenvalues of A_i. As A_i is Hurwitz, $\vartheta(A_i) < 0$. According to the Perron-Frobenius theorem [26], there exists a strictly positive eigenvector χ_i such that $A_i\chi_i = \vartheta(A_i)\chi_i$. We can see that Assumption 1 in Section 2 is satisfied for all modes of such a switched linear system. The following result can be obtained as a special case of Theorem 1.

Corollary 1. Consider the switched system (2) satisfying Assumption 2, where $f_i(x) = A_i x$, $\forall i \in \mathcal{M}$, with A_i being Metzler, Hurwitz, and irreducible; the initial states $x(0) \in \mathbb{R}^n_+$. It holds that $x(t) \in \mathbb{R}^n_+$, $\forall t \geqslant 0$. At each switching instant t_{ik}, we choose $z_i(t_{ik})$ as in (5). The origin is asymptotically stable if Eq. (7) holds for every switching instant.

Remark 4. For the stability analysis of switched positive linear systems, most existing methods in literature rely on the co-positive Lyapunov function [24, 26, 30], which can be constructed because of the nonnegativity of states. However, Corollary 1 provides an alternative clue that follows the Perron-Frobenius theorem and is based on the positive eigenvector of A_i. These eigenvectors can also be found conveniently in linear systems.

3.2 Conditions based on initial state, ray, and time

A disadvantage of Theorem 1 is that the states need to be checked all the time. With the help of the dynamic behavior of the rays (11) and (12), we shall give several different stability conditions that depend on the initial state, the rays, and the time, as shown below.

Corollary 2. Consider the switched system (2) with each mode satisfying Assumptions 1 and 2, $\gamma_{\chi_i} < 0$, $\forall i \in \mathcal{M}$, and the initial state $x(0) \in \mathbb{R}^n_+$. At each t_ι, let

$$z_{\sigma(t_\iota)}(t_\iota) = \psi_{\sigma(t_\iota)}(z_{\sigma(t_{\iota-1})}(t_\iota)), \tag{15}$$

$$z_{\sigma(0)}(0) = \psi_{\sigma(0)}(x(0)). \tag{16}$$

Suppose that the mode $\sigma(t_\iota)$ is switched on for the kth time at $t = t_\iota$, that is, $t_\iota = t_{\sigma(t_\iota)k}$. The origin is asymptotically stable if

$$t_\iota - t_{\iota-1} > \max_{s\in\mathcal{N}} \left(\frac{r_s}{\lambda_{\sigma(t_{\iota-1})s}p} \left(\frac{1}{\psi^{-1}_{\sigma(t_\iota)}\left(z_{\sigma(t_\iota)s}(t_{\sigma(t_\iota)(k-1)})\right)^{\frac{p}{r_s}}} - \frac{1}{z_{\sigma(t_{\iota-1})s}(t_{\iota-1})^{\frac{p}{r_s}}} \right), \tau \right), \text{ for } p > 0, \tag{17}$$

$$t_\iota - t_{\iota-1} > \max_{s\in\mathcal{N}} \left(\frac{1}{\lambda_{\sigma(t_{\iota-1})s}} \ln \left(\frac{\psi^{-1}_{\sigma(t_\iota)}\left(z_{\sigma(t_\iota)s}(t_{\sigma(t_\iota)(k-1)})\right)}{z_{\sigma(t_{\iota-1})s}(t_{\iota-1})} \right), \tau \right), \text{ for } p = 0. \tag{18}$$

Proof. Because $z_{\sigma(0)}(0) = \psi_{\sigma(0)}(x(0))$, we have

$$x(t) \leqslant z_{\sigma(0)}(t), \quad \forall t \in [0, t_1).$$

This together with $z_{\sigma(t_1)}(t_1) = \psi_{\sigma(t_1)}(z_{\sigma(0)}(t_1))$ leads to $x(t_1) \leqslant z_{\sigma(t_1)}(t_1)$. Therefore,

$$x(t) \leqslant z_{\sigma(t_1)}(t), \quad \forall t \in [t_1, t_2).$$

By induction, we have

$$x(t) \leqslant z_{\sigma(t_\iota)}(t), \quad \forall t \in [t_\iota, t_{\iota+1}).$$

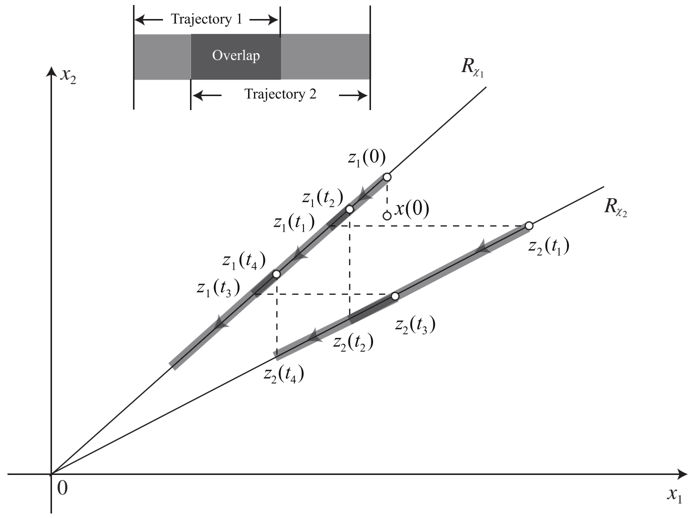

Figure 2 An illustrative switching law of Corollary 2.

Therefore, the boundedness and convergence of $z_i(t)$ also implies the boundedness and convergence of $x(t)$.

Letting $t = t_l$, $t_{ik} = t_{l-1}$ in (11) and (12), and substituting (17) into (11) for $p > 0$ or substituting (18) into (12) for $p = 0$, we obtain

$$\psi_{\sigma(t_\iota)}\left(z_{\sigma(t_{\iota-1})}(t_\iota)\right) < z_{\sigma(t_\iota)}(t_{\sigma(t_\iota)(k-1)}). \tag{19}$$

Because $z_{\sigma(t_\iota)}(t_{\sigma(t_\iota)k}) = z_{\sigma(t_\iota)}(t_\iota) = \psi_{\sigma(t_\iota)}(z_{\sigma(t_{\iota-1})}(t_\iota))$, we have

$$z_{\sigma(t_\iota)}(t_{\sigma(t_\iota)k}) < z_{\sigma(t_\iota)}(t_{\sigma(t_\iota)(k-1)}).$$

This is the same as condition (7) in Theorem 1. Asymptotical stability can be achieved by following Theorem 1.

Algorithm 1 shows how to check the stability by using Corollary 2.

Algorithm 1 Stability checking algorithm $\mathcal{S}_1$

Step 1: Given a switching law $\sigma(t)$ and the initial state $x(0)$, let $z_{\sigma(0)}(0) = \psi_{\sigma(0)}(x(0))$, and let l take the value 1. We define m vectors $\Theta_i \in \mathbb{R}^n$, $i \in \mathcal{M}$, and let $\Theta_i = 0\ \forall i \in \mathcal{M}$.
Step 2: Calculate $z_{\sigma(t_{\iota-1})}(t_\iota)$ according to (11) or (12).
Step 3: Calculate $z_{\sigma(t_\iota)}(t_\iota)$ according to (15). If $\Theta_{\sigma(t_\iota)} = 0$, go to Step 5.
Step 4: If $z_{\sigma(t_\iota)}(t_\iota) < \Theta_{\sigma(t_\iota)}$, go to Step 5; else stop the algorithm.
Step 5: Let $\Theta_{\sigma(t_\iota)} = z_{\sigma(t_\iota)}(t_\iota)$, and let $l = l + 1$. Go to Step 2.

Remark 5. On comparing (15) and (16) with (5), we find that unlike Theorem 1, in Corollary 2, $z_{\sigma(t_{\iota-1})}(t_\iota)$ (not $x(t_\iota)$) is mapped onto the ray $R_{\chi_{\sigma(t_\iota)}}$ at the time t_ι. Figure 2 illustrates $\mathcal{S}_1$ when a switched system has two state variables and two modes; the switching sequence is $1 \to 2 \to 1 \to \cdots$. Let $z_1(0) = \psi_1(x(0))$, $z_2(t_1) = \psi_2(z_1(t_1))$, $z_1(t_2) = \psi_1(z_2(t_2))$, and so on, as proposed by (15) and (16). The value of $z_{\sigma(t_{\iota-1})}(t)$ can be calculated a priori conveniently based on the dynamics of the ray $R_{\chi_{\sigma(t_{\iota-1})}}$; this allows us not to check $x(t)$ all the time. We can find from (8) that for the two cases of $p > 0$ and $p = 0$, the solution difference $z_i(t_1) - z_i(t_2)$ at any two instants t_1 and t_2 with $t_1 \geqslant t_2$ satisfies (11) and (12), respectively. This further leads to the conditions (17) and (18).

Remark 6. Given a switching law with the prescribed switching sequence and switching time, the algorithm $\mathcal{S}_1$ provides an explicit procedure to check the stability a priori and off-line by using the behaviors of the invariant rays. For $t \in [t_{\iota-1}, t_\iota)$, $z_{\sigma(t_{\iota-1})}(t)$ can be calculated based on $R_{\chi_{\sigma(t_{\iota-1})}}$ (Steps 2 and 3); then we can check whether t_ι satisfies the conditions (17) and (18) (Step 4). To check any time instant before which the mode i is activated for k times, we require only the information of $z_i(t_{ik})$.

Therefore, the values of at-the-most m vectors need to be preserved; Θ_i is just used to store $z_i(t_{ik})$ (Step 5).

Corollary 3. Consider the switched system (2) with each mode satisfying Assumptions 1 and 2; $\gamma_{\chi_i} < 0$, $\forall i \in \mathcal{M}$, and the initial states $x(0) \in \mathbb{R}_+^n$. At each t_ι, let

$$z_{\sigma(t_\iota)}(t_\iota) = \psi_{\sigma(t_\iota)}(z_{\sigma(t_{\iota-1})}(t_\iota)), \tag{20}$$

$$z_{\sigma(0)}(0) = \psi_{\sigma(0)}(x(0)). \tag{21}$$

The origin is asymptotically stable if

$$t_\iota - t_{\iota-1} > \max_{s\in\mathcal{N}} \left(\frac{r_s}{\lambda_{\sigma(t_{\iota-1})s}p} \left(\frac{1}{\psi^{-1}_{\sigma(t_\iota)}\left(z_{\sigma(t_{\iota-1})s}(t_{\iota-1})\right)^{\frac{p}{r_s}}} - \frac{1}{z_{\sigma(t_{\iota-1})s}(t_{\iota-1})^{\frac{p}{r_s}}} \right), \tau \right), \text{ for } p > 0, \tag{22}$$

$$t_\iota - t_{\iota-1} > \max_{s\in\mathcal{N}} \left(\frac{1}{\lambda_{\sigma(t_{\iota-1})s}} \ln \left(\frac{\psi^{-1}_{\sigma(t_\iota)}\left(z_{\sigma(t_{\iota-1})s}(t_{\iota-1})\right)}{z_{\sigma(t_{\iota-1})s}(t_{\iota-1})} \right), \tau \right), \text{ for } p = 0. \tag{23}$$

Proof. Substituting (22) into (11) for $p > 0$ or substituting (23) into (12) for $p = 0$ yields

$$\psi_{\sigma(t_\iota)}\left(z_{\sigma(t_{\iota-1})}(t_\iota)\right) < z_{\sigma(t_{\iota-1})}(t_{\iota-1}). \tag{24}$$

Further, we have

$$z_{\sigma(t_\iota)}(t_\iota) < z_{\sigma(t_{\iota-1})}(t_{\iota-1}). \tag{25}$$

Note that condition (25) is included by (7) as a special case; stability is achieved by using Theorem 1. Asymptotical stability also follows from Theorem 1. This completes the proof.

Algorithm 2 shows how to check the stability by using Corollary 3.

Algorithm 2 Stability checking algorithm $\mathcal{S}_2$

Step 1: Given a switching law $\sigma(t)$ and the initial state $x(0)$, let $z_{\sigma(0)}(0) = \psi_{\sigma(0)}(x(0))$ and let l take the value 1. Define a vector $\Theta \in \mathbb{R}^n$, and let $\Theta = z_{\sigma(0)}(0)$.
Step 2: Calculate $z_{\sigma(t_{\iota-1})}(t_\iota)$ according to (11) or (12).
Step 3: Calculate $z_{\sigma(t_\iota)}(t_\iota)$ according to (15). If $\Theta_{\sigma(t_\iota)} = 0$, go to Step 5.
Step 4: Let $\Theta = z_{\sigma(t_\iota)}(t_\iota)$, and let $l = l + 1$. Go to Step 2.

Remark 7. On comparing Corollary 2 with Corollary 3 (and comparing $\mathcal{S}_1$ and $\mathcal{S}_2$), we find that Corollary 2 compensates for the difference between z_i and other modes in the whole period $[t_{i(k-1)}, t_{ik})$ such that $z_i(t_{ik}) < z_i(t_{i(k-1)})$. However, Corollary 3 compensates for the difference among the z_i values $\forall i \in \mathcal{M}$ in each interval $[t_\iota, t_{\iota+1})$; $z_{\sigma(t_\iota)}(t_\iota)$ decreases over each time instant t_ι, $\iota = 1, 2, \ldots$, as shown in Figure 3. For $t \in [t_{\iota-1}, t_\iota)$, because $z_{\sigma(t_{\iota-1})}(t_{\iota-1})$ is known, the value of $z_{\sigma(t_{\iota-1})}(t)$ can be computed. Then, the value of t_ι can be checked by using conditions (22) and (23). One benefit of this method is that for any $t \in [t_\iota, t_{\iota+1})$, only the information of $z_{\sigma(t_\iota)}(t_\iota)$ needs to be known. The value of only one vector needs to be preserved (in Θ of $\mathcal{S}_2$).

Note that the algorithms $\mathcal{S}_1$ and $\mathcal{S}_2$ check the values of $z_{\sigma(t_{\iota-1})}(t_\iota)$ and $z_{\sigma(t_\iota)}(t_\iota)$ at each t_ι; this guarantees the convergence of the states for each switching instant. Therefore, we provide some insights into the implementation issue of both algorithms (see Remark 8).

Remark 8. Because the Zeno phenomenon is excluded as indicated in Section 2, three kinds of switching laws can be considered. Although in the practice, the first situation is often encountered, the algorithms $\mathcal{S}_1$ and $\mathcal{S}_2$ are available in all these three situations.

(1) Finite switchings happen in finite time intervals. In this case, both algorithms stop after a finite number of steps. The asymptotical stability of the system can be guaranteed, and the states will asymptotically converge to a small region around the origin in this finite time interval.

(2) Finite switchings happen in infinite time intervals. In this case, both algorithms also stop after a finite number of steps. Let us suppose that there is no switching after $t = t^\star$. Each mode is individually

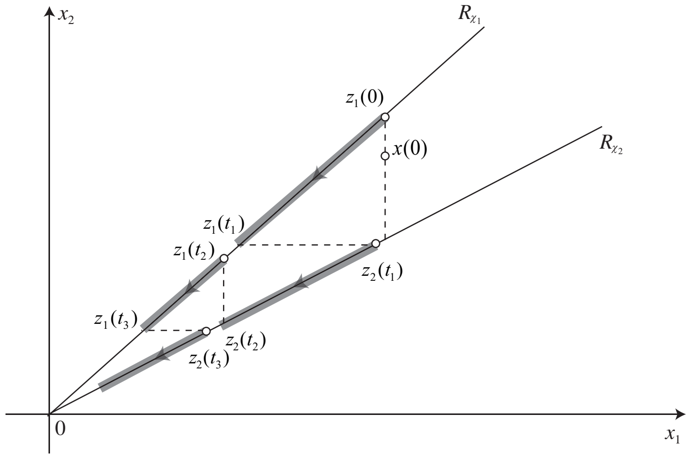

Figure 3 An illustrative switching law of Corollary 3.

asymptotically stable; therefore, the asymptotical stability of the system in the whole infinite time interval can also be guaranteed.

(3) Infinite switchings happen in infinite time interval. In this case, both algorithms will never stop. This can be avoided using a "periodic" switching law that often appears in practice. In each period $[kT,(k+1)T)$ when $k=0,1,\ldots$ with the period time length $T>0$, the switching sequence and the switching intervals are the same as those in other periods. The two algorithms only need to check the period $[0,T)$, and need to stop at $t=T$. Because $z_i(t)$, $\forall i\in\mathcal{M}$ monotonously decreases along the invariant ray R_{χ_i}, if the asymptotical stability is verified in $[0,T)$, it is also guaranteed in the whole infinite time interval.

3.3 Conditions based on initial state and time

Similar to the classical multiple Lyapunov function methods in switched systems [11, 14], Theorem 1 needs to check the states all the time. Corollaries 2 and 3 also require some information about z_i on the rays. One will expect to check the stability a priori without requiring any information about $z_{\sigma(t)}(t)$ and $x(t)$.

Definition 5 ([11])**.** If there exists a positive number τ_a such that

$$N_\sigma(\bar{t}_1,\bar{t}_2)\leqslant N_0+\frac{\bar{t}_2-\bar{t}_1}{\tau_a},\qquad \forall\bar{t}_2\geqslant\bar{t}_1\geqslant 0, \tag{26}$$

where $N_0>0$ denotes the chattering bound, and $N_\sigma(\bar{t}_1,\bar{t}_2)$ denotes the number of switchings of σ over the interval $[\bar{t}_1,\bar{t}_2)$. Then, the positive constant τ_a is called the average dwell time (a.d.t.) of σ over $[\bar{t}_1,\bar{t}_2)$.

Note that if the condition specified in (26) is satisfied $\forall\bar{t}_2\geqslant\bar{t}_1\geqslant 0$, then such an a.d.t. is available globally over the time interval $[0,\infty)$. If $\bar{t}_1$ and $\bar{t}_2$ are specified, then τ_a is only available for $[\bar{t}_1,\bar{t}_2)$.

Let $z_{\sigma(t_\iota)}(t_\iota)=\psi_{\sigma(t_\iota)}(z_{\sigma(t_{\iota-1})}(t_\iota))$. For the sake of clarity, let $i=\sigma(t_\iota)$, $j=\sigma(t_{\iota-1})$. Then, we have

$$\frac{z_{is}}{z_{js}}=\underbrace{\left(\frac{\chi_{jl}}{\chi_{il}}\right)^{\frac{r_s}{r_l}}\frac{\chi_{is}}{\chi_{js}}}_{\mu_{ijs}},\quad \forall s\in\mathcal{N}, \tag{27}$$

where $l\in\mathcal{N}$ such that $z_{il}=z_{jl}$. Because χ_i and χ_j can be fixed a priori, the value of μ_{ijs} can also be fixed a priori. We define

$$\mu\triangleq\max_{i,j\in\mathcal{M},i\neq j}\left(\max_{s\in\mathcal{N}}\mu_{ijs}\right). \tag{28}$$

If all modes share a common invariant ray, then $\mu = 1$. Also, we define

$$\lambda \triangleq \max_{i\in\mathcal{M}} \left(\max_{s\in\mathcal{N}} \lambda_{is} \right), \tag{29}$$

where λ_{is} is defined in (8).

Theorem 2. Consider the switched system (2) with each mode satisfying Assumptions 1 and 2; $\gamma_{\chi_i} < 0$, $\forall i \in \mathcal{M}$, and the initial state $x(0) \in \mathbb{R}^n_+$. At each t_ι, let $z_{\sigma(t_\iota)}(t_\iota)$ and $z_{\sigma(0)}(0)$ satisfy (20) and (21). The origin is asymptotically stable if

- For $p > 0$, an a.d.t. τ_a is available for $[kT, (k+1)T)$ where $T > 0$ and $k = 0, 1, \ldots$ such that

$$\tau_a > \max_{s\in\mathcal{N}} \left(\frac{T \ln\left(\mu^{\frac{p}{r_s}}\right)}{\ln\left(\frac{1-\frac{\lambda pT}{r_s} z_{\sigma(0)s}(0)^{\frac{p}{r_s}}}{\mu^{\frac{pN_0}{r_s}}}\right)} \right); \tag{30}$$

- For $p = 0$, τ_a is available for $[0, \infty)$ such that

$$\tau_a > -\frac{\ln \mu}{\lambda}. \tag{31}$$

Before proving Theorem 2, it is necessary to obtain the key idea behind it. Recall that the "dwell time" method for switched systems often rely on a common constant upper bound of the ratios between any of the two Lyapunov functions, that is, $V_i \leqslant \mu V_j$, $\mu \geqslant 1$, $\forall i, j \in \mathcal{M}$ [11]. Therefore, if the assumptions of Theorem 1 are relaxed, we can find a common constant upper bound for the ratios between z_i and z_j. If this can be done, then based on the dynamic behavior of z_i, we can design the switching law based on the dwell time.

Considering the point $x \in \mathbb{R}^n_+$, we shall try to find the ratio between $z_i(x)$ and $z_j(x)$. For the mode i, let $z_{is} = \psi_{is}(x)$, $\forall s \in \mathcal{N}$. From the definition of ψ_i in (4), we have $z_{is} \geqslant x_s$, and there exists at least a number $l \in \mathcal{N}$ such that $z_{il} = x_l$. Also, note that $z_{il} = \varepsilon^{r_l}\chi_{il}$; therefore,

$$z_{is} = \varepsilon^{r_s}\chi_{is} = \left(\frac{z_{il}}{\chi_{il}}\right)^{\frac{r_s}{r_l}} \chi_{is} = \left(\frac{x_l}{\chi_{il}}\right)^{\frac{r_s}{r_l}} \chi_{is}. \tag{32}$$

Similarly, for the mode j,

$$z_{js} = \left(\frac{x_v}{\chi_{jv}}\right)^{\frac{r_s}{r_v}} \chi_{js}, \tag{33}$$

where $v \in \mathcal{N}$ such that $z_{jv} = x_v$. Consequently,

$$\frac{z_{is}}{z_{js}} = \left(\frac{x_l}{\chi_{il}}\right)^{\frac{r_s}{r_l}} \left(\frac{\chi_{jv}}{x_v}\right)^{\frac{r_s}{r_v}} \frac{\chi_{is}}{\chi_{js}}. \tag{34}$$

We can see from (34) that the ratio $\frac{z_{is}}{z_{js}}$ largely depends on the value of x, and this ratio changes as x changes. This makes the upper bound of $\frac{z_{is}}{z_{js}}$ difficult to determine unless $l = v$ (in this case, such an upper bound depends on the values of χ_i and χ_j). This upper bound of the ratios is expected to be satisfied between any two modes that are activated successively in the given or even arbitrary switching sequences. Unfortunately, it is not possible to guarantee that the trajectory of $x(t)$ can always satisfy the condition $l = v$.

However, inspired by Corollary 3, we let $z_{\sigma(t_\iota)}(t_\iota)$ satisfy (20), and we try to find the ratio between $z_{\sigma(t_\iota)}$ and $z_{\sigma(t_{\iota-1})}$. This leads to the setting of Theorem 2.

Proof of Theorem 2. For the case $p > 0$, the main idea of achieving asymptotical stability is to choose a period time length T such that $z_{\sigma(t)}(t)$ is guaranteed to decrease over each time instant kT, $k = 0, 1, \ldots$, which is the starting time of each period in which the local a.d.t. is available, that is $z_{\sigma((k+1)T)}((k+1)T) < z_{\sigma(kT)}(kT)$.

Consider the interval $[0,T)$. Let us suppose that there are ς switching times in $[0,T)$. It follows from (11) that $\forall s\in\mathcal{N}$,

$$\frac{1}{z_{\sigma(t_\varsigma)s}(T)^{\frac{p}{r_s}}}-\frac{1}{z_{\sigma(t_\varsigma)s}(t_s)^{\frac{p}{r_s}}}\geqslant\frac{-\lambda p}{r_s}(T-t_s),$$
$$\vdots$$
$$\frac{1}{z_{\sigma(0)s}(t_1^-)^{\frac{p}{r_s}}}-\frac{1}{z_{\sigma(0)s}(0)^{\frac{p}{r_s}}}\geqslant\frac{-\lambda pt_1}{r_s}.$$

Further, we have

$$\begin{aligned}\frac{1}{z_{\sigma(t_\varsigma)s}(T^-)^{\frac{p}{r_s}}}&\geqslant\frac{1}{\mu^{\frac{p\varsigma}{r_s}}z_{\sigma(0)s}(0)^{\frac{p}{r_s}}}-\frac{\lambda p}{r_s}(T-t_s)-\sum_{j=0}^{\varsigma-1}\frac{1}{\mu^{\frac{p(\varsigma-j)}{r_s}}}\frac{\lambda p}{r_s}(t_{j+1}-t_j)\\&\geqslant\frac{1}{\mu^{\frac{p\varsigma}{r_s}}z_{\sigma(0)s}(0)^{\frac{p}{r_s}}}+\frac{\lambda_0pT}{\mu^{\frac{p\varsigma}{r_s}}r_s}.\end{aligned}\tag{35}$$

Therefore,

$$\frac{1}{z_{\sigma(t_\varsigma)s}(T)^{\frac{p}{r_s}}}-\frac{1}{z_{\sigma(0)s}(0)^{\frac{p}{r_s}}}\geqslant\underbrace{\left(\frac{1}{\mu^{\frac{p}{r_s}(N_0+\frac{T}{\tau_a})}}-1\right)\frac{1}{z_{\sigma(0)s}(0)^{\frac{p}{r_s}}}+\frac{\lambda_0pT}{\mu^{\frac{p}{r_s}(N_0+\frac{T}{\tau_a})}r_s}}_{\Phi_s}.\tag{36}$$

Condition (30) ensures that $\Phi_s>0$, $\forall s\in\mathcal{N}$.

By induction, we can conclude that for $k=1,2,\ldots$,

$$\frac{1}{z_{\sigma((k+1)T)s}((k+1)T)^{\frac{p}{r_s}}}-\frac{1}{z_{\sigma(0)s}(0)^{\frac{p}{r_s}}}\geqslant(k+1)\Phi_s.\tag{37}$$

Therefore, $\lim_{k\to\infty}z_{\sigma(kT)}(kT)=0$. Since $z_{\sigma(t)}(t)$ is always bounded in each interval $[kT,(k+1)T)$, $k=0,1,\ldots$, we can conclude that $\lim_{t\to\infty}z_{\sigma(t)}(t)=0$. This also implies that $\lim_{t\to\infty}x(t)=0$. The asymptotical stability is guaranteed at the origin.

For the case $p=0$, the result follows from the well-known a.d.t. condition [11]. This completes the proof.

Remark 9. The initial condition $x(0)$ needs to be known for the case $p>0$, but this condition is not required for $p=0$. For the case $p>0$ and $\mu=1$, the asymptotical stability can be guaranteed under arbitrary switching. However, in the presence of multiple invariant rays, we can see from (30) that $T\uparrow\Rightarrow\tau_a\uparrow$. The required a.d.t increases and the switching becomes slow when the time interval in which the a.d.t. is available becomes long. Therefore, it is difficult to apply the a.d.t. to the overall time interval $[0,\infty)$ in this case. This is essentially a range a.d.t. values. Different selections of T lead to different τ_a values while the asymptotical stability is always maintained. Theorem 2 can also be extended to other variations of a.d.t. such as the persistent dwell time [13], and mode-dependent dwell time [15].

Remark 10. The result of Theorem 2 is consistent with that of Corollaries 2 and 3, that is, if the switching is slow enough, then the systems is asymptotically stable at the origin. However, Corollaries 2 and 3 impose restrictions on each switching interval. Theorem 2 relies on the a.d.t. and an upper bound of the ratio between $z_{\sigma(t_\iota)}$ and $z_{\sigma(t_{\iota-1})}$. Because Corollaries 2 and 3 require some information of $z_{\sigma(t)}(t)$, their computation burdens are heavier than those of Theorem 2.

Remark 11. The result obtained using Theorem 2 is different from the result obtained from [34], where we consider a class of SPNSs with each mode being cooperative and homogeneous of degree 0 w.r.t. δ_ε^1. A key assumption in [34] is that for every $i\in\mathcal{M}$, there exists a vector $v_i\gg0$ such that $f_i(v_i)\ll0$; the a.d.t. scheme is developed based on such a set of vectors. The method used for choosing v_i still needs further investigation. Moreover, the case of a higher-order homogeneous degree is not considered in [34].

3.4 Discussion on robustness

The methods and procedures proposed in the previous subsections can be potentially extended to the systems with uncertainties in both the mode dynamics and switching scheme, as explained below:

- For mode uncertainties. In this case, the dynamics of the mode i can be written as $\dot{x} = f_i(x)+g_i(x)$, where g_i denotes the uncertainties in the mode i. In [8], the researchers have shown that if $g_i(x) \in \mathbb{R}^n_+$, then for the mode i with f_i satisfying Assumption 1 and the initial states belonging to $\mathbb{R}^n_+$, $\mathbb{R}^n_+$ is still a forward invariant set. However, the invariant ray may not exist for the uncertain mode even if f_i satisfies Assumption 1, and $\gamma_{\chi_i} < 0$. The invariant ray-based approach is still available if an additional assumption is imposed on g_i, which is based on the homogeneous system theory [39]. If g_i is the higher-order homogeneous term as compared with f_i, that is, g_i is homogeneous of degree $(p+k) \in \mathbb{R}^+$ w.r.t. δ^r_ε, where $k > 0$, then the asymptotical stability of the system $\dot{x} = f_i(x)$ implies the asymptotical stability of mode i. This still allows us to analyze $\dot{x} = f_i(x)$ by using the proposed methods without considering the uncertainties.
- For switching uncertainties. In this case, the switching condition is perturbed, which makes the actual switching law deviate from the nominal law. We consider two typical switching laws: the time-dependent switching law and the state-dependent switching law. For the time-dependent switching law, the uncertainties bring differences between each actual switching time and the nominal law, as described in [17]. If the upper bound of these difference is known, then we can modify the conditions of Corollaries 2 and 3 and Theorem 2 by taking into account such a bound and by lengthening or shortening the nominal activating periods of the modes. For the state-dependent switching, the nominal set of states that trigger each switching changes into a new set of states. If we know the maximum ranges of these changes in the state space, we can modify the conditions of Theorem 1 by taking into account these ranges and reestablishing the relationship between $z_i(t_{i(k+1)})$ and $z_i(t_{ik})$.

4 An example of chemical reactions

4.1 Model setting

In this subsection, we consider a class of dissipative cyclic chemical reactions as in [8]

$$\alpha X_1 \to \alpha X_2 \to \cdots \to \alpha X_n \to \alpha X_1,$$

where X_i, $i \in \mathcal{N}$, denotes the ith chemical component, and α is a natural number that denotes the stoichiometric coefficient. This means that in the cyclic reactions, the reactant of the ith component becomes the product of the $(i+1)$th component.

We define $x \triangleq (x_1\ x_2\ \cdots\ x_n)^{\mathrm{T}}$, where x_i denotes the concentration of X_i. Let us suppose that all reactions are at a given constant temperature. According to the mass action principle [40], the dynamical behavior of x obeys the following differential equation:

$$\dot{x} = Cr(x) - d_{\sigma(t)} r(x), \tag{38}$$

where $r(x) \triangleq (x_1^\alpha\ x_2^\alpha\ \cdots\ x_n^\alpha)^{\mathrm{T}}$ with α being the power of the states, and

$$C \triangleq \begin{pmatrix} -1 & 0 & \cdots & 1 \\ 1 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & -1 \end{pmatrix}, \quad d_\sigma \triangleq \begin{pmatrix} a_{\sigma 1} & 0 & \cdots & b_{\sigma 1} \\ b_{\sigma 2} & a_{\sigma 2} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & b_{\sigma n} & a_{\sigma n} \end{pmatrix}, \tag{39}$$

where the reaction rate constants of all reactions are kept equal to 1 to simplify the notation and the calculations. The switching dissipation term $d_{\sigma(t)}$ models the extraction of some chemicals from the reactor, thus $\forall i \in \mathcal{N}$, $a_i > 0$, $0 < b_i < 1$. The switching function $\sigma : [0, \infty) \to \mathcal{M}$ implies that the dissipative behavior may switch among the m different modes. It is clear that the system (38) is a switched system with all the modes satisfying Assumption 1 in Section 2.

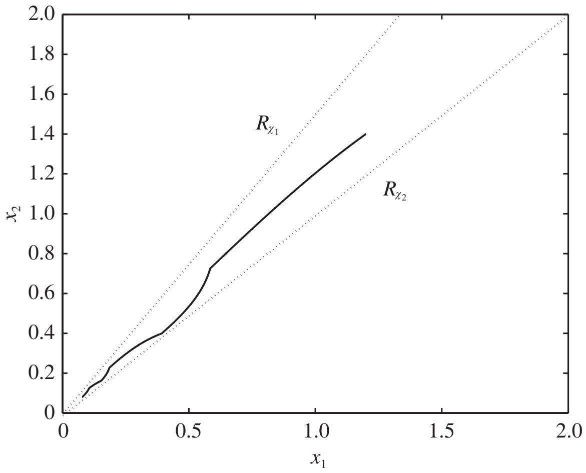

Figure 4 The state trajectory.

4.2 Simulation

In the simulation, let $n = 2$, $m = 2$, and $\alpha = 2$. Then, the chemical reaction process can be described as

$$2X_1 \leftrightarrows 2X_2,$$

and

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -x_1^2 + x_2^2 - a_{\sigma 1}x_1^2 - b_{\sigma 1}x_2^2 \\ -x_2^2 + x_1^2 - a_{\sigma 2}x_2^2 - b_{\sigma 2}x_1^2 \end{pmatrix}, \tag{40}$$

where the switching function $\sigma : [0, \infty) \to \{1, 2\}$. It is clear that the homogeneous degree $p = 1$.

For the mode 1, let us suppose $a_{11} = \frac{9}{4}$, $b_{11} = 0$, $a_{12} = \frac{1}{9}$, and $b_{12} = 0$. For the mode 2, we suppose $a_{21} = 1$, $b_{21} = 0$, and $a_{22} = 1$, $b_{22} = 0$. We obtain two points $\chi_1 = (1\ \frac{3}{2})^{\mathrm{T}}$ and $\chi_2 = (1\ 1)^{\mathrm{T}}$, which lead to two invariant rays R_{χ_1}, R_{χ_2} for two modes, and $\gamma_{\chi_1} = \gamma_{\chi_2} = -1$. Suppose that the initial states are $x(0) = (1.2\ 1.4)^{\mathrm{T}}$ and $\sigma(0) = 1$, and the switching sequence is periodical, that is, $1 \to 2 \to 1 \to \cdots$. We also suppose that the prescribed switching instants are $t_1 = 0.5$ s, $t_2 = 1.5$ s, $t_3 = 3$ s, $t_4 = 4.5$ s, and $t_5 = 6$ s (in the simulation, we consider only a finite number of switchings in the finite time interval, which is enough to illustrate the theoretical results). We shall check the stability under the given switching law.

First, we illustrate Corollary 2 and $\mathcal{S}_1$. From the definition of ψ_i and (16), it follows that $z_1(0) = (1.2\ 1.8)^{\mathrm{T}}$; let $\Theta_1 = z_1(0)$. According to (11), one has $z_1(t_1) = (0.75\ 1.125)^{\mathrm{T}}$; this together with (15) yields $z_2(t_1) = (1.125\ 1.125)^{\mathrm{T}}$; let $\Theta_2 = z_2(t_1)$. Similarly, we have $z_2(t_2) = (0.5294\ 0.5294)^{\mathrm{T}}$ and $z_1(t_2) = (0.5294\ 0.7941)^{\mathrm{T}}$; we also have $z_1(t_2) < \Theta_1$, which means that $t_2 - t_1$ satisfies condition (17). Also, let $\Theta_1 = z_1(t_2)$. At the instant t_3, we have $z_1(t_3) = (0.2951\ 0.4426)^{\mathrm{T}}$ and $z_2(t_3) = (0.4426\ 0.4426)^{\mathrm{T}}$; therefore, $z_2(t_3) < \Theta_2$, and condition (17) holds. Let $\Theta_2 = z_2(t_3)$. By repeating the above procedures, we obtain that condition (17) is also satisfied for t_4 and t_5. Note that the above procedures can be followed off-line without requiring any real-time state information.

Second, we illustrate Corollary 3 and $\mathcal{S}_2$. Based on the calculations for Corollary 2, we have $z_{\sigma(t_i)}(t_i) < z_{\sigma(t_{i-1})}(t_{i-1})$ for $i = 1, 2, \ldots, 5$. This makes condition (22) hold at each switching instant.

Finally, we illustrate Theorem 2. We have $\lambda = -\frac{2}{3}$, $\mu = \frac{3}{2}$. Let us suppose that $N_0 = 0.01$, $T = 3$. Condition (30) requires that $\tau_a > 0.9973$ s, which is satisfied by the prescribed switching instants.

The state trajectory in the interval $[0, 7)$ s is shown in Figure 4, from which we can see that the states are always positive, and asymptotical stability is achieved.

One advantage of the invariant ray-based method over the classical multiple Lyapunov functions technique is that the states x do not have to be checked. Moreover, the dynamics of z_i is known, and its solution can be calculated conveniently a priori; therefore, the values of z_i also do not have to be checked in real time. This significantly facilitates the stability checking process.

5 Conclusion

This paper establishes several stability conditions for a class of SPNSs by using the tool of invariant rays. These conditions can be used for stability checking as well as for stabilization design. By using the methods in [16], Theorem 2 could be potentially extended to the case when there exist some unstable modes, that is, $\gamma_{\chi_i} > 0$ for $i \in \mathcal{M}$. The key idea is to achieve stability by a tradeoff among the stable and unstable modes. For an unstable mode, there may exist multiple invariant rays [9]; we could use any of these rays for stability analysis.

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