SCIENCE CHINA
Information Sciences

• RESEARCH PAPER •

November 2019, Vol. 62 212201:1–212201:11
https://doi.org/10.1007/s11432-018-9846-y

Finite-time and fixed-time consensus problems for second-order multi-agent systems with reduced state information

Huifen HONG[1], He WANG[1], Zhenling WANG[2] & Wenwu YU[1*]

[1]*School of Mathematics, Southeast University, Nanjing* 210096*, China;*
[2]*Institute of Complexity Science, Qingdao University, Qingdao* 266071*, China*

Received 9 December 2018/Accepted 4 March 2019/Published online 18 September 2019

Abstract This paper studies the fixed-time consensus (FixTC) and connectivity-preserving finite-time consensus (FinTC) protocol designs for second-order multi-agent systems using output information only. Herein, a distributed FixTC protocol based on the Lyapunov stability and bi-limit homogeneity approaches is proposed with the aid of an auxiliary system. Then, when the graph is state-dependent, i.e., the agents have limited sensing and communication ranges, a connectivity-preserving FinTC is proposed by designing a mechanism suitable for this purpose. Theoretical analysis and several simulations are presented to verify the effectiveness of the proposed protocols.

Keywords MASs, finite-time consensus, fixed-time consensus, connectivity-preserving mechanism, reduced state information

Citation Hong H F, Wang H, Wang Z L, et al. Finite-time and fixed-time consensus problems for second-order multi-agent systems with reduced state information. Sci China Inf Sci, 2019, 62(11): 212201, https://doi.org/10.1007/s11432-018-9846-y

1 Introduction

Distributed consensus for multi-agent systems (MASs) refers to drive a cluster of agents to reach an agreement on state by designing a distributed control protocol. This has become a popular research topic owing to its broad applications in a variety of areas, such as swarming of social foraging agents, flocking, rendezvous, and formation control [1–7].

Existing consensus results roughly fall into two categories according to convergence rate: asymptotic consensus [8] and finite-time consensus (FinTC) [9–16]. In numerous practical engineering applications, FinTC demonstrates some advantageous properties like higher accuracy and better disturbance rejection [17]; therefore, it is usually demanded that agents should achieve consensus in finite time. First, FinTC was investigated in [9] for first-order MASs, where the normalized and signed gradient-based discontinuous consensus protocols were designed. Ref. [10] proposed both continuous and discontinuous finite-time control protocols for first-order MASs to achieve consensus. Wang et al. [11] investigated FinTC for second-order linear MASs using the homogeneity method. In [12], the FinTC tracking control for MASs with double-integrator dynamics and bounded disturbances was studied via the terminal sliding-mode approach. Li et al. [13] investigated the robust FinTC problem for second-order MASs with and without a leader using the backstepping method. Yu et al. [15] studied FinTC for more complicated second-order MASs with uncertain disturbances and nonlinear dynamics by designing a decoupled distributed

* Corresponding author (email: wwyu@seu.edu.cn)

sliding-mode control protocol. The bipartite FinTC problem for first/second-order MASs on a directed cooperative-competitive network was investigated in [16], and the settling time was precisely estimated and formulated.

Although FinTC was achieved in the aforementioned cases, the settling time relies on the initial states and may reach infinity as the initial states grow. To overcome this issue, the fixed-time stability concept was proposed in [18], wherein the convergence time for a system was uniformly bounded about the initial states. Ref. [19] presented the fixed-time stability theory for nonlinear differential equations according to the Lyapunov stability approach. Some two-term nonlinear-control protocols were proposed in [20, 21] for first-order MASs to ensure prescribed-time consensus. A new class of fixed-time consensus (FixTC) protocols were designed in [22] for integrator MASs with nonlinear dynamics and uncertainties simultaneously, and the role of each term in controllers was analyzed. Ref. [23] studied the FixTC problem for nonlinear MASs under general directed topologies. Some FixTC protocols for MASs with second-order dynamics were proposed in [24–26] from different viewpoints. Concretely, the FixTC tracking problem was studied in [24, 25] by designing a new sliding surface and an observer-based controller, respectively. Ref. [26] studied FixTC for nonlinear MASs without leader with the aid of the bi-limit homogeneity method. Tian et al. [27] studied FixTC tracking of MASs where agents have high-order integrator dynamics using the sliding mode and bi-limit homogeneity approaches.

Connectivity-preserving FinTC is another interesting topic. In many practical engineering applications, the agents have limited communication and sensing ranges, which implies that the communication graphs for the MASs depend upon the distance between any pair of agents. To guarantee consensus, a connectivity-preserving mechanism is essential, which has recently been studied. Connectivity-preserving consensus for first-order linear MASs was investigated by using a special potential function [28] and a bounded navigation function [29], respectively. Ref. [30] proposed a connectivity-preserving mechanism for leader-follower MASs, which can ensure agents reach rendezvous. Su et al. [31] studied the rendezvous for second-order MASs, where the control protocol has the ability to maintain connectivity. A robust connectivity-preserving leader-follower consensus problem for nonlinear and disturbed MASs was studied in [32]. In the above results, agents were all able to reach consensus asymptotically. Recently, some finite-time connectivity-preserving consensus protocols have been proposed. The connectivity-preserving FinTC problems for first-order MASs were investigated with unknown Lipschitz terms [33] and bounded disturbances [34], respectively. Hong et al. [35] investigated the connectivity-preserving FinTC for second-order nonlinear MASs by designing a new piecewise smooth potential function.

As discussed above, it can be seen that most existing FixTC results for second-order MASs use both relative velocity and relative position information in these control protocols. However, the full state is usually unavailable in practical situations; for instance, velocity may be measured inaccurately or there may be no velocity sensors in the MASs. Therefore, the consensus problem for second-order MASs with reduced state information is meaningful and practical. Recently, Tian et al. [36] proposed a FixTC for second-order leader-follower MASs utilizing only output information by designing a fixed-time distributed observer. On the other hand, results concerning second-order connectivity-preserving FinTC with only relative position information have not been reported yet.

Motivated by the foregoing discussion and inspired by the auxiliary system approach [37], this study investigates two consensus problems for second-order MASs with reduced state information, i.e., using only relative position states. The first is the FixTC problem (in which the graph is time-invariant) and the second is the connectivity-preserving FinTC problem with limited communication capabilities. The novelties of this paper are summarized as follows. First, according to the Lyapunov stability theory and the bi-limit homogeneity approach, a distributed FixTC protocol is designed with reduced state measurements. Second, finite-time stability theory and homogeneity theory are utilized to design and analyze a distributed connectivity-preserving FinTC protocol using only output states. The designed control protocol can preserve the initial edges and cause the MAS to achieve FinTC.

The rest of the paper is arranged as follows. The mathematical preliminaries are given in Section 2. The problem statement and main theoretical results concerning FixTC and connectivity-preserving FinTC protocol designs are presented in Section 3. Section 4 includes numerical simulations to corroborate the theoretical results. Finally, Section 5 concludes this paper.

2 Preliminaries

Notations: $\mathbb{R}^n$ is a Euclidian space with n dimensions. The superscript T denotes transposition; let $[N]$ denote the set $\{1, 2, \ldots, N\}$; symbol $\|\cdot\|$ denotes the Euclidean norm. Let $\xi^{[k]} = \mathrm{sign}(\xi)||\xi||^k$, where $\xi \in \mathbb{R}^n$ and

$$\mathrm{sign}(\xi) = \begin{cases} \dfrac{\xi}{||\xi||}, & ||\xi|| \neq 0, \\ 0, & ||\xi|| = 0 \end{cases}$$

is the signum function.

2.1 Graph theory

An undirected communication network can be represented by an undirected time-varying graph $\mathbb{G}(t) = (\mathbb{V}, \mathbb{E}(t), \mathbb{A}(t))$, where $\mathbb{V} = \{\nu_1, \nu_2, \ldots, \nu_N\}$ is the agent set. $\mathbb{E}(t) \in \mathbb{V}\times\mathbb{V}$ and $\mathbb{A}(t) = (a_{ij}(t))_{N\times N}$ stand for the edge set and the corresponding adjacency matrix at time t, respectively. Herein, $a_{ij}(t) = a_{ji}(t) \geqslant 0$ is the edge weight. Typically, at time instant t, an undirected edge $\epsilon_{ij}(t)$ can be described by an unordered pair of nodes (ν_i, ν_j), indicating an undirected information flow between ν_i and ν_j. Furthermore, $a_{ij}(t) > 0 \Leftrightarrow \epsilon_{ij}(t) \in \mathbb{E}(t)$. Define the Laplacian matrix at time instant t as $L(t) = (l_{ij}(t))_{N\times N}$, where $l_{ij}(t) = -a_{ij}(t),\ j \neq i,\ l_{ii} = \sum_{j=1, j\neq i}^{N} a_{ij}(t)$. Hence, $L(t)1_N = 0, \forall t$.

2.2 Nonsmooth differential equations

Consider the following nonlinear system:

$$\dot{z}(t) = \omega(z(t)), \tag{1}$$

where $z = (z_1, z_2, \ldots, z_m)^{\mathrm{T}}$ is the state vector and $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^{\mathrm{T}} : \mathbb{R}^m \to \mathbb{R}^m$ is a vector field. When $\omega(z(t))$ is non-smooth, the solutions of (1) are understood in the Filippov sense [38] which is defined by an absolutely continuous solution $z : [0, T] \to \mathbb{R}^m$ such that $\dot{z}(t) \in \mathcal{K}[\omega](z(t))$ for almost all $t \in [0, T]$, where $\mathcal{K}[\omega] : \mathbb{R}^m \to 2^{\mathbb{R}^m}$ is the Filippov set-valued map, defined by

$$\mathcal{K}[f](z) \triangleq \bigcap_{\delta>0} \bigcap_{u(S)=0} \overline{\mathrm{co}}\{f(B_\delta(z)\backslash S)\},$$

where u denotes the Lebesgue measure in $\mathbb{R}^m$, $\overline{\mathrm{co}}$ denotes the convex closure, and $B_\delta(z)$ denotes the open ball centered at z with radius δ.

For a locally Lipschitz function $W : \mathbb{R}^m \to \mathbb{R}$, the notation $\partial W(z)$ denotes the generalized gradient of W at z, and $\mathcal{L}_\omega W(z) = \cap_{\xi\in\partial W(z)} \xi^{\mathrm{T}} \mathcal{K}[\omega](z)$ denotes the set-valued Lie derivative of W with respect to ω at z.

Lemma 1 ([38])**.** If $\omega(z(t))$ is measurable and locally essentially bounded, then there exist Fillippov solutions of (1) for any initial values.

$\omega(z)$ is called homogeneous of degree $\sigma \in \mathbb{R}$ with dilation $(r_1, \ldots, r_m)$, if $\omega_i(\epsilon^{r_1} z_1, \ldots, \epsilon^{r_m} z_m) = \epsilon^{\sigma+r_i}\omega_i(z_1, \ldots, z_m)$, where $r_i > 0$ and $i \in [m]$.

Lemma 2 ([39])**.** If $\omega(z)$ is homogeneous of degree $\sigma < 0$ with a dilation $(r_1, \ldots, r_m)$, where $\omega(z)$ is continuous and its asymptotically stable equilibrium is zero, then the solutions of (1) can reach zero in finite time.

The following definition and lemma clarify when the fixed-time stability can be obtained for system (1).

Definition 1 (Homogeneity in the ℓ-limit ($\ell = 0$ or ∞) [18])**.** A vector field $\omega(z) : \mathbb{R}^m \to \mathbb{R}^m$ is said to be homogeneous in the ℓ-limit with associated triple $(r_\ell, \sigma_\ell, \omega_\ell)$, where $r_\ell = (r_{\ell,1}, \ldots, r_{\ell,m})$ is the weight with $r_{\ell,i} > 0$, $\sigma_\ell \in \mathbb{R}$ is the degree, and $\omega_\ell = (\omega_{\ell 1}, \ldots, \omega_{\ell m})^{\mathrm{T}}$ is the approximating vector field in the

ℓ-limit. If ω is continuous, for each $i \in [m]$, $\omega_{\ell i}$ is continuous and not identically zero, $\sigma_\ell + r_{\ell,i} \geqslant 0$, and, for each compact set $C \subsetneqq \mathbb{R}^m \setminus \{0\}$, the equation $\lim_{\varepsilon\to\ell} \max_{z\in C} |\frac{\omega_i(\varepsilon^{r_{\ell,1}} z_1,\ldots,\varepsilon^{r_{\ell,m}} z_m)}{\varepsilon^{\sigma_\ell + r_{\ell,i}}} - \omega_{\ell i}(z)| = 0$ holds.

Definition 2 ([18]). If a vector field $\omega : \mathbb{R}^m \to \mathbb{R}^m$ is homogeneous in the 0-limit and homogeneous in the ∞-limit simultaneously, it is called homogeneous in the bi-limit.

Lemma 3 ([18]). Suppose the vector field $\omega : \mathbb{R}^m \to \mathbb{R}^m$ is homogeneous in the bi-limit with associated triples $(r_\infty, \sigma_\infty, \omega_\infty)$ and $(r_0, \sigma_0, \omega_0)$. If the origins of the systems

$$\dot{z} = \omega(z), \quad \dot{z} = \omega_\infty(z), \quad \dot{z} = \omega_0(z)$$

are globally asymptotically stable and $\sigma_\infty > 0 > \sigma_0$, then all solutions of the system (1) converge to the origin in fixed time.

3 Main results

3.1 Problem statement

The following second-order MAS consisting of N agents is considered:

$$\dot{p}_i(t) = v_i(t), \quad \dot{v}_i(t) = u_i(t), \quad i \in [N], \tag{2}$$

where $p_i(t) \in \mathbb{R}^n$ and $v_i(t) \in \mathbb{R}^n$ represent the position and the velocity states of the ith agent, respectively, and $u_i(t) \in \mathbb{R}^n$ is a control input to be designed later. Denote $p(t) = (p_1^{\mathrm{T}}, \ldots, p_N^{\mathrm{T}})^{\mathrm{T}}$ and $v(t) = (v_1^{\mathrm{T}}, \ldots, v_N^{\mathrm{T}})^{\mathrm{T}}$.

Definition 3. FinTC is said to be achieved by the second-order MAS (2), if there exists a finite time $T(p(0), v(0)) > 0$, such that $\lim_{t\to T(p(0),v(0))} ||p_i(t) - p_j(t)|| = 0$, $\lim_{t\to T(p(0),v(0))} ||v_i(t) - v_j(t)|| = 0$, and $p_i(t) = p_j(t)$, $v_i(t) = v_j(t)$, $\forall t \geqslant T$, $i, j \in [N]$. Furthermore, if there exists $T_{\max}$ such that $T(p(0), v(0)) \leqslant T_{\max}$ for all $(p(0)^{\mathrm{T}}, v(0)^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{2nN}$, then the second-order MAS (2) is said to achieve FixTC.

In the following, fully distributed control protocols will be designed to make the MAS (2) reach FixTC without using velocity information when the considered network is time-invariant. Then, if the graph is state-dependent, a fully distributed control protocol will be further designed without employing velocity measurements which can preserve the initial edges and reach FinTC.

The following lemma is a standard result of an undirected graph and is useful for deducing theoretical results.

Lemma 4 ([40]). The equation $\sum_{i=1}^N \sum_{j=1}^N q_{ij} y_i^{\mathrm{T}} \phi(x_i - x_j) = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N q_{ij} (y_i - y_j)^{\mathrm{T}} \phi(x_i - x_j)$ holds for $y_i, x_i \in \mathbb{R}^n$, where $Q = [q_{ij}] \in \mathbb{R}^{N\times N}$ is a symmetric matrix and $\phi(\cdot)$ is an odd function.

3.2 FixTC protocol under a fixed topology

In this subsection, we assume that $\mathcal{G}(t)$ is time-invariant and connected. Our goal is to propose a FixTC protocol without using velocity information.

This protocol is designed as follows:

$$\begin{aligned} u_i(t) =& -l_1 \sum_{j=1}^N a_{ij} (p_i - p_j)^{[\alpha_1]} - l_2 \sum_{j=1}^N a_{ij} (p_i - p_j)^{[\alpha_2]} + \gamma \dot{z}_i(t), \\ \dot{z}_i(t) =& -c_1 z_i^{[\beta_1]} - c_2 z_i^{[\beta_2]} - l_1 \sum_{j=1}^N a_{ij} (p_i - p_j)^{[\alpha_1]} - l_2 \sum_{j=1}^N a_{ij} (p_i - p_j)^{[\alpha_2]}, \quad i \in [N], \end{aligned} \tag{3}$$

where $l_1, l_2, \gamma, c_1, c_2$ are some positive feedback gains, $z_i(t)$ is an auxiliary variable, $\alpha_1 \in (0, 1), \alpha_2 > 1$ and $\beta_i = \frac{2\alpha_i}{1+\alpha_i}, i = 1, 2$.

Theorem 1. Assume that the undirected time-invariant communication graph $\mathbb{G}$ is connected; then, FixTC can be reached for the MAS (2) with control protocol (3).

Proof. Step 1: First, we prove that the system (2) with (3) is globally asymptotically stable.

Choose a Lyapunov candidate function:

$$V(t) = \frac{l_1}{1+\alpha_1}\sum_{i,j=1}^{N} a_{ij}||p_i - p_j||^{1+\alpha_1} + \frac{l_2}{1+\alpha_2}\sum_{i,j=1}^{N} a_{ij}||p_i - p_j||^{1+\alpha_2} + \sum_{i=1}^{N}||v_i - \gamma z_i||^2 + \sum_{i=1}^{N}\gamma||z_i||^2,$$

which is a positive definite function with respect to $p_i - p_j, i \neq j$, z_i, and $v_i - \gamma z_i, i, j \in [N]$.

Differentiating $V(t)$ gives

$$\begin{aligned}\dot{V}(t) =& l_1\sum_{i,j=1}^{N} a_{ij}(v_i - v_j)^{\mathrm{T}}(p_i - p_j)^{[\alpha_1]} + l_2\sum_{i,j=1}^{N} a_{ij}(v_i - v_j)^{\mathrm{T}}(p_i - p_j)^{[\alpha_2]}\\ &+ 2\sum_{i=1}^{N}(v_i - \gamma z_i)^{\mathrm{T}}(\dot{v}_i - \gamma\dot{z}_i) + 2\sum_{i=1}^{N}\gamma z_i^{\mathrm{T}}\dot{z}_i\\ =& 2l_1\sum_{i,j=1}^{N} a_{ij}v_i^{\mathrm{T}}(p_i - p_j)^{[\alpha_1]} + 2l_2\sum_{i,j=1}^{N} a_{ij}v_i^{\mathrm{T}}(p_i - p_j)^{[\alpha_2]}\\ &+ 2\sum_{i=1}^{N}(v_i - \gamma z_i)^{\mathrm{T}}\bigg(-l_1\sum_{j=1}^{N} a_{ij}(p_i - p_j)^{[\alpha_1]} - l_2\sum_{j=1}^{N} a_{ij}(p_i - p_j)^{[\alpha_2]}\bigg)\\ &+ 2\sum_{i=1}^{N}\gamma z_i^{\mathrm{T}}\bigg(-c_1 z_i^{[\beta_1]} - c_2 z_i^{[\beta_2]} - l_1\sum_{j=1}^{N} a_{ij}(p_i - p_j)^{[\alpha_1]} - l_2\sum_{j=1}^{N} a_{ij}(p_i - p_j)^{[\alpha_2]}\bigg)\\ =& -2\gamma\sum_{i=1}^{N}(c_1||z_i||^{1+\beta_1} + c_2||z_i||^{1+\beta_2}).\end{aligned}$$

Herein, Lemma 4 is used to derive the second equation. Define the invariant set $\Omega = \{(p_1^{\mathrm{T}}, \ldots, p_N^{\mathrm{T}}, v_1^{\mathrm{T}}, \ldots, v_N^{\mathrm{T}}, z_1^{\mathrm{T}}, \ldots, z_N^{\mathrm{T}}) | \dot{V}(t) = 0\}$. From $\dot{V}(t) = 0$, one obtains $z_i = 0, i \in [N]$. Because the graph is undirected and connected according to (3), $z_i = 0$ means $p_1 = p_2 = \cdots = p_N$. Furthermore, we find that $v_1 = v_2 = \cdots = v_N$. Thus, by invoking the nonsmooth LaSalle invariance principle [41], $p_i - p_j \to 0$, $v_i - v_j \to 0, z_i \to 0$ as $t \to \infty$ for $i, j \in [N]$.

Step 2: Let $\tilde{p}_i = p_i - \bar{p}$, $\tilde{v}_i = v_i - \bar{v}$ and $\tilde{z}_i = z_i$, where $\bar{p} = \frac{1}{N}\sum_{j=1}^{N} p_j$ and $\bar{v} = \frac{1}{N}\sum_{j=1}^{N} v_j$. We can obtain the error system of (2) with (3) by transforming the equilibrium point to the origin:

$$\begin{aligned}\dot{\tilde{p}}_i(t) =& \tilde{v}_i,\\ \dot{\tilde{v}}_i(t) =& -\gamma c_1\tilde{z}_i^{[\beta_1]} - l_1(1+\gamma)\sum_{j=1}^{N} a_{ij}(\tilde{p}_i - \tilde{p}_j)^{[\alpha_1]}\\ &- \gamma c_2\tilde{z}_i^{[\beta_2]} - l_2(1+\gamma)\sum_{j=1}^{N} a_{ij}(\tilde{p}_i - \tilde{p}_j)^{[\alpha_2]} - \gamma c_1\frac{1}{N}\sum_{j=1}^{N}\tilde{z}_j^{[\beta_1]} - \gamma c_2\frac{1}{N}\sum_{j=1}^{N}\tilde{z}_j^{[\beta_2]},\\ \dot{\tilde{z}}_i(t) =& -c_1\tilde{z}_i^{[\beta_1]} - c_2\tilde{z}_i^{[\beta_2]} - l_1\sum_{j=1}^{N} a_{ij}(\tilde{p}_i - \tilde{p}_j)^{[\alpha_1]} - l_2\sum_{j=1}^{N} a_{ij}(\tilde{p}_i - \tilde{p}_j)^{[\alpha_2]}, \quad i \in [N].\end{aligned} \tag{4}$$

It is possible to check that Eq. (4) is homogeneous in the bi-limit, where its approximating systems in the 0-limit and in the ∞-limit, respectively, are listed as follows:

$$\begin{aligned}\dot{\tilde{p}}_i(t) =& \tilde{v}_i,\\ \dot{\tilde{v}}_i(t) =& -\gamma c_1\tilde{z}_i^{[\beta_1]} - l_1(1+\gamma)\sum_{j=1}^{N} a_{ij}(\tilde{p}_i - \tilde{p}_j)^{[\alpha_1]} - \gamma c_1\frac{1}{N}\sum_{j=1}^{N}\tilde{z}_j^{[\beta_1]},\\ \dot{\tilde{z}}_i(t) =& -c_1\tilde{z}_i^{[\beta_1]} - l_1\sum_{j=1}^{N} a_{ij}(\tilde{p}_i - \tilde{p}_j)^{[\alpha_1]}, \quad i \in [N],\end{aligned} \tag{5}$$

$$\begin{aligned}\dot{\tilde{p}}_i(t) =&\tilde{v}_i,\\ \dot{\tilde{v}}_i(t) =&-\gamma c_2\tilde{z}_i^{[\beta_2]}-l_2(1+\gamma)\sum_{j=1}^N a_{ij}(\tilde{p}_i-\tilde{p}_j)^{[\alpha_2]}-\gamma c_2\frac{1}{N}\sum_{j=1}^N\tilde{z}_j^{[\beta_2]},\\ \dot{\tilde{z}}_i(t) =&-c_2\tilde{z}_i^{[\beta_2]}-l_1\sum_{j=1}^N a_{ij}(\tilde{p}_i-\tilde{p}_j)^{[\alpha_2]},\quad i\in[N].\end{aligned}\tag{6}$$

It can be shown that the approximating system in the 0-limit (5) with variables $(\tilde{p}_1^{\mathrm{T}},\ldots,\tilde{p}_N^{\mathrm{T}},\tilde{v}_1^{\mathrm{T}},\ldots,\tilde{v}_N^{\mathrm{T}},\tilde{z}_1^{\mathrm{T}},\ldots,\tilde{z}_N^{\mathrm{T}})^{\mathrm{T}}$ is homogeneous of degree $\sigma_0=\alpha_1-1<0$ with dilation

$$(\underbrace{2,\ldots,2}_{nN},\underbrace{1+\alpha_1,\ldots,1+\alpha_1}_{nN},\underbrace{1+\alpha_1,\ldots,1+\alpha_1}_{nN});$$

meanwhile, the approximating system in the ∞-limit (6) with variables $(\tilde{p}_1^{\mathrm{T}},\ldots,\tilde{p}_N^{\mathrm{T}},\tilde{v}_1^{\mathrm{T}},\ldots,\tilde{v}_N^{\mathrm{T}},\tilde{z}_1^{\mathrm{T}},\ldots,\tilde{z}_N^{\mathrm{T}})^{\mathrm{T}}$ is homogeneous of degree $\sigma_\infty=\alpha_2-1>0$ with dilation

$$(\underbrace{2,\ldots,2}_{nN},\underbrace{1+\alpha_2,\ldots,1+\alpha_2}_{nN},\underbrace{1+\alpha_2,\ldots,1+\alpha_2}_{nN}).$$

From Step 1, one can conclude that the error system (4) is globally asymptotically stable. Following a similar procedure in Step 1, it is easy to prove that systems (5) and (6) are all globally asymptotically stable. From the above analysis and Lemma 3, we can conclude that the MAS (2) with control protocol (3) can reach FixTC.

Corollary 1. Assume that the undirected time-invariant graph $\mathbb{G}$ is connected; then FixTC can be reached for the MAS (2) using control protocol (7):

$$\begin{aligned}u_i(t) =&-\sum_{j=1}^N(l_1a_{ij}(p_i-p_j)^{[\alpha_1]}+l_2a_{ij}(p_i-p_j)^{[\alpha_2]})+\gamma\dot{z}_i(t),\\ \dot{z}_i(t) =&-\sum_{j=1}^N(c_1a_{ij}(z_i-z_j)^{[\beta_1]}+c_2a_{ij}(z_i-z_j)^{[\beta_2]}+l_1a_{ij}(p_i-p_j)^{[\alpha_1]}+l_2a_{ij}(p_i-p_j)^{[\alpha_2]}),\end{aligned}\tag{7}$$

where l_1,l_2,γ,c_1,c_2 are some positive feedback gains, $z_i(t)$ is an auxiliary variable, $\alpha_1\in(0,1),\alpha_2>1$ and $\beta_i=\frac{2\alpha_i}{1+\alpha_i},i=1,2$.

Proof. This proof follows a similar procedure to that used in Theorem 1 and can be skipped for brevity.

Remark 1. Both control protocols (3) and (7) are fully distributed because each controller only uses information from itself and its neighbors. Under control protocol (3), agent needs to only exchange position information with its neighbors, whereas control protocol (7) requires the communication of the corresponding position and auxiliary variables. In [36], a differentiator was utilized to estimate relative velocities within a fixed time. However, neighbors' input information had to be known in that case. Unlike [36], the auxiliary variable z_i introduced here depends only on relative state information and compensates for relative velocity variation.

Remark 2. Herein, the final consensus value will be further analyzed. In fact, according to (2) and (3), one obtains the final consensus value as $\bar{p}(t)=\frac{1}{N}\sum_{i=1}^N p_i(t)=\frac{1}{N}\sum_{i=1}^N(p_i(0)+t(v_i(0)-\gamma z_i(0)))$, and $\bar{v}(t)=\frac{1}{N}\sum_{i=1}^N v_i(t)=\frac{1}{N}\sum_{i=1}^N(v_i(0)-\gamma z_i(0))$. If one chooses $z_i(0)=0,i\in[N]$, the final consensus value will not be affected by the initial values of auxiliary variable $z_i(0)$.

3.3 Connectivity-preserving FinTC protocol under a state-dependent topology

In accordance with the literature [33–35], we consider state-dependent communication in this subsection, which means that any two agents can only communicate with each other when their distance falls within a certain range.

Assumption 1. There is a connection (an undirected edge) between two agents when and only when the distance between them is within a certain range R, i.e., for the neighbors of agent i at time t, $\mathcal{N}_i(t) = \{j \big| ||p_i(t) - p_j(t)|| < R\}$.

Design the finite-time connectivity-preserving consensus protocol as follows:

$$\begin{aligned} u_i(t) =& -l\sum_{j=1}^{N} \varpi_{ij}(t)(p_i - p_j)^{[\alpha]} + \gamma \dot{z}_i(t), \\ \dot{z}_i(t) =& -cz_i^{[\beta]} - l\sum_{j=1}^{N} \varpi_{ij}(t)(p_i - p_j)^{[\alpha]}, \quad i \in [N], \end{aligned} \tag{8}$$

where l, γ, c are some positive feedback gains, $\alpha \in (0,1)$, $\beta = \frac{\alpha}{1+\alpha}$, and $\varpi_{ij}(t) = \varpi(||p_i(t) - p_j(t)||)$ is designed as

$$\varpi_{ij}(t) = \begin{cases} \psi(||p_i(t) - p_j(t)||), & ||p_{ij}(0)|| < R \text{ and } ||p_{ij}(t)|| < R, \\ \ell, & ||p_{ij}(0)|| \geqslant R \text{ and } ||p_{ij}(t)|| < R, \\ 0, & \text{otherwise}, \end{cases}$$

where $p_{ij}(t) = p_i(t) - p_j(t)$ and $\psi(s) : [0, R) \to [0, \infty)$ is an artificial potential function which satisfies the following properties.

(1) $\psi(s)$ is continuous on $[0, R)$. $\psi(s) = \ell$ when $s \in [0, \tau R]$, where $\ell \geqslant 1$ is a positive constant; $\psi(s) \geqslant \ell$ when $s \in [\tau R, R)$ with $\tau \in (0,1)$.

(2) $\int_0^r \psi(s)s^\alpha \mathrm{d}s \to \infty$ when $r \to R$.

Remark 3. The first property of $\psi(s)$ is to make homogeneous condition satisfied which requires that $\psi(s)$ is a constant in the region $[0, \tau R]$. Furthermore, the condition $\psi(s) \geqslant \psi(\tau R) \geqslant 1$ for $s \in [0, R)$ ensures that the weight will not be too small or else the consensus rate will be slow. The second property of $\psi(s)$ ensures that the initial edges will not be lost as time evolves. The following function is one example that was originally used in [35]:

$$\psi(s) = \begin{cases} \dfrac{R^{\alpha+1}}{R^{\alpha+1} - (\tau R)^{\alpha+1}}, & 0 \leqslant s \leqslant \tau R, \\ \dfrac{R^{\alpha+1}}{R^{\alpha+1} - s^{\alpha+1}}, & \tau R < s < R. \end{cases} \tag{9}$$

Remark 4. Note that the control protocol (8) is discontinuous because $\varpi_{ij}(t)$ is discontinuous. Therefore, the solutions of system (2) with control protocol (8) are understood in the Filippov sense [38]. Since the right-hand side of (2) with (8) is measurable and locally essentially bounded, there exist Filippov solutions of (2) based on Lemma 1.

The following energy function is used in this subsection:

$$W(t) = l\sum_{i,j=1}^{N} \int_0^{||p_i - p_j||} \varpi(s)s^\alpha \mathrm{d}s + \sum_{i=1}^{N} ||v_i - \gamma z_i||^2 + \sum_{i=1}^{N} \gamma ||z_i||^2.$$

Theorem 2. If $\mathbb{G}(0)$ is connected and $W(0)$ is finite, then for MAS (2) with control protocol (8), the following statements hold.

(1) The initial edges will always be preserved for $t \geqslant 0$.

(2) The FinTC can be achieved.

Proof. (1) T_1 is assumed to be the first finite-time instant satisfying $\lim_{t \to T_1^-} ||p_i(t) - p_j(t)|| = R$ for some $(i,j) \in \mathcal{E}(0)$. That is, the initial edges will always be maintained on the time interval $[0, T_1)$. Let ω be the right-hand side of system (2) with (8). The set-valued Lie derivative of $W(t)$ is calculated as follows:

$$\mathcal{L}_\omega W(t) = \mathcal{K}\left\{ l\sum_{i,j=1}^{N} \varpi_{ij}(t)(v_i - v_j)^{\mathrm{T}}(p_i - p_j)^{[\alpha]} + 2\sum_{i=1}^{N}(v_i - \gamma z_i)^{\mathrm{T}}(\dot{v}_i - \gamma \dot{z}_i) + 2\sum_{i=1}^{N} \gamma z_i^{\mathrm{T}} \dot{z}_i \right\}$$

$$
\begin{aligned}
=&\mathcal{K}\Bigg\{2l\sum_{i,j=1}^{N}\varpi_{ij}(t)v_i^{\mathrm{T}}(p_i-p_j)^{[\alpha]}+2\sum_{i=1}^{N}(v_i-\gamma z_i)^{\mathrm{T}}\Bigg(-l\sum_{j=1}^{N}\varpi_{ij}(t)(p_i-p_j)^{[\alpha]}\Bigg)\\
&+2\sum_{i=1}^{N}\gamma z_i^{\mathrm{T}}\Bigg(-cz_i^{[\beta]}-l\sum_{j=1}^{N}\varpi_{ij}(t)(p_i-p_j)^{[\alpha]}\Bigg)\Bigg\}\\
=&-2\gamma c\sum_{i=1}^{N}||z_i||^{1+\beta}. \qquad (10)
\end{aligned}
$$

Herein, Lemma 4 is utilized to derive the second equation. Based on Eq. (10), one has $\mathcal{L}_\omega W(t)\leqslant 0$, from which it follows that W does not increase. On the contrary, if there exist agents i' and j' satisfying $(i',j')\in\mathcal{E}(0)$ and $\lim_{t\to T_1^-}||p_{i'j'}(t)||=R$, then $\int_0^{||p_{i'j'}(t)||}\varpi(s)s^\alpha\mathrm{d}s=\int_0^{||p_{i'j'}(t)||}\psi(s)s^\alpha\mathrm{d}s\to+\infty$ based on the second property of ψ. This means $W(t)\to\infty$ as $t\to T_1$, which contradicts the fact that $\mathcal{L}_\omega W(t)\leqslant 0$. Therefore, the initial edges will be always be preserved for $t\geqslant 0$.

(2) Define the invariant set $S=\{(p_1^{\mathrm{T}},\ldots,p_N^{\mathrm{T}},v_1^{\mathrm{T}},\ldots,v_N^{\mathrm{T}},z_1^{\mathrm{T}},\ldots,z_N^{\mathrm{T}})|0\in\mathcal{L}_fW(t)\}$. Note that $\dot{W}(t)=0$ implies $z_i=0,i\in[N]$. Because the graph is always connected, from (8), $z_i=0$ means that $p_1=p_2=\cdots=p_N$. Furthermore, we obtain that $v_1=v_2=\cdots=v_N$. Therefore, by invoking the invariance principle for discontinuous dynamic systems [42], we obtain $p_i-p_j\to 0$, $v_i-v_j\to 0$, $z_i\to 0$ when $t\to\infty$ for $i,j\in[N]$.

Let $\tilde{p}_i=p_i-\bar{p}$, $\tilde{v}_i=v_i-\bar{v}$ and $\tilde{z}_i=z_i$, where $\bar{p}=\frac{1}{N}\sum_{j=1}^{N}p_j$ and $\bar{v}=\frac{1}{N}\sum_{j=1}^{N}v_j$. We obtain the error system of (2) with (8) by transforming the equilibrium point to the origin:

$$
\begin{aligned}
\dot{\tilde{p}}_i(t)=&\tilde{v}_i,\\
\dot{\tilde{v}}_i(t)=&-\gamma c\tilde{z}_i^{[\beta]}-l(1+\gamma)\sum_{j=1}^{N}\varpi_{ij}(t)(\tilde{p}_i-\tilde{p}_j)^{[\alpha]}-\gamma c\frac{1}{N}\sum_{j=1}^{N}\tilde{z}_j^{[\beta]},\\
\dot{\tilde{z}}_i(t)=&-c\tilde{z}_i^{[\beta]}-l\sum_{j=1}^{N}\varpi_{ij}(t)(\tilde{p}_i-\tilde{p}_j)^{[\alpha]},\quad i\in[N].
\end{aligned}
\qquad (11)
$$

Let $\bar{\Omega}=\{(p_1^{\mathrm{T}},\ldots,p_N^{\mathrm{T}},v_1^{\mathrm{T}},\ldots,v_N^{\mathrm{T}},z_1^{\mathrm{T}},\ldots,z_N^{\mathrm{T}})|||p_i(t)-p_j(t)||\leqslant\tau R,i,j\in[N]\}$. From the above analysis, it is apparent that the error system (11) is globally asymptotically stable. Thus, there exists a finite time T such that $(p_1^{\mathrm{T}},\ldots,p_N^{\mathrm{T}},v_1^{\mathrm{T}},\ldots,v_N^{\mathrm{T}},z_1^{\mathrm{T}},\ldots,z_N^{\mathrm{T}})\in\bar{\Omega}$ for all $t\geqslant T$. Moreover, $\varpi_{ij}(t)=\psi(\tau R)$ for $t\geqslant T$ based on the definition of $\varpi_{ij}(t)$. Then, the error system (11) with variables $(\tilde{p}_1^{\mathrm{T}},\ldots,\tilde{p}_N^{\mathrm{T}},\tilde{v}_1^{\mathrm{T}},\ldots,\tilde{v}_N^{\mathrm{T}},\tilde{z}_1^{\mathrm{T}},\ldots,\tilde{z}_N^{\mathrm{T}})^{\mathrm{T}}$ is homogeneous of degree $\sigma=\alpha-1<0$ with dilation

$$
(\underbrace{2,\ldots,2}_{nN},\underbrace{1+\alpha,\ldots,1+\alpha}_{nN},\underbrace{1+\alpha,\ldots,1+\alpha}_{nN}).
$$

Then, based on Lemma 2, FinTC can be achieved for MAS (2) with (8).

Remark 5. In the proof of Theorem (2), the initial energy $W(0)$ must be finite, implying that there do not exist any two neighboring agents with a geometrical distance infinitely close to R at $t=0$.

4 Simulations

To verify the two control protocols (3) and (8), two numerical examples will be presented in this section.

Example 1. In this example, the graph is set time-invariant, as shown in Figure 1. Consider the MAS (2) with the FixTC control protocol (3) and choose the control parameters $l_1=2,l_2=3,c_1=c_2=\gamma=1,\alpha_1=2/3,\alpha_2=5/4,\beta_1=4/5$, and $\beta_2=10/9$. The initial values are chosen as $p(0)=(-3,1,5,-9,-5)^{\mathrm{T}},v(0)=(3,-4,-2,2,5)^{\mathrm{T}}$, and $z(0)=(0,0,0,0,0)^{\mathrm{T}}$. The trajectories of the state are illustrated in Figure 2 and the control inputs are shown in Figure 3.

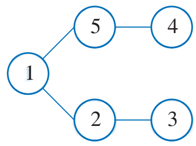

Figure 1 (Color online) Connected graph $\mathbb{G}(0)$.

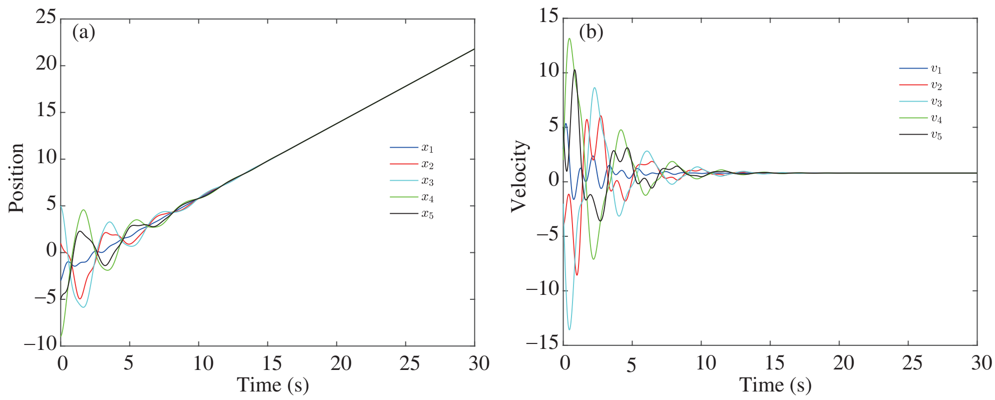

Figure 2 (Color online) Trajectories of (a) positions and (b) velocities under control protocol (3).

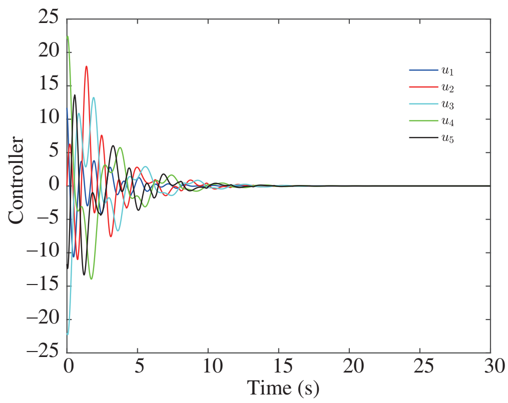

Figure 3 (Color online) Evolution of control inputs (3).

Example 2. In this example, it is assumed that the graph is state-dependent and that the sensing range is $R = 3$. Consider the MAS (2) with the connectivity-preserving FinTC control protocol (8) and choose the control parameters $l = 1.5, c = \gamma = 1$, $\alpha = 1/2, \beta = 2/3$. Furthermore, let $\tau = 0.2$. The initial values are chosen as $p(0) = (-3, -1, 1, -7, -5)^{\mathrm{T}}, v(0) = (3, -4, -2, 2, 5)^{\mathrm{T}}$, and $z(0) = (0, 0, 0, 0, 0)^{\mathrm{T}}$. The initial graph is shown in Figure 1. The trajectories of the states are depicted in Figure 4 and the control inputs are given in Figure 5. From Figure 6, one can observe that the original connections are preserved with control protocol (8) because the distances for adjacent agents in graph $\mathbb{G}(0)$ are always less than the sensing range.

5 Conclusion

Herein, two consensus problems for second-order MASs without velocity measurements were considered. The first was the FixTC problem with a time-invariant graph. A distributed FixTC protocol was designed without using velocity measurements based on the Lyapunov stability theory and the bi-limit homogeneity approach. The other problem was the connectivity-preserving FinTC problem with a state-dependent graph. By invoking the finite-time stability and homogeneity theories, a distributed connectivity-preserving FinTC protocol was designed using only output state information, which can preserve the initial edges and make the MAS reach FinTC. One main advantage of the homogeneity-

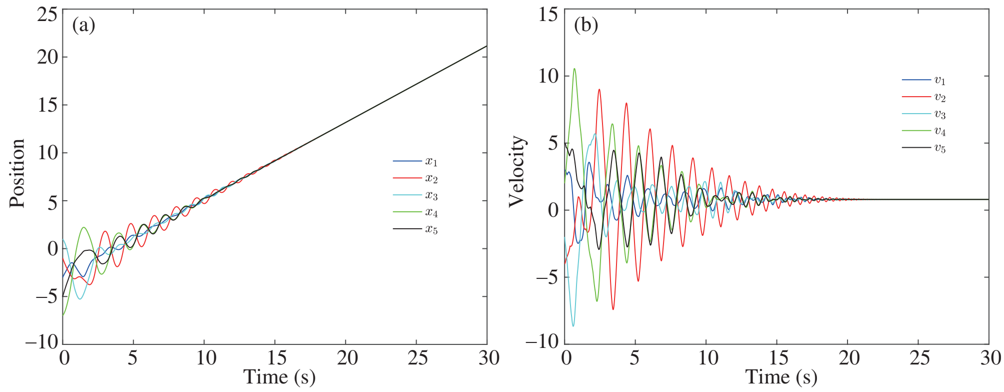

Figure 4 (Color online) Trajectories of (a) positions and (b) velocities under control protocol (8).

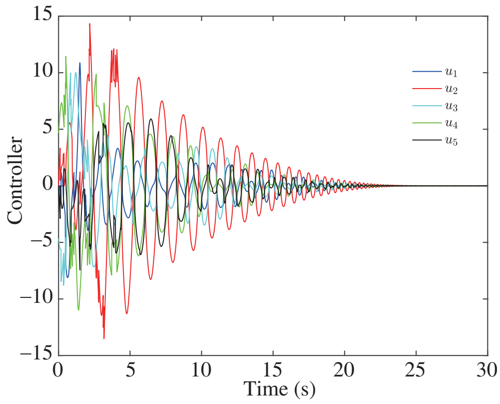

Figure 5 (Color online) Evolution of control inputs (8).

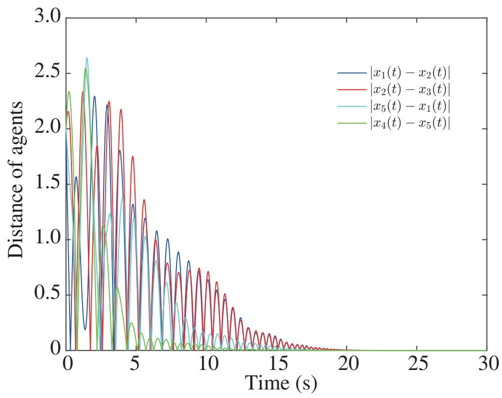

Figure 6 (Color online) The evolution of distances for adjacent agents in $\mathbb{G}(0)$ with control protocol (8).

based distributed controller designed here is the concise choice of parameters. However, the time could not be estimated easily. Future work will focus on precise convergence time estimation and the design of distributed FixTC protocols for nonlinear second- and higher-order MASs with reduced state information.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant No. 61673107), National Ten Thousand Talent Program for Young Top-notch Talents (Grant No. W2070082), Cheung Kong Scholars Programme of China for Young Scholars (Grant No. Q2016109), Jiangsu Provincial Key Laboratory of Networked Collective Intelligence (Grant No. BM2017002), and Scientific Research Foundation of Graduate School of Southeast University (Grant No. YBJJ1718).

References

1 Gazi V, Passino K M. Stability analysis of social foraging swarms. IEEE Trans Syst Man Cybern B, 2004, 34: 539–557
2 Olfati-Saber R. Flocking for multi-agent dynamic systems: algorithms and theory. IEEE Trans Automat Contr, 2006, 51: 401–420
3 Dimarogonas D V, Kyriakopoulos K J. On the rendezvous problem for multiple nonholonomic agents. IEEE Trans Automat Contr, 2007, 52: 916–922
4 Fax J A, Murray R M. Information flow and cooperative control of vehicle formations. IEEE Trans Automat Contr, 2004, 49: 1453–1464
5 Lin Z L. Control design in the presence of actuator saturation: from individual systems to multi-agent systems. Sci China Inf Sci, 2019, 62: 026201
6 Yu Y G, Zeng Z W, Li Z K, et al. Event-triggered encirclement control of multi-agent systems with bearing rigidity. Sci China Inf Sci, 2017, 60: 110203
7 Yu W W, Wang H, Hong H F, et al. Distributed cooperative anti-disturbance control of multi-agent systems: an overview. Sci China Inf Sci, 2017, 60: 110202
8 Yu W W, Wen G H, Chen G R, et al. Distributed Cooperative Control of Multi-agent Systems. Singapore: Wiley/Higher Education Press, 2016
9 Cortés J. Finite-time convergent gradient flows with applications to network consensus. Automatica, 2006, 42: 1993–

2000
10 Liu X Y, Lam J, Yu W W, et al. Finite-time consensus of multiagent systems with a switching protocol. IEEE Trans Neural Netw Learn Syst, 2016, 27: 853–862
11 Wang X L, Hong Y G. Finite-time consensus for multi-agent networks with second-order agent dynamics. IFAC Proc Vol, 2008, 41: 15185–15190
12 Khoo S Y, Xie L H, Man Z H. Robust finite-time consensus tracking algorithm for multirobot systems. IEEE/ASME Trans Mechatron, 2009, 14: 219–228
13 Li S H, Du H B, Lin X Z. Finite-time consensus algorithm for multi-agent systems with double-integrator dynamics. Automatica, 2011, 47: 1706–1712
14 Du H B, He Y G, Cheng Y Y. Finite-time synchronization of a class of second-order nonlinear multi-agent systems using output feedback control. IEEE Trans Circ Syst I, 2014, 61: 1778–1788
15 Yu W W, Wang H, Cheng F, et al. Second-order consensus in multiagent systems via distributed sliding mode control. IEEE Trans Cybern, 2017, 47: 1872–1881
16 Wang H, Yu W W, Wen G H, et al. Finite-time bipartite consensus for multi-agent systems on directed signed networks. IEEE Trans Circ Syst I, 2018, 65: 4336–4348
17 Du H B, Li S H, Qian C J. Finite-time attitude tracking control of spacecraft with application to attitude synchronization. IEEE Trans Automat Contr, 2011, 56: 2711–2717
18 Andrieu V, Praly L, Astolfi A. Homogeneous approximation, recursive observer design, and output feedback. SIAM J Control Opt, 2008, 47: 1814–1850
19 Polyakov A. Nonlinear feedback design for fixed-time stabilization of linear control systems. IEEE Trans Automat Contr, 2012, 57: 2106–2110
20 Parsegov S E, Polyakov A E, Shcherbakov P S. Fixed-time consensus algorithm for multi-agent systems with integrator dynamics. IFAC Proc Vol, 2013, 46: 110–115
21 Zuo Z Y, Tie L. A new class of finite-time nonlinear consensus protocols for multi-agent systems. Int J Control, 2014, 87: 363–370
22 Hong H F, Yu W W, Wen G H, et al. Distributed robust fixed-time consensus for nonlinear and disturbed multiagent systems. IEEE Trans Syst Man Cybern Syst, 2017, 47: 1464–1473
23 Wang H, Yu W W, Wen G H, et al. Fixed-time consensus of nonlinear multi-agent systems with general directed topologies. IEEE Trans Circ Syst II Exp Briefs, in press. doi. 10.1109/TCSII.2018.2886298
24 Zuo Z Y. Nonsingular fixed-time consensus tracking for second-order multi-agent networks. Automatica, 2015, 54: 305–309
25 Fu J, Wang J Z. Fixed-time coordinated tracking for second-order multi-agent systems with bounded input uncertainties. Syst Control Lett, 2016, 93: 1–12
26 Hong H F, Yu W W, Fu J J, et al. A novel class of distributed fixed-time consensus protocols for second-order nonlinear and disturbed multi-agent systems. IEEE Trans Netw Sci Eng, in press. doi:10.1109/TNSE.2018.2873060
27 Tian B L, Zuo Z Y, Wang H. Leader-follower fixed-time consensus of multi-agent systems with high-order integrator dynamics. Int J Control, 2017, 90: 1420–1427
28 Ji M, Egerstedt M. Distributed coordination control of multiagent systems while preserving connectedness. IEEE Trans Robot, 2007, 23: 693–703
29 Dimarogonas D V, Johansson K H. Decentralized connectivity maintenance in mobile networks with bounded inputs. In: Proceedings of IEEE International Conference on Robotics and Automation, Pasadena, 2008. 1507–1512
30 Gustavi T, Dimarogonas D V, Egerstedt M, et al. Sufficient conditions for connectivity maintenance and rendezvous in leader-follower networks. Automatica, 2010, 46: 133–139
31 Su H S, Wang X F, Chen G R. Rendezvous of multiple mobile agents with preserved network connectivity. Syst Control Lett, 2010, 59: 313–322
32 Feng Z, Sun C, Hu G Q. Robust connectivity preserving rendezvous of multirobot systems under unknown dynamics and disturbances. IEEE Trans Control Netw Syst, 2017, 4: 725–735
33 Cao Y C, Ren W, Casbeer D W, et al. Finite-time connectivity-preserving consensus of networked nonlinear agents with unknown Lipschitz terms. IEEE Trans Automat Contr, 2016, 61: 1700–1705
34 Dong J G. Finite-time connectivity preservation rendezvous with disturbance rejection. Automatica, 2016, 71: 57–61
35 Hong H F, Yu W W, Fu J J, et al. Finite-time connectivity-preserving consensus for second-order nonlinear multi-agent systems. IEEE Trans Control Netw Syst, in press. doi. 10.1109/TCNS.2018.2808599
36 Tian B L, Lu H C, Zuo Z Y, et al. Fixed-time leader-follower output feedback consensus for second-order multiagent systems. IEEE Trans Cybern, in press. doi. 10.1109/TCYB.2018.2794759
37 Zheng Y S, Zhu Y R, Wang L. Finite-time consensus of multiple second-order dynamic agents without velocity measurements. Int J Syst Sci, 2014, 45: 579–588
38 Filippov A F. Differential Equations With Discontinuous Right-Hand Side, Mathematics and Its Applications (Soviet Series). Boston: Kluwer, 1988
39 Bhat S P, Bernstein D S. Finite time stability of homogeneous systems. In: Proceedings of the 1997 American Control Conference, Albuquerque, 1997. 2513–2514
40 Ren W, Beard R W. Distributed Consensus in Multi-Vehicle Cooperative Control. London: Springer-Verlag, 2008
41 Rouche N, Habets P, Laloy M. Stability Theory by Liapunov's Direct Method. New York: Springer-Verlag, 1977
42 Alvarez J, Orlov I, Acho L. An invariance principle for discontinuous dynamic systems with application to a coulomb friction oscillator. J Dyn Sys Meas Control, 2000, 122: 687–690