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Bounds and constructions of optimal optical orthogonal codes with low correlation zone

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Dear editor,
In optical code division multiple access (CDMA) systems, optical orthogonal codes (OOC) should exhibit the following correlations to reduce multiple access interference: the non-trivial correlation between a sequence and its shift sequence and the correlation between sequence and shift sequence of other sequences should be as low as possible [1,2]. Herein, the shift values of a sequence are randomly distributed throughout the sequence period. In quasi-synchronous CDMA systems, the time delay of the system is not very large and is generally distributed only in areas smaller than the sequence period. The purpose of reducing multiple access interference can be achieved using an address code that has a very small correlation function in the zone near zero. Therefore, a two-dimensional optical orthogonal code with zero cross-correlation zone is introduced [3].

In this study, the concept of optical orthogonal code with low correlation zone (LCZ) is proposed, the theoretical bounds on the code including sequence length, sequence weight, sequence set size, sequence correlation, and LCZ length are derived, and two methods of constructing OOCs with LCZ are given.

Preliminaries. The concept of OOC with LCZ is proposed as follows.

Definition 1. Let N, ω, Z, λ_a, and λ_c be positive integers. An (N, ω, Z, λ_a, λ_c) OOC with LCZ ((N, ω, Z, λ_a, λ_c)-LCZ OOC) is a family C of $\{0,1\}$-sequences (codewords) of length N and Hamming weight ω satisfying the two correlation properties: (1) For any $\boldsymbol{x}=\{x(i)\}_{i=0}^{N-1}\in C$ and any integer τ, $0<\tau\leqslant Z$, $0<Z<N$, $R_{\boldsymbol{x},\boldsymbol{x}}(\tau)=\Sigma_{t=0}^{N-1}x(t)x(t+\tau)\leqslant\lambda_a$; (2) For any $\boldsymbol{x}\neq\boldsymbol{y}$, $\boldsymbol{x}=\{x(i)\}_{i=0}^{N-1}$, $\boldsymbol{y}=\{y(i)\}_{i=0}^{N-1}\in C$, and any integer τ, $0\leqslant\tau\leqslant Z$, $0<Z<N$, $R_{\boldsymbol{x},\boldsymbol{y}}(\tau)=\Sigma_{t=0}^{N-1}x(t)y(t+\tau)\leqslant\lambda_c$, where $i+\tau$ is addition modulo N.

Let (N, ω, Z, λ)-LCZ OOC denote an (N, ω, Z, λ_a, λ_c)-LCZ OOC owing to the property that $\lambda_a=\lambda_c=\lambda$. The term OOC with LCZ (LCZ OOC) is also used if parameters need not be listed.

When $Z=N-1$, (N, ω, Z, λ_a, λ_c)-LCZ OOC is ordinary (N, ω, λ_a, λ_c)-OOC. Hence, LCZ OOC is an extension of OOC.

Frequency hopping sequences (FHSs) that control the carrier frequencies play an important role in the frequency hopping communication system. The LCZ OOC is easily obtained by FHS set. Let F be a frequency slot set of size q. Let A be an FHS set with family size M and sequence length N over F. Let $h(a,b)=1$ for $a=b$ and $h(a,b)=0$ for $a\neq b$. Let $\boldsymbol{a}=\{a(i)\}_{i=0}^{N-1}$, $\boldsymbol{b}=\{b(i)\}_{i=0}^{N-1}\in A$, the periodic Hamming correlation function between them at time delay τ is $H_{\boldsymbol{a},\boldsymbol{b}}(\tau)=\Sigma_{i=0}^{N-1}h(a(i),b(i+\tau))$, where $i+\tau$ is addition modulo N. The set A is called low hit zone FHS set if for any τ, $0<\tau\leqslant Z$, $Z=1,2,\ldots,N-1$, and $\boldsymbol{a}\in A$, $H_{\boldsymbol{a},\boldsymbol{a}}(\tau)\leqslant H_a$, and if for any τ, $0\leqslant\tau\leqslant Z$, $Z=1,2,\ldots,N-1$,

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$\boldsymbol{a}, \boldsymbol{b} \in A$, and $\boldsymbol{a} \neq \boldsymbol{b}$, $H_{\boldsymbol{a},\boldsymbol{b}}(\tau) \leqslant H_c$, denoted by (N, q, M, Z, H_a, H_c)-LHZ FHSS.

Gong [4] proposed the concept of interleaved sequences. The interleaving technique is an important tool for sequence design. Thus, this interleaving technique is used to construct LCZ OOC in this study.

Let $\boldsymbol{a} = \{a(i)\}_{i=0}^{N-1}$ be a $\{0,1\}$-binary sequence, and $E = \{\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_{M-1}\}$ be the shift sequence set, $\boldsymbol{e}_j = (e_j(0), e_j(1))$, where $e_j(k) = 0, 1, \ldots, N-1$, $j = 0, 1, \ldots, M-1$, $k = 0, 1$. The sequence $\boldsymbol{x}_j = \{x_j(i)\}_{i=0}^{2N-1}$ is called the interleaved sequence associated with $\boldsymbol{a}$ and $\boldsymbol{e}_j$, denoted by $S^{\boldsymbol{e}_j}(\boldsymbol{a})$ if $x_j(i) = a(i_1 + e_j(i_2))$, $i = 2i_1 + i_2$, $i_1 = 0, 1, \ldots, N-1$, $i_2 = 0, 1$, where $i_1 + e_j(i_2)$ is addition modulo N. The sequence $\boldsymbol{a}$ is called the component sequence and $\boldsymbol{e}_j$ is called the shift sequence.

Let $\boldsymbol{e}_j = (e_j(0), e_j(1))$, $\boldsymbol{e}_l = (e_l(0), e_l(1))$, $d_0 = e_j(0) - e_l(0)$, $d_1 = e_j(1) - e_l(1)$, $d_2 = e_j(0) - e_l(1)$, $d_3 = e_j(1) - e_l(0) - 1$. $\boldsymbol{e}_j$ and $\boldsymbol{e}_l$ are inequivalent if $d_0 \neq d_1$ and $d_2 \neq d_3$, and $S^{\boldsymbol{e}_j}(\boldsymbol{a})$ and $S^{\boldsymbol{e}_l}(\boldsymbol{a})$ are shift inequivalent if $\boldsymbol{e}_j$ and $\boldsymbol{e}_l$ are inequivalent.

Bounds for LCZ OOC. A lower bound on the correlations of LCZ OOC is given in the following theorem.

Theorem 1. If C is $(N, \omega, Z, \lambda_a, \lambda_c)$-LCZ OOC with size M, then C has a lower bound as follows:

$$Nz\lambda_a + N(M-1)(z+1)\lambda_c \geqslant M\omega^2(z+1) - N\omega,$$

where z is an any integer satisfying $0 \leqslant z \leqslant Z$.

Let $Q(z) = \Sigma_{\boldsymbol{x},\boldsymbol{y}\in\mathcal{C}}\Sigma_{\tau=0}^{z} R_{\boldsymbol{x},\boldsymbol{y}}(\tau)$. From Definition 1, we have that $Q(z) = \Sigma_{\boldsymbol{x}\in\mathcal{C}} R_{\boldsymbol{x},\boldsymbol{x}}(0) + \Sigma_{\boldsymbol{x}\in\mathcal{C}}\Sigma_{\tau=1}^{z} R_{\boldsymbol{x},\boldsymbol{x}}(\tau) + \Sigma_{\boldsymbol{x},\boldsymbol{y}\in\mathcal{C},\boldsymbol{x}\neq\boldsymbol{y}}\Sigma_{\tau=0}^{z} R_{\boldsymbol{x},\boldsymbol{y}}(\tau) \leqslant M\omega + Mz\lambda_a + M(M-1)(z+1)\lambda_c$.

Let C be $(N, \omega, Z, \lambda_a, \lambda_c)$-LCZ OOC with size M, $C = \{\boldsymbol{x}_j | j = 0, 1, \ldots, M-1\}$, $\boldsymbol{x}_j = \{x_j(i)\}_{i=0}^{N-1}$, $j = 0, 1, \ldots, M-1$, $\boldsymbol{b}_k = \{x_j(k)\}_{j=0}^{M-1}$, $k = 0, 1, \ldots, N-1$, $m_k = \Sigma_{j=0}^{M-1} x_j(k)$, and $m_{k+\tau} = \Sigma_{j=0}^{M-1} x_j(k+\tau)$. Then $Q(z) = \Sigma_{k=0}^{N-1}\Sigma_{\tau=0}^{z} m_k m_{k+\tau}$, and $\Sigma_{k=0}^{N-1} m_k = M\omega$. From [5], we know that $Q(z) \geqslant \frac{(z+1)(M\omega)^2}{N}$. Therefore, Theorem 1 is established.

Corollary 1. Let C be $(N, \omega, Z, \lambda_a, \lambda_c)$-LCZ OOC with size M, and $\lambda_m = \max\{\lambda_a, \lambda_c\}$. For any positive integer z, $0 \leqslant z \leqslant Z$, we have

$$\lambda_m \geqslant \frac{M\omega^2(z+1) - N\omega}{N(Mz + M - 1)}.$$

For a given set of positive integers N, ω, Z, λ_a, and λ_c, the largest possible size of a $(N, \omega, Z, \lambda_a, \lambda_c)$-LCZ OOC is denoted by $\Phi(N, \omega, Z, \lambda_a, \lambda_c)$. To determine the exact value of $\Phi(N, \omega, Z, \lambda_a, \lambda_c)$ and construct a specific LCZ OOC that attains the maximum size is of interest. An upper bound on the maximum possible code size of LCZ OOC is given from Theorem 1 in the following theorem.

Theorem 2. Let C be $(N, \omega, Z, \lambda_a, \lambda_c)$-LCZ OOC. An upper bound on the maximum possible code size of C is given as follows:

$$\Phi(N, \omega, Z, \lambda_a, \lambda_c) \leqslant \left\lfloor \frac{N(\omega + Z\lambda_a - (Z+1)\lambda_c)}{(Z+1)(\omega^2 - N\lambda_c)} \right\rfloor.$$

Throughout this study, $\lfloor x \rfloor$ denotes the largest integer not exceeding x.

Corollary 2. Let $\Phi(N, \omega, Z, \lambda)$ denote the largest possible size of an (N, ω, Z, λ)-LCZ OOC, then

$$\Phi(N, \omega, Z, \lambda) \leqslant \left\lfloor \frac{N(\omega - \lambda)}{(Z+1)(\omega^2 - N\lambda)} \right\rfloor.$$

The LCZ OOC achieving this upper bound on the maximum possible code size is called optimal LCZ OOC.

Theorem 3. Let C and C' be optimal LCZ OOC and optimal OOC, respectively, with same length, Hamming weight, and correlation. Let $M(C)$ and $M(C')$ be the sizes of C and C', respectively. Then, $M(C) \geqslant M(C')$.

Theorem 3 is established by setting $M(C) = \lfloor \frac{N(\omega-\lambda)}{(Z+1)(\omega^2 - N\lambda)} \rfloor$, $M(C') = \lfloor \frac{\omega-\lambda}{\omega^2 - N\lambda} \rfloor$, and $\frac{N}{z+1} \geqslant 1$.

Under the condition of the same length, weight and correlation, the size of LCZ OOC is bigger than that of traditional OOC.

Construction of LCZ OOC. We propose two constructions of LCZ OOCs: by LHZ FHSS and by interleaving technique.

Construction 1. Step 1. Select a set $A = \{\boldsymbol{a}_j = \{a_j(i)\}_{i=0}^{L-1} | j = 0, 1, \ldots, M-1\}$ that is an optimal (L, q, M, Z, λ)-LHZ FHSS over Z_q, where $\gcd(L, q) = 1$, Z_q is the additive group of integers modulo q.

Step 2. Construct the set $X = \{\boldsymbol{x}_j = \{x_j(i)\}_{i=0}^{Lq-1} | j = 0, 1, \ldots, M-1\}$ as follows:

$$x_j(i) = x_j(i_1, i_2) = \begin{cases} 1, \ i_2 = a_j(i_1), \\ 0, \ \text{otherwise}, \end{cases}$$

where $i = qi_1 + i_2$, $i_1 = 0, 1, \ldots, L-1$, $i_2 = 0, 1, \ldots, q-1$.

Theorem 4. The set X obtained in Construction 1 is an optimal (Lq, L, Z, λ)-LCZ OOC with size M.

Let integer $\tau = q\tau_1 + \tau_2$, $\tau_1 = 0, 1, \ldots, L-1$, $\tau_2 = 0, 1, \ldots, q-1$, $\boldsymbol{x}_j, \boldsymbol{x}_k \in X$, $j, k = 0, 1, \ldots, M-1$, then $R_{\boldsymbol{x}_j\boldsymbol{x}_k}(\tau) = H_{\boldsymbol{a}_j, \boldsymbol{a}_{k-\tau_2}}(\tau_1)$, where $\boldsymbol{a}_k-\tau_2=\{a_k(i_1)-\tau_2\}_{i_1=0}^{L-1}$. When $\tau\in[0,Z]$, $\tau_1\in[0,Z]$, and $\tau_2=0$. For $\tau\in[0,Z]$ and j, $k=0,1,\ldots,M-1$, we have $R_{\boldsymbol{x}_j\boldsymbol{x}_k}(\tau)=H_{\boldsymbol{a}_j,\boldsymbol{a}_k}(\tau_1)\leqslant\lambda$. Therefore, X obtained in Construction 1 is a (Lq,L,Z,λ)-LCZ OOC with size M. Because A is optimal (L,q,M,Z,λ)-LHZ-FHSS, we have $\lambda=\lceil\frac{(MZ+M-q)L}{(MZ+M-1)q}\rceil$, i.e., $M=\lfloor\frac{Lq(L-\lambda)}{(Z+1)(L^2-Lq\lambda)}\rfloor$ from [5]. From Corollary 2, we know that X is optimal. Therefore, Theorem 4 is established.

Construction 2. Step 1. Select (N,k,λ)-difference set D. Let $\boldsymbol{a}=\{a(i)\}_{i=0}^{N-1}$ be the characteristic sequence of D, i.e., $a(i)=1$ if $i\in D$, $a(i)=0$ if $i\notin D$,$i=0,1,\ldots,N-1$.

Step 2. Let c, g and L be three positive integers, where $0<L<N$.

When L is even, let $M=\lfloor\frac{N-2}{L}\rfloor$, then

$$\boldsymbol{e}_j=\left(c-\frac{L}{2}j,c+\frac{L}{2}j+g\right),\ j=0,1,\ldots,M-1,$$

where $0\leqslant c\leqslant N-1$, $\frac{L}{2}+1\leqslant g\leqslant N-ML+\frac{L}{2}$, and $g\neq\frac{N+1-lL}{2}$, $l=0,1,\ldots,2M-2$ if N is odd.

When L is odd, let $M=\lfloor\frac{N-1}{L}\rfloor$, then

$$\boldsymbol{e}_j=\begin{cases}\left(c-\dfrac{L}{2}j,c+\dfrac{L}{2}j+g\right), & j\text{ is even},\\ \left(c+\dfrac{L}{2}j+g-\dfrac{1}{2},c-\dfrac{L}{2}j+\dfrac{1}{2}\right), & j\text{ is odd},\end{cases}$$

where $0\leqslant c\leqslant N-1$, $\frac{L+1}{2}+1\leqslant g\leqslant N-ML+\frac{L+1}{2}$, and $g\neq\frac{N+1-lL}{2}$, $l=0,1,\ldots,2M-2$, $j=0,1,\ldots,M-1$.

Construct shift sequence set $E=\{\boldsymbol{e}_j|j=0,1,\ldots,M-1\}$.

Step 3. Using interleaving techniques on the component sequence $\boldsymbol{a}$ and shift sequence set E, we obtain a set $X=\{\boldsymbol{x}_j=S^{\boldsymbol{e}_j}(\boldsymbol{a})|j=0,1,\ldots,M-1\}$.

Theorem 5. The set X obtained in Construction 2 is a $(2N,2k,L-1,2\lambda)$-LCZ OOC with size M.

From Theorem 2 in [6], we have $\min_{e_i\neq e_j\in E}\{d_0,d_1\}\geqslant\frac{L}{2}$ and $\min_{e_i,e_j\in E}\{d_2,d_3\}\geqslant\frac{L-1}{2}$. Therefore, $\tau_1<\min\{d_i|i=0,1,2,3\}$ if $\tau=2\tau_1+\tau_2\leqslant L-1$. From Theorem 3 in [6], we know that e_i and e_j are inequivalent for $e_i,e_j\in E$. Using Property 1 in [7], we deduce that the set X obtained in Construction 2 is a $(2N,2k,L-1,2\lambda)$-LCZ OOC with size M. Therefore, Theorem 5 is established.

Conclusion. In this study, the concept of LCZ OOC, which is the extension of ordinary OOC is proposed. The theoretical bounds on LCZ OOC involving sequence length, sequence Hamming weight, sequence set size, LCZ length, and maximum correlation value are derived. Using the theoretical bounds on LCZ OOC, we obtained the result that the size of LCZ OOC is bigger than that of ordinary OOC under the conditions of the same length, Hamming weight, and correlation. Additionally, LCZ OOCs can be constructed based on frequency hopping sequence set and interleaving technology.

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References

1 Salehi J A. Code division multiple-access techniques in optical fiber networks. I. Fundamental principles. IEEE Trans Commun, 1989, 37: 824–833
2 Salehi J A, Brackett C A. Code division multiple-access techniques in optical fiber networks. II. Systems performance analysis. IEEE Trans Commun, 1989, 37: 834–842
3 Ji J H, Luo Z, Zheng H X, et al. A new family of two-dimensional optical orthogonal codes with zero cross-correlation zone. In: Proceedings of 2015 Optoelectronics Global Conference (OGC 2015), 2015. 1–2
4 Gong G. Theory and applications of q-ary interleaved sequences. IEEE Trans Inform Theor, 1995, 41: 400–411
5 Peng D Y, Fan P Z, Lee M H. Lower bounds on the periodic Hamming correlations of frequency hopping sequences with low hit zone. Sci China Ser F-Inf Sci, 2006, 49: 208–218
6 Chen X Y, Xu C Q. Construction of shift distinct sequence sets with low or zero correlation zone based on interleaving technique. Acta Electron Sin, 2013, 41: 890–896
7 Zhou Z C, Tang X H, Gong G. A new class of sequences with zero or low correlation zone based on interleaving technique. IEEE Trans Inform Theor, 2008, 54: 4267–4273