SCIENCE CHINA
Information Sciences

• RESEARCH PAPER •

October 2019, Vol. 62 202205:1–202205:16
https://doi.org/10.1007/s11432-018-9755-7

Boundedness and stability of highly nonlinear hybrid neutral stochastic systems with multiple delays

Mingxuan SHEN[1,3], Chen FEI[2], Weiyin FEI[3*] & Xuerong MAO[4]

[1]*School of Science, Nanjing University of Science and Technology, Nanjing* 210094*, China;*
[2]*Glorious Sun School of Business and Management, Donghua University, Shanghai* 200051*, China;*
[3]*School of Mathematics and Physics, Anhui Polytechnic University, Wuhu* 241000*, China;*
[4]*Department of Mathematics and Statistics, University of Strathclyde, Glasgow* G1 1XH*, UK*

Received 8 November 2018/Accepted 12 December 2018/Published online 16 August 2019

Abstract This paper reports the boundedness and stability of highly nonlinear hybrid neutral stochastic differential delay equations (NSDDEs) with multiple delays. Without imposing linear growth condition, the boundedness and exponential stability of the exact solution are investigated by Lyapunov functional method. In particular, using the M-matrix technique, the mean square exponential stability is obtained. Finally, three examples are presented to verify our results.

Keywords hybrid delay systems, neutral stochastic systems, multiple delays, highly nonlinear

Citation Shen M X, Fei C, Fei W Y, et al. Boundedness and stability of highly nonlinear hybrid neutral stochastic systems with multiple delays. Sci China Inf Sci, 2019, 62(10): 202205, https://doi.org/10.1007/s11432-018-9755-7

1 Introduction

Since continuous-time Markov chains were introduced to describe stochastic systems which undergo abrupt changes in their parameters and structures, hybrid stochastic differential equations (SDEs) have been widely used in engineering and science. Moreover, stochastic differential delay equations (SDDEs) are often used to investigate systems whose evolution depends on not only the present state but also the past state [1–5]. Boundedness and stability are two of the most basic issues in analyzing hybrid SDDEs. However, most of the papers in this field only considered systems whose coefficients are bounded by linear functions [6–11]. Recently, some significant results are obtained for highly nonlinear stochastic delay systems. For example, Hu et al. [12] investigated the boundedness and stability of hybrid SDDEs without the linear growth condition, while Hu et al. [13] studied the robust stability and boundedness of nonlinear hybrid SDDEs by the method of M-matrix. Fei et al. [14] established delay dependent criteria for highly nonlinear hybrid SDDEs under the polynomial growth condition. Moreover, Fei et al. [15] discussed structured robust stability and boundedness under highly nonlinear condition by introducing a new Lyapunov function.

Many stochastic systems which not only depend on present and past states but also involve derivatives with delays are modeled by neutral stochastic differential delay equations (NSDDEs). Although extensive literature can be found in this area, we mention a few of them herein [16–23]. Also, many researchers have made efforts in regard to the stability of highly nonlinear neutral stochastic systems. For example, Luo et al. [24] established criteria for exponential stability of neutral SDEs that exhibit a time-dependent delay. Furthermore, Song and Shen [25] investigated stability criteria that not only covers a large class

* Corresponding author (email: wyfei@ahpu.edu.cn)

of highly nonlinear neutral stochastic functional differential equations but also handles general stability issues. Moreover, the stability of NSDDEs with Markov switching, also known as hybrid NSDDEs, is studied by many researchers. For example, Kolmanovskii et al. [26] established a fundamental theory for hybrid NSDDEs and discussed the boundedness and stability of the systems. Mao et al. [27] investigated almost sure stability of a class hybrid NSDDEs. Furthermore, Li and Deng [28] established a criterion for general decay rate of almost sure stability of hybrid NSDDEs with lévy noise using Lyapunov functional and M-matrix techniques. However, all the functions $V(x,t,i)$ in these references are required to have the same degree for each $i \in S$, which may not be satisfied by hybrid NSDDEs that have different structures in different modes. Moreover, neutral stochastic systems may not depend only on single delay. Zhao et al. [29] provided motivation with regards to the study of stability of linear neutral systems with multiple delays. Chen et al. [30] investigated delay dependent exponential stability of multiple time-varying delays neutral stochastic systems. For more information on the stability of neutral systems with multiple delays, we refer the reader to [31–33]. However, it is crucial to observe that all the multiple time delays systems discussed in these papers are linear systems. Considering [12,24,25,28,30], herein, we extend the linear multiple delays neutral stochastic systems to highly nonlinear hybrid NSDDEs. By applying the Lyapunov functional technique, a sufficient condition for the stability of highly nonlinear NSDDEs with multiple delays is established under mild conditions. In particular, mean square exponential stability is investigated using M-matrix method.

The important features of this paper are as follows:

• In establishing the theory on the boundedness and stability of hybrid NSDDEs with highly nonlinear coefficients, multiple delays are taken into consideration.

• New mathematical techniques are developed to handle multiple delays NSDDEs under highly nonlinear growth condition. For example, a general Lyapunov function is introduced.

• H_∞ stability and almost sure exponential stability are discussed.

The remaining sections of this paper are structured as follows. In Section 2, some preliminary definitions, assumptions and a key lemma are presented. Section 3 mainly discusses the existence and uniqueness of solutions of highly nonlinear NSDDEs with multiple delays. Also, using the Lyapunov functional and M-matrix methods, the boundedness and stability criteria for hybrid NSDDEs are discussed. In Section 4, three examples are given to illustrate the applicability of our results. Finally, a conclusion is drawn in Section 5.

2 Preliminary

Let $B(t) = (B_1(t), \dots, B_d(t))^{\mathrm{T}}$ be a d-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, P)$, where the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfies the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $\{r(t), t \geqslant 0\}$ be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \dots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$. (For the detail of Γ we refer the reader to [5, page 47].) With regard to what follows, we assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$.

Let $C(\mathbb{R}^n \times [-\tau, \infty); \mathbb{R}_+)$ denote the family of all continuous functions from $\mathbb{R}^n \times [-\tau, \infty)$ to $\mathbb{R}_+$. Let

$$f : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \times S \times \mathbb{R}_+ \to \mathbb{R}^n \quad \text{and} \quad g : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \times S \times \mathbb{R}_+ \to \mathbb{R}^{n\times d}$$

be Borel measurable functions. Let $\delta_l : \mathbb{R}_+ \to [0, \tau_l]$, $l = 1, \dots, m$ denote the variable time delay such that $\dot{\delta}_l(t) := \mathrm{d}\delta_l(t)/\mathrm{d}t \leqslant \bar{\delta}_l < 1$ for all $t \geqslant 0$.

Herein, we consider the following neutral hybrid NSDDE with multiple delays:

$$\begin{aligned} \mathrm{d}[X(t) - \Lambda(X(t-\delta_1(t)), r(t), t)] = {} & f(X(t), X(t-\delta_1(t)), \dots, X(t-\delta_m(t)), r(t), t)\mathrm{d}t \\ & + g(X(t), X(t-\delta_1(t)), \dots, X(t-\delta_m(t)), r(t), t)\mathrm{d}B(t) \end{aligned} \tag{1}$$

on $t \geqslant 0$ with initial data

$$\{X(t) : -\tau \leqslant t \leqslant 0\} = \xi \in C([-\tau, 0]; \mathbb{R}^n) \quad \text{and} \quad r(0) = i_0 \in S, \tag{2}$$

where $\tau = \max\{\tau_l : l = 1, \ldots, m\}$, $\Lambda : \mathbb{R}^n \times S \times \mathbb{R}_+ \to \mathbb{R}^n$, and $C([-\tau, 0]; \mathbb{R}^n)$ denotes the family of continuous functions ξ from $[-\tau, 0] \to \mathbb{R}^n$ with the norm $\|\xi\| = \sup_{-\tau \leqslant s \leqslant 0} |\xi(s)|$. We also assume that $f(0, \ldots, 0, i, t) = g(0, \ldots, 0, i, t) = \Lambda(0, i, t) = 0$.

Let $C^{2,1}(\mathbb{R}^n \times S \times \mathbb{R}_+; \mathbb{R}_+)$ denote the family of nonnegative functions $V(x, i, t)$ on $\mathbb{R}^n \times S \times \mathbb{R}_+$, which are continuously twice differentiable in x and once in t, and define an operator $LV : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \times S \times \mathbb{R}_+ \to R$ by

$$\begin{aligned}
& LV(x - \Lambda(y_1, i, t), y_1, \ldots, y_m, i, t) \\
& = V_t(x - \Lambda(y_1, i, t), i, t) + V_x(x - \Lambda(y_1, i, t), i, t) f(x, y_1, \ldots, y_m, i, t) \\
& \quad + \frac{1}{2}\mathrm{trace}\big[g^{\mathrm{T}}(x, y_1, \ldots, y_m, i, t) V_{xx}(x - \Lambda(y_1, i, t), i, t) g(x, y_1, \ldots, y_m, i, t)\big] \\
& \quad + \sum_{j=1}^{N} \gamma_{ij} V(x - \Lambda(y_1, i, t), j, t),
\end{aligned}$$

where $V_t(x, i, t) = \frac{\partial V(x,i,t)}{\partial t}$, $V_x(x, i, t) = \left(\frac{\partial V(x,i,t)}{\partial x_1}, \ldots, \frac{\partial V(x,i,t)}{\partial x_n}\right)$, and $V_{xx}(x, i, t) = \left(\frac{\partial^2 V(x,i,t)}{\partial x_k \partial x_l}\right)_{n \times n}$.

Assumption 1. For each integer $H \geqslant 1$, we assume there exists a positive constant $\bar{K}_H$ such that

$$\begin{aligned}
& |f(x, y_1, \ldots, y_m, i, t) - f(\bar{y}, \bar{y}_1, \ldots, \bar{y}_m, i, t)|^2 \vee |g(x, y_1, \ldots, y_m, i, t) - g(\bar{x}, \bar{y}_1, \ldots, \bar{y}_m, i, t)|^2 \\
& \leqslant \bar{K}_H \left(|x - \bar{x}|^2 + \sum_{l=1}^{m} |y_l - \bar{y}_l|^2 \right)
\end{aligned}$$

for those $x, \bar{x}, y_l, \bar{y}_l \in \mathbb{R}^n$ with $|x| \vee |\bar{x}| \vee |y_l| \vee |\bar{y}_l| \leqslant H$ and all $(i, t) \in S \times \mathbb{R}_+$.

Assumption 2. We assume there exist constants $\kappa_i \in (0, 1)$ such that

$$|\Lambda(a, i, t) - \Lambda(b, i, t)| \leqslant \kappa_i |a - b| \tag{3}$$

for all $a, b \in \mathbb{R}^n$, and $\Lambda(0, i, t) = 0$. Furthermore, we assume that $\kappa = \max\{\kappa_i, i = 1, \ldots, N\}$.

We now state Lemma 1 which plays crucial role in this paper.

Lemma 1. Define the quasi polynomial function $U(x) = a_h|x|^{\beta_h} + \cdots + a_1|x|^{\beta_1}$, $x \in \mathbb{R}^n$, where $|x|$ is the Euclidean norm of x, $a_i \geqslant 0$, $i = 1, \ldots, h-1$, $\beta_h \geqslant \beta_{h-1} \geqslant \cdots \geqslant \beta_1 \geqslant 0$ and $a_h > 0$. Let $\tau > 0$ and δ be a differentiable function from $\mathbb{R}_+ \to [0, \tau]$ such that $\mathrm{d}\delta(t)/\mathrm{d}t \leqslant \bar{\delta} < 1$. Assume $|\Lambda(X(t), r(t), t)| \leqslant \kappa |X(t)|$, $X(t) : [-\tau, \infty) \to \mathbb{R}^n$ is a continuous function, and $r(t)$ is a Markov chain on the state space S, where $X(t) = \xi(t)$, $t \in [-\tau, 0]$, $0 < \kappa < 1$. Fixing $\varepsilon > 0$ arbitrarily, we have the following conclusions:

(i)

$$\int_0^{\mathrm{T}} \mathrm{e}^{\varepsilon t} U(X(t - \delta(t))) \mathrm{d}t \leqslant \frac{\mathrm{e}^{\varepsilon\tau}}{1 - \bar{\delta}} \int_{-\tau}^{0} U(\xi(t)) \mathrm{d}t + \frac{\mathrm{e}^{\varepsilon\tau}}{1 - \bar{\delta}} \int_0^{\mathrm{T}} \mathrm{e}^{\varepsilon t} U(X(t)) \mathrm{d}t, \quad \forall T > 0; \tag{4}$$

(ii)

$$\int_0^{\mathrm{T}} \mathrm{e}^{\varepsilon t} U(X(t) - \Lambda(X(t - \delta(t)), r(t), t)) \mathrm{d}t \leqslant \frac{\mathrm{e}^{\varepsilon\tau}}{1 - \bar{\delta}} \int_{-\tau}^{0} U(\xi(t)) \mathrm{d}t + C_\tau \int_0^{\mathrm{T}} \mathrm{e}^{\varepsilon t} U(X(t)) \mathrm{d}t, \quad \forall T > 0,$$

where $C_\tau = \max\{\frac{\kappa \mathrm{e}^{\varepsilon\tau}}{1 - \bar{\delta}} + (1 - \kappa)^{1 - \beta_h}, \frac{\mathrm{e}^{\varepsilon\tau}}{1 - \bar{\delta}} + 1\}$.

Proof. For $p \geqslant 0$, it is easy to show that

$$\begin{aligned}
\int_0^{\mathrm{T}} \mathrm{e}^{\varepsilon t} |X(t - \delta(t))|^p \mathrm{d}t & \leqslant \mathrm{e}^{\varepsilon\tau} \int_0^{\mathrm{T}} \mathrm{e}^{\varepsilon(t - \delta(t))} |X(t - \delta(t))|^p \mathrm{d}t \\
& \leqslant \frac{\mathrm{e}^{\varepsilon\tau}}{1 - \bar{\delta}} \int_{-\tau}^{T} \mathrm{e}^{\varepsilon s} |X(s)|^p \mathrm{d}s \\
& \leqslant \frac{\mathrm{e}^{\varepsilon\tau}}{1 - \bar{\delta}} \int_{-\tau}^{0} |\xi(s)|^p \mathrm{d}s + \frac{\mathrm{e}^{\varepsilon\tau}}{1 - \bar{\delta}} \int_0^{\mathrm{T}} \mathrm{e}^{\varepsilon s} |X(s)|^p \mathrm{d}s.
\end{aligned} \tag{5}$$

By the definition of $U(x)$, we obtain

$$\begin{aligned}
&\int_0^{\mathrm{T}} \mathrm{e}^{\varepsilon t} U(X(t-\delta(t)))\mathrm{d}t \\
&= \int_0^{T} \mathrm{e}^{\varepsilon t}(a_h|X(t-\delta(t))|^{\beta_h} + \cdots + a_1|X(t-\delta(t))|^{\beta_1})\mathrm{d}t \\
&\leqslant \frac{\mathrm{e}^{\varepsilon\tau}}{1-\bar{\delta}} \int_{-\tau}^{0} a_h|\xi(s)|^{\beta_h} + \cdots + a_1|\xi(s)|^{\beta_1}\mathrm{d}s + \frac{\mathrm{e}^{\varepsilon\tau}}{1-\bar{\delta}} \int_0^{\mathrm{T}} \mathrm{e}^{\varepsilon s}(a_h|X(s)|^{\beta_h} + \cdots + a_1|X(s)|^{\beta_1})\mathrm{d}s \\
&= \frac{\mathrm{e}^{\varepsilon\tau}}{1-\bar{\delta}} \int_{-\tau}^{0} U(\xi(s))\mathrm{d}s + \frac{\mathrm{e}^{\varepsilon\tau}}{1-\bar{\delta}} \int_0^{\mathrm{T}} \mathrm{e}^{\varepsilon s} U(X(s))\mathrm{d}s,
\end{aligned}$$

establishing (i). Next, we establish the assertion (ii). For $p \geqslant 1$, we apply the inequality

$$(u+v)^p \leqslant (1+\varpi)^{p-1}(u^p + \varpi^{1-p}v^p) \quad \forall u,v \geqslant 0,\ p \geqslant 1,\ \varpi > 0,$$

it is easy to see that

$$|X(t) - \Lambda(X(t-\delta(t)), r(t), t)|^p \leqslant (1+\varpi)^{p-1}(|X(t)|^p + \varpi^{1-p}|\Lambda(X(t-\delta(t)), r(t), t)|^p).$$

Setting $\varpi = \frac{\kappa}{1-\kappa}$, we derive

$$\begin{aligned}
|X(t) - \Lambda(X(t-\delta(t)), r(t), t)|^p &\leqslant (1-\kappa)^{1-p}|X(t)|^p + \kappa^{1-p}|\Lambda(X(t-\delta(t)), r(t), t)|^p \\
&\leqslant (1-\kappa)^{1-p}|X(t)|^p + \kappa|X(t-\delta(t))|^p,
\end{aligned} \tag{6}$$

which, together with (5), shows that

$$\begin{aligned}
&\int_0^{\mathrm{T}} \mathrm{e}^{\varepsilon t}|X(t) - \Lambda(X(t-\delta(t)), r(t), t)|^p\mathrm{d}t \\
&\leqslant (1-\kappa)^{1-p} \int_0^{T} \mathrm{e}^{\varepsilon t}|X(t)|^p\mathrm{d}t + \kappa \int_0^{T} \mathrm{e}^{\varepsilon t}|X(t-\delta(t))|^p\mathrm{d}t \\
&\leqslant \frac{\kappa\mathrm{e}^{\varepsilon\tau}}{1-\bar{\delta}} \int_{-\tau}^{0} |\xi(t)|^p\mathrm{d}t + \frac{\kappa\mathrm{e}^{\varepsilon\tau}}{1-\bar{\delta}} \int_0^{\mathrm{T}} \mathrm{e}^{\varepsilon t}|X(t)|^p\mathrm{d}t + (1-\kappa)^{1-p} \int_0^{T} \mathrm{e}^{\varepsilon t}|X(t)|^p\mathrm{d}t \\
&= \frac{\kappa\mathrm{e}^{\varepsilon\tau}}{1-\bar{\delta}} \int_{-\tau}^{0} |\xi(t)|^p\mathrm{d}t + \left[\frac{\kappa\mathrm{e}^{\varepsilon\tau}}{1-\bar{\delta}} + (1-\kappa)^{1-p}\right] \int_0^{\mathrm{T}} \mathrm{e}^{\varepsilon t}|X(t)|^p\mathrm{d}t.
\end{aligned} \tag{7}$$

Noting that $0 < 1-\kappa < 1$ and $1-p \leqslant 0$, we have $(1-\kappa)^{1-p} \leqslant (1-\kappa)^{1-\beta_h}$, $\forall p \leqslant \beta_h$. Thus, from (7), the following inequality holds for all $1 \leqslant p \leqslant \beta_h$:

$$\begin{aligned}
&\int_0^{\mathrm{T}} \mathrm{e}^{\varepsilon t}|X(t) - \Lambda(X(t-\delta(t)), r(t), t)|^p\mathrm{d}t \\
&\leqslant \frac{\kappa\mathrm{e}^{\varepsilon\tau}}{1-\bar{\delta}} \int_{-\tau}^{0} |\xi(t)|^p\mathrm{d}t + \left[\frac{\kappa\mathrm{e}^{\varepsilon\tau}}{1-\bar{\delta}} + (1-\kappa)^{1-\beta_h}\right] \int_0^{\mathrm{T}} \mathrm{e}^{\varepsilon t}|X(t)|^p\mathrm{d}t.
\end{aligned} \tag{8}$$

For $0 \leqslant p < 1$, by the elementary inequality $(u+v)^p \leqslant u^p + v^p$, $\forall u,v \geqslant 0$, we obtain

$$\begin{aligned}
\int_0^{\mathrm{T}} \mathrm{e}^{\varepsilon t}|X(t) - \Lambda(X(t-\delta(t)), r(t), t)|^p\mathrm{d}t &\leqslant \frac{\kappa^p\mathrm{e}^{\varepsilon\tau}}{1-\bar{\delta}} \int_{-\tau}^{0} |\xi(t)|^p\mathrm{d}t + \left(\frac{\kappa^p\mathrm{e}^{\varepsilon\tau}}{1-\bar{\delta}} + 1\right) \int_0^{\mathrm{T}} \mathrm{e}^{\varepsilon t}|X(t)|^p\mathrm{d}t \\
&\leqslant \frac{\mathrm{e}^{\varepsilon\tau}}{1-\bar{\delta}} \int_{-\tau}^{0} |\xi(t)|^p\mathrm{d}t + \left(\frac{\mathrm{e}^{\varepsilon\tau}}{1-\bar{\delta}} + 1\right) \int_0^{\mathrm{T}} \mathrm{e}^{\varepsilon t}|X(t)|^p\mathrm{d}t.
\end{aligned}$$

Thus, together with (8) and the definition of $U(x)$, we see that

$$\int_0^{\mathrm{T}} \mathrm{e}^{\varepsilon t} U(X(t) - \Lambda(X(t-\delta(t)), r(t), t))\mathrm{d}t$$

$$
\begin{aligned}
&= \int_0^{\mathrm{T}} \mathrm{e}^{\varepsilon t}(a_h|X(t)-\Lambda(X(t-\delta(t)),r(t),t)|^{\beta_h}+\cdots+a_1|X(t)-\Lambda(X(t-\delta(t)),r(t),t)|^{\beta_1})\mathrm{d}t \\
&\leqslant \frac{\mathrm{e}^{\varepsilon\tau}}{1-\bar{\delta}}\int_{-\tau}^{0} a_h|\xi(t)|^{\beta_h}+\cdots+a_1|\xi(t)|^{\beta_1}\mathrm{d}t + C_\tau\int_0^{\mathrm{T}}\mathrm{e}^{\varepsilon t}(a_h|X(t)|^{\beta_h}+\cdots+a_1|X(t)|^{\beta_1})\mathrm{d}t \\
&= \frac{\mathrm{e}^{\varepsilon\tau}}{1-\bar{\delta}}\int_{-\tau}^{0} U(\xi(t))\mathrm{d}t + C_\tau\int_0^{\mathrm{T}}\mathrm{e}^{\varepsilon t}U(X(t))\mathrm{d}t.
\end{aligned}
$$

Hence, the proof is complete.

Assumption 3. We assume that there exist three functions $V\in C^{2,1}(\mathbb{R}^n\times S\times\mathbb{R}_+;\mathbb{R}_+)$, $U_1,U_2\in U$, as well as nonnegative constants, a_1,a_2,c_l, where $l=1,\ldots,m$ and $a_2>\sum_{l=1}^m c_l$, such that

$$
U_1(x,t)\leqslant V(x,i,t)\leqslant U_2(x,t),\quad \forall(x,i,t)\in\mathbb{R}^n\times S\times\mathbb{R}_+, \tag{9}
$$

and

$$
LV(x-\Lambda(y_1,i,t),y_1,\ldots,y_m,i,t)\leqslant a_1-a_2U_2(x,t)+\sum_{l=1}^{m}c_l(1-\bar{\delta}_l)U_2(y_l,t-\delta_l(t)),
$$

for all $x,y_1,\ldots,y_m\in\mathbb{R}^n$ and $(i,t)\in S\times\mathbb{R}_+$.

3 Boundedness and stability

Using the notations and assumptions introduced in the previous section, we first establish the existence and uniqueness of a solution of the system (1) in this section.

Theorem 1. Suppose Assumptions 1–3 hold; then for any initial data given by (2), we have the following assertions:

(i) There exists a unique global solution $X(t)$ to the the hybrid NSDDE (1) on $t\in[-\tau,\infty)$.

(ii) The solution obtained in (i) has the property that

$$
\limsup_{t\to\infty}\frac{1}{t}\int_0^t EU_2(X(s),s)\mathrm{d}s\leqslant\frac{a_1}{a_2-\sum_{l=1}^m c_l}. \tag{10}
$$

Proof. As the coefficients of the hybrid NSDDE (1) are locally Lipschitz continuous, it follows that for any given initial data (2) and $r_0\in S$ arbitrarily, there exists a unique maximal local solution $X(t)$ on $t\in[-\tau,\bar{\tau}_e)$ (e.g., [5, Theorem 7.12]), where $\bar{\tau}_e$ is the explosion time. Defining $Z(t)=X(t)-\Lambda(X(t-\delta_1(t)),r(t),t)$, it is easy to show that $|Z(0)|\leqslant\|\xi\|+|\Lambda(\xi,r(0),0)|\leqslant(1+\kappa)\|\xi\|$. Let $k_0>0$ be a sufficiently large integer such that $\|\xi\|<k_0$. For each integer $k>k_0$, we define the stopping time $\sigma_k=\inf\{t\in[0,\bar{\tau}_e):|X(t)|\geqslant k\}$ and $\inf\emptyset=\infty$. Clearly, σ_k is increasing as $k\to\infty$, and $\bar{\sigma}_\infty=\lim_{k\to\infty}\sigma_k\leqslant\bar{\tau}_e$. Hence, the assertion (i) will follow if we can show that $\bar{\sigma}_\infty=\infty$ a.s.

By the generalized Itô formula (e.g., [5, Theorem 1.45]) and Assumption 3, we obtain

$$
\begin{aligned}
&EV(Z(t\wedge\sigma_k),r(t\wedge\sigma_k),t\wedge\sigma_k)\\
&=V(Z(0))+E\int_0^{t\wedge\sigma_k}LV(X(s),X(s-\delta_1(s)),\ldots,X(s-\delta_m(s)),r(s),s)\mathrm{d}s\\
&\leqslant V(Z(0))+a_1t-a_2E\int_0^{t\wedge\sigma_k}U_2(X(s),s)\mathrm{d}s\\
&\quad+\sum_{l=1}^m c_l(1-\bar{\delta}_l)E\int_0^{t\wedge\sigma_k}U_2(X(s-\delta_l(s)),s-\delta_l(s))\mathrm{d}s,
\end{aligned}\tag{11}
$$

but

$$
\int_0^{t\wedge\sigma_k}U_2(X(s-\delta_l(s)),s-\delta_l(s))\mathrm{d}s\leqslant\frac{1}{1-\bar{\delta}_l}\int_{-\tau_l}^{0}U_2(X(s),s)\mathrm{d}s+\frac{1}{1-\bar{\delta}_l}\int_0^{t\wedge\sigma_k}U_2(X(s),s)\mathrm{d}s.
$$

Thus, we have

$$EV(Z(t\wedge\sigma_k),r(t\wedge\sigma_k),t\wedge\sigma_k)\leqslant K_1+a_1t-\left(a_2-\sum_{l=1}^{m}c_l\right)E\int_0^{t\wedge\sigma_k}U_2(X(s),s)\mathrm{d}s,$$

where $K_1=V(Z(0))+\sum_{l=1}^m c_l\int_{-\tau_l}^0 U_2(X(s),s)\mathrm{d}s$. Using the hypothesis $a_2>\sum_{l=1}^m c_l$ and Assumption 3, we have

$$EU_1(Z(t\wedge\sigma_k),t\wedge\sigma_k)\leqslant K_1+a_1t.$$

Now, we define $\nu_k=\inf_{|z|\geqslant(1-\kappa)k,t\geqslant0}U_1(z,t)$. Then, by (3), we have

$$\begin{aligned}&|X(\sigma_k)-\Lambda(X(\sigma_k-\delta_1(\sigma_k)),r(\sigma_k),\sigma_k)|I_{\{\sigma_k\leqslant t\}}\geqslant(k-|\Lambda(X(\sigma_k-\delta_1(\sigma_k)),r(\sigma_k),\sigma_k)|)I_{\{\sigma_k\leqslant t\}}\\&\geqslant(k-\kappa|X(\sigma_k-\delta_1(\sigma_k))|)I_{\{\sigma_k\leqslant t\}}\geqslant(1-\kappa)kI_{\{\sigma_k\leqslant t\}}.\end{aligned}$$

By the definition of $U_1(z,t)$ and σ_k, we can obtain

$$\begin{aligned}K_1+a_1t\geqslant&E[U_1(Z(t\wedge\sigma_k),t\wedge\sigma_k)I_{\{\sigma_k\leqslant t\}}]\\\geqslant&E\left[\inf_{|z|\geqslant(1-\kappa)k,t\geqslant0}U_1(z,t)I_{\{\sigma_k\leqslant t\}}\right]=\nu_kP\{\sigma_k\leqslant t\}.\end{aligned}$$

Note that $\nu_k\to\infty$ as $k\to\infty$. Then, letting $k\to\infty$ in the last inequality, we have that $P\{\bar\sigma_\infty\leqslant t\}=0$, which implies $\bar\sigma_\infty>t$ a.s. Also, letting $t\to\infty$, we get that $\bar\sigma_\infty=\infty$ a.s, which implies the assertion (i).

We now show the assertion (ii). From (11), it follows that

$$\left(a_2-\sum_{l=1}^{m}c_l\right)E\int_0^{t\wedge\sigma_k}U_2(X(s),s)\mathrm{d}s\leqslant K_1+a_1t.$$

Dividing both sides of the last inequality by $a_2-\sum_{l=1}^m c_l$, we then have

$$E\int_0^{t\wedge\sigma_k}U_2(X(s),s)\mathrm{d}s\leqslant\frac{K_1}{a_2-\sum_{l=1}^m c_l}+\frac{a_1t}{a_2-\sum_{l=1}^m c_l}.$$

Letting $k\to\infty$ and using the well-known Fubini theorem, we obtain

$$\int_0^t EU_2(X(s),s)\mathrm{d}s\leqslant\frac{K_1}{a_2-\sum_{l=1}^m c_l}+\frac{a_1t}{a_2-\sum_{l=1}^m c_l}.\tag{12}$$

Dividing both sides of inequality (12) by t and letting $t\to\infty$, we obtain (10). Thus, the proof is complete.

Next, we establish the criteria for the stability of the solution obtained in Theorem 1.

Theorem 2. Suppose Assumptions 1–3 hold; then for any given initial data (2), the unique global solution to (1) has the property that

$$\limsup_{t\to\infty}EU_1(X(t)-\Lambda(X(t-\delta_1(t)),r(t),t),t)<\frac{a_1}{\varepsilon},\tag{13}$$

where $0<\varepsilon<1$ is sufficiently small such that

$$a_2-\varepsilon C_{\tau_1}-\sum_{l=1}^m c_l\mathrm{e}^{\varepsilon\tau_l}>0,\tag{14}$$

where $C_{\tau_1}=\max\{\frac{\kappa\mathrm{e}^{\varepsilon\tau_1}}{1-\delta_1}+(1-\kappa)^{1-\beta_h},\frac{\mathrm{e}^{\varepsilon\tau_1}}{1-\delta_1}+1\}$. Moreover, if $a_1=0$, then the solution satisfies

$$\limsup_{t\to\infty}\frac{\log U_1(X(t)-\Lambda(X(t-\delta_1(t)),r(t),t),t)}{t}\leqslant-\varepsilon\quad\text{a.s.},\tag{15}$$

and

$$\int_0^\infty U_2(X(t),t)\mathrm{d}t<\infty\quad\text{a.s.}\tag{16}$$

Proof. Applying the generalized Itô formula to $\mathrm{e}^{\varepsilon(t\wedge\sigma_k)}V(Z(t\wedge\sigma_k),r(t\wedge\sigma_k),t\wedge\sigma_k)$, we obtain

$$\begin{aligned}&E\mathrm{e}^{\varepsilon(t\wedge\sigma_k)}V(Z(t\wedge\sigma_k),r(t\wedge\sigma_k),t\wedge\sigma_k)\\&=V(Z(0),r(0),0)+E\int_0^{t\wedge\sigma_k}\varepsilon\mathrm{e}^{\varepsilon s}V(Z(s),r(s),s)\mathrm{d}s\\&\quad+E\int_0^{t\wedge\sigma_k}\mathrm{e}^{\varepsilon s}LV(Z(s),X(s-\delta_1(s)),\ldots,X(s-\delta_m(s)),r(s),s)\mathrm{d}s.\end{aligned}$$

Also, from Assumption 3, we have

$$\begin{aligned}&E\mathrm{e}^{\varepsilon(t\wedge\sigma_k)}U_1(Z(t\wedge\sigma_k))\\&\leqslant V(Z(0),r_0,0)+\varepsilon E\int_0^{t\wedge\sigma_k}\mathrm{e}^{\varepsilon s}U_2(Z(s),s)\mathrm{d}s+\frac{a_1}{\varepsilon}\mathrm{e}^{\varepsilon t}\\&\quad-a_2E\int_0^{t\wedge\sigma_k}\mathrm{e}^{\varepsilon s}U_2(X(s),s)\mathrm{d}s+\sum_{l=1}^m c_l(1-\bar{\delta}_l)E\int_0^{t\wedge\sigma_k}\mathrm{e}^{\varepsilon s}U_2(X(s-\delta_l(s)),s-\delta_l(s))\mathrm{d}s.\end{aligned}$$

Noting that

$$\begin{aligned}&\int_0^{t\wedge\sigma_k}\mathrm{e}^{\varepsilon s}U_2(X(s-\delta_l(s)),s-\delta_l(s))\mathrm{d}s\\&\leqslant \mathrm{e}^{\varepsilon\tau_l}\int_0^{t\wedge\sigma_k}\mathrm{e}^{\varepsilon(s-\delta_l(s))}U_2(X(s-\delta_l(s)),s-\delta_l(s))\mathrm{d}s\\&\leqslant\frac{\mathrm{e}^{\varepsilon\tau_l}}{1-\bar{\delta}_l}\int_{-\tau_l}^0U_2(X(s),s)\mathrm{d}s+\frac{\mathrm{e}^{\varepsilon\tau_l}}{1-\bar{\delta}_l}\int_0^{t\wedge\sigma_k}\mathrm{e}^{\varepsilon s}U_2(X(s),s)\mathrm{d}s,\end{aligned}$$

and by Lemma 1, we obtain

$$\begin{aligned}E\mathrm{e}^{\varepsilon(t\wedge\sigma_k)}U_1(Z(t\wedge\sigma_k),(t\wedge\sigma_k))&\leqslant K_2+\frac{a_1}{\varepsilon}\mathrm{e}^{\varepsilon t}+\varepsilon C_{\tau_1}E\int_0^{t\wedge\sigma_k}\mathrm{e}^{\varepsilon s}U_2(X(s),s)\mathrm{d}s\\&\quad-a_2E\int_0^{t\wedge\sigma_k}\mathrm{e}^{\varepsilon s}U_2(X(s),s)\mathrm{d}s+\sum_{j=1}^m c_l\mathrm{e}^{\varepsilon\tau_l}E\int_0^{t\wedge\sigma_k}\mathrm{e}^{\varepsilon s}U_2(X(s),s)\mathrm{d}s,\end{aligned}$$

where $K_2=V(Z(0),r_0,0)+\frac{\varepsilon\mathrm{e}^{\varepsilon\tau_1}}{1-\bar{\delta}}\int_{-\tau_1}^0U_2(\xi(s),s)\mathrm{d}s+\sum_{l=1}^m c_l\mathrm{e}^{\varepsilon\tau_l}\int_{-\tau_l}^0U_2(\xi(s),s)\mathrm{d}s$. By condition (14) and letting $k\to\infty$, we have

$$E\mathrm{e}^{\varepsilon t}U_1(Z(t),t)\leqslant K_2+\frac{a_1}{\varepsilon}\mathrm{e}^{\varepsilon t},\tag{17}$$

which shows that

$$\limsup_{t\to\infty}EU_1(X(t)-\Lambda(X(t-\delta_1(t)),r(t),t),t)<\frac{a_1}{\varepsilon},$$

and

$$\limsup_{0\leqslant t<\infty}EU_1(X(t)-\Lambda(X(t-\delta_1(t)),r(t),t),t)<\infty.$$

If $a_1=0$, Eq. (17) yields

$$E\mathrm{e}^{\varepsilon t}U_1(Z(t),t)\leqslant K_2,\tag{18}$$

which implies that

$$\limsup_{t\to\infty}\frac{\log EU_1(X(t)-\Lambda(X(t-\delta_1(t)),r(t),t),t)}{t}\leqslant-\varepsilon.$$

Moreover, if $a_1=0$, from (12) we can then obtain that

$$\int_0^tEU_2(X(s),s)\mathrm{d}s\leqslant\frac{K_1}{a_2-\sum_{l=1}^mc_l}.$$

By letting $t \to \infty$ in the last inequality, we obtain

$$\int_0^{\infty} EU_2(X(s), s)\mathrm{d}s \leqslant \frac{K_1}{a_2 - \sum_{l=1}^{m} c_l}. \tag{19}$$

Furthermore, using the well-known Fubini theorem, we have

$$E\int_0^{\infty} U_2(X(s), s)\mathrm{d}s \leqslant \frac{K_1}{a_2 - \sum_{l=1}^{m} c_l},$$

which implies (16). Finally, we show that (15) holds. By the generalized Itô formula (e.g., [5, Theorem 1.45]), we have that for any $t \geqslant 0$,

$$\begin{aligned} \mathrm{e}^{\varepsilon t} V(Z(t), r(t), t) = V(Z(0), r_0, 0) &+ \int_0^t \varepsilon \mathrm{e}^{\varepsilon s} V(Z(s), r(s), s)\mathrm{d}s \\ &+ \int_0^t \mathrm{e}^{\varepsilon s} LV(Z(s), X(s - \delta_1(s)), \ldots, X(s - \delta_m(s)), r(s), s)\mathrm{d}s + M(t), \end{aligned}$$

where $M(t)$ is a local martingale with the initial value $M(0) = 0$. For $a_1 = 0$, by the same argument as before, we obtain

$$\mathrm{e}^{\varepsilon t} U_1(Z(t), t) \leqslant K_2 + M(t).$$

Using the nonnegative semimartingale convergence theorem (e.g., [8, Theorem 1.3.9]), we immediately obtain that

$$\limsup_{t\to\infty} \mathrm{e}^{\varepsilon t} U_1(Z(t), t) < \infty \quad \text{a.s.}$$

Therefore, there exists a finite positive random variable η such that

$$\sup_{0\leqslant t<\infty} \mathrm{e}^{\varepsilon t} U_1(Z(t), t) < \varsigma \quad \text{a.s.}, \tag{20}$$

which implies that

$$\limsup_{t\to\infty} \frac{\log U_1(Z(t), t)}{t} < -\varepsilon \quad \text{a.s.}$$

Hence, the assertion (15) can be obtained. Thus, the proof is complete.

Remark 1. It is important to note that Theorems 1 and 2 are not only applicable in handling NSDDEs that have different parameters in different modes, but can also be applied to NSDDEs that have different structures in different modes. This will be illustrated in Examples 1 and 2.

Remark 2. We also remark that Theorems 1 and 2 are extendable to more general class of equations with neutral parts that include more than one delay.

Corollary 1. Suppose all the conditions of Theorem 2 hold. If there is a pair of positive constants λ and $p \geqslant 1$ such that

$$\lambda |x|^p \leqslant U_1(x, t), \quad (x, t) \in \mathbb{R}^d \times \mathbb{R}_+, \tag{21}$$

then the solution satisfies

$$\limsup_{t\to\infty} E|X(t)|^p < \infty.$$

Moreover, if $a_1 = 0$ and $\kappa_1 \mathrm{e}^{\varepsilon\tau_1} < 1$, then we have

$$\int_0^{\infty} E|X(t)|^p \mathrm{d}t < \infty, \tag{22}$$

$$\limsup_{t\to\infty} \frac{\log(E|X(t)|^p)}{t} < -\varepsilon, \tag{23}$$

and

$$\limsup_{t\to\infty} \frac{\log |X(t)|}{t} < -\frac{\varepsilon}{p} \quad \text{a.s.} \tag{24}$$

Proof. The proof is similar to the proof of (6). We have that for any $t > \tau$,

$$\begin{aligned}\sup_{0\leqslant s\leqslant t} E|X(s)|^p &\leqslant \sup_{0\leqslant s\leqslant t}(1-\kappa_1)^{1-p}E|Z(s)|^p + \sup_{0\leqslant s\leqslant t}\kappa_1 E|X(s-\delta_1(s))|^p \\ &\leqslant \sup_{0\leqslant s\leqslant t}(1-\kappa_1)^{1-p}E|Z(s)|^p + \kappa_1\|\xi\|^p + \sup_{0\leqslant s\leqslant t}\kappa_1 E|X(s)|^p. \end{aligned} \tag{25}$$

Thus, we obtain

$$(1-\kappa_1)\sup_{0\leqslant s\leqslant t} E|X(s)|^p \leqslant (1-\kappa_1)^{1-p}\sup_{0\leqslant s\leqslant t} E|Z(s)|^p + \kappa_1\|\xi\|^p.$$

Letting $t\to\infty$, we have

$$(1-\kappa_1)\sup_{0\leqslant s<\infty} E|X(s)|^p \leqslant (1-\kappa_1)^{1-p}\sup_{0\leqslant s<\infty} E|Z(s)|^p + \kappa_1\|\xi\|^p, \tag{26}$$

which, with (13) and (21), shows that

$$\limsup_{t\to\infty} E|X(t)|^p \leqslant \frac{1}{\lambda(1-\kappa_1)^p}\limsup_{t\to\infty} EU_1(Z(t),t) + \frac{\kappa_1}{\lambda(1-\kappa_1)}\|\xi\|^p < \infty.$$

For $a_1 = 0$, using (9), (19) and (21), we obtain

$$\int_0^\infty E|X(t)|^p\mathrm{d}t \leqslant \frac{1}{\lambda}\int_0^\infty EU_1(X(t),t)\mathrm{d}t \leqslant \frac{1}{\lambda}\int_0^\infty EU_1(X(t),t)\mathrm{d}t \leqslant \frac{K_1}{\lambda(a_2-\sum_{l=1}^m c_l)} < \infty.$$

Recalling (18) and (25) we see that for any $t > 0$,

$$\begin{aligned}\sup_{0\leqslant s\leqslant t}\mathrm{e}^{\varepsilon s}E|X(s)|^p &\leqslant (1-\kappa_1)^{1-p}\sup_{0\leqslant s\leqslant t}\mathrm{e}^{\varepsilon s}E|Z(s)|^p + \kappa_1\sup_{0\leqslant s\leqslant t}\mathrm{e}^{\varepsilon s}E|X(s-\delta_1(s))|^p \\ &\leqslant \frac{K_2(1-\kappa_1)^{1-p}}{\lambda} + \mathrm{e}^{\varepsilon\tau_1}\kappa_1\|\xi\|^p + \mathrm{e}^{\varepsilon\tau_1}\kappa_1\sup_{0\leqslant s\leqslant t}\mathrm{e}^{\varepsilon s}E|X(s)|^p,\end{aligned}$$

which implies

$$\sup_{0\leqslant s\leqslant t}\mathrm{e}^{\varepsilon s}E|X(s)|^p \leqslant \frac{K_2(1-\kappa_1)^{1-p}}{\lambda(1-\kappa_1\mathrm{e}^{\varepsilon\tau_1})} + \frac{\mathrm{e}^{\varepsilon\tau_1}\kappa_1}{1-\kappa_1\mathrm{e}^{\varepsilon\tau_1}}\|\xi\|^p.$$

By letting $t\to\infty$, the assertion (23) is obtained. Employing a similar argument as (26), we have

$$(1-\kappa_1)\sup_{0\leqslant t<\infty}|X(t)|^p \leqslant (1-\kappa_1)^{1-p}\sup_{0\leqslant t<\infty}|Z(t)|^p + \kappa_1\|\xi\|^p.$$

By inequalities (20) and (21), we have

$$\sup_{0\leqslant t<\infty}|X(t)|^p \leqslant \frac{(1-\kappa_1)^{1-p}}{\lambda(1-\kappa_1)}\sup_{0\leqslant t<\infty}U_1(Z(t),t) + \frac{\kappa_1}{1-\kappa_1}\|\xi\|^p < \infty \quad \text{a.s.}$$

From inequality (20), for any $t > \tau$, we have that

$$\begin{aligned}\sup_{0\leqslant s\leqslant t}\mathrm{e}^{\varepsilon s}|X(s)|^p &\leqslant (1-\kappa_1)^{1-p}\sup_{0\leqslant s\leqslant t}\mathrm{e}^{\varepsilon s}|Z(s)|^p + \kappa_1\mathrm{e}^{\varepsilon s}\sup_{0\leqslant s\leqslant t}|X(s-\delta_1(s))|^p \\ &\leqslant \frac{(1-\kappa_1)^{1-p}}{\lambda}\sup_{0\leqslant s<\infty}\mathrm{e}^{\varepsilon t}U_1(Z(s),s) + \mathrm{e}^{\varepsilon\tau_1}\kappa_1\|\xi\|^p + \mathrm{e}^{\varepsilon\tau_1}\kappa_1\sup_{0\leqslant s<\infty}\mathrm{e}^{\varepsilon s}|X(s)|^p \\ &\leqslant \frac{(1-\kappa_1)^{1-p}}{\lambda}\varsigma + \mathrm{e}^{\varepsilon\tau_1}\kappa_1\|\xi\|^p + \mathrm{e}^{\varepsilon\tau_1}\kappa_1\sup_{0\leqslant s\leqslant t}\mathrm{e}^{\varepsilon s}|X(s)|^p,\end{aligned}$$

which implies

$$\sup_{0\leqslant s\leqslant t}\mathrm{e}^{\varepsilon s}|X(s)|^p \leqslant \frac{\varsigma(1-\kappa_1)^{1-p}}{\lambda(1-\kappa_1\mathrm{e}^{\varepsilon\tau_1})} + \frac{\mathrm{e}^{\varepsilon\tau_1}\kappa_1}{1-\kappa_1\mathrm{e}^{\varepsilon\tau_1}}\|\xi\|^p < \infty \quad \text{a.s.}$$

Letting $t\to\infty$ in the last inequality yields the assertion (24). Thus, the proof is complete.

The following criterion is very convenient for the mean square exponential stability because the main condition is explicitly related to the coefficients f and g. Before we give the criterion, it is necessary to make Assumptions 4 and 5.

Assumption 4. We assume that there is a constant $q > 2$. Assume that there exist $\alpha_{i1} \in \mathbb{R}$, and $\alpha_{i2l}, \beta_{i3}, \beta_{i4l} \in \mathbb{R}_+$, $l = 1, \ldots, m$ such that

$$(x - \Lambda(y_1, i, t))^{\mathrm{T}} f(x, y_1, \ldots, y_m, i, t) + \frac{1}{2}|g(x, y_1, \ldots, y_m, i, t)|^2$$
$$\leqslant \beta_{i1}|x|^2 + \sum_{l=1}^{m} \beta_{i2l}|y_l|^2 - \beta_{i3}|x|^q + \sum_{l=1}^{m} \beta_{i4l}|y_l|^q$$

for each $i \in S$.

Assumption 5. We assume

$$\mathcal{A} = -\mathrm{diag}(2\beta_{11}, \ldots, 2\beta_{N1}) - \Gamma - \hat{\Gamma}$$

is a nonsingular M-matrix, where $\hat{\Gamma} = (|\gamma_{ij}|\kappa_i)_{N\times N}$. Set

$$(\lambda_1, \ldots, \lambda_N)^{\mathrm{T}} = \mathcal{A}^{-1}(1, \ldots, 1)^{\mathrm{T}};$$

then $\lambda_i > 0$ for all $i \in S$.

Theorem 3. Let Assumptions 4 and 5 hold, and let $b_1 = \min_{i\in S}\lambda_i$, $b_2 = \max_{i\in S}\lambda_i$, $\theta_{21} = \max_{i\in S}\{2\lambda_i\beta_{i21} + \sum_{j\in S}\kappa_i|\gamma_{ij}|\lambda_j + \sum_{j\in S, j\neq i}\kappa_i^2\gamma_{ij}\lambda_j\}$, $\theta_{2l} = \max_{i\in S}\{2\lambda_i\beta_{i2l}\}$, where $l = 2, \ldots, m$, $\theta_3 = \min_{i\in S}\{2\lambda_i\beta_{i3}\}$, $\theta_{4l} = \max_{i\in S}\{2\lambda_i\beta_{i4l}\}$, where $l = 1, \ldots, m$, and $c_l = \max\{\frac{\theta_{2l}}{1-\delta_l}, \frac{\theta_{4l}}{\theta_3(1-\delta_l)}\}$, where $l = 1, \ldots, m$. Suppose

$$1 > \sum_{l=1}^{m} c_l; \tag{27}$$

then we have that

$$\limsup_{t\to\infty} \frac{\log(E|X(t)|^2)}{t} < -\varepsilon, \tag{28}$$

and

$$\limsup_{t\to\infty} \frac{\log|X(t)|}{t} < -\frac{\varepsilon}{2} \quad \text{a.s.} \tag{29}$$

Proof. Let $V(x, i, t) = \lambda_i|x|^2$, $U_1(x, t) = b_1|x|^2$, $U_2(x, t) = b_2|x|^2 + b_2\theta_3|x|^q$. Clearly,

$$U_1(x, t) \leqslant V(x, i, t) \leqslant U_2(x, t).$$

Now, we compute $LV(z, y_1, \ldots, y_m, i, t)$. For any $i \in S$,

$$\begin{aligned}
LV(z, y_1, \ldots, y_m, t, i) &= 2\lambda_i\left[(x - \Lambda(y_1, i, t))^{\mathrm{T}} f(x, y_1, \ldots, y_m, i, t) + \frac{1}{2}|g(x, y_1, \ldots, y_m, i, t)|^2\right] \\
&\quad + \sum_{j\in S}\lambda_j\gamma_{ij}(x - \Lambda(y_1, i, t))^{\mathrm{T}}(x - \Lambda(y_1, i, t)) \\
&\leqslant 2\lambda_i\beta_{i1}|x|^2 + 2\lambda_i\sum_{l=1}^{m}\beta_{i2l}|y_l|^2 - 2\lambda_i\beta_{i3}|x|^q + 2\lambda_i\sum_{j=1}^{m}\beta_{i4l}|y_l|^q \\
&\quad + \sum_{j\in S}\lambda_j\gamma_{ij}|x|^2 - 2\sum_{j\in S}\lambda_j\gamma_{ij}x^{\mathrm{T}}\Lambda(y_1, i, t) + \sum_{j\in S}\lambda_j\gamma_{ij}|\Lambda(y_1, i, t)|^2 \\
&\leqslant \left[2\lambda_i\beta_{i1} + \sum_{j\in S}\lambda_j\gamma_{ij} + \sum_{j\in S}\kappa_i\lambda_j|\gamma_{ij}|\right]|x|^2 + 2\lambda_i\sum_{l=1}^{m}\beta_{i2l}|y_l|^2 \\
&\quad + \left[\sum_{j\in S}\kappa_i|\gamma_{ij}|\lambda_j + \sum_{j\in S, j\neq i}\kappa_i^2\gamma_{ij}\lambda_j\right]|y_1|^2 - 2\lambda_i\beta_{i3}|x|^q + 2\lambda_i\sum_{l=1}^{m}\beta_{i4l}|y_l|^q.
\end{aligned}$$

Furthermore, by definitions of θ_{21}, θ_{2l}, θ_3 and θ_{4l}, we have

$$\begin{aligned} LV(x-\Lambda(y_1,i,t),y_1,\ldots,y_m,t,i) &\leqslant -|x|^2+\sum_{l=1}^{m}\theta_{2l}|y_l|^2-\theta_3|x|^q+\sum_{l=1}^{m}\theta_{4l}|y_l|^q \\ &\leqslant -\frac{1}{b_2}U_2(x,t)+\frac{1}{b_2}\sum_{l=1}^{m}c_l(1-\bar{\delta}_l)U_2(y_l,t-\delta_l(t)). \end{aligned}$$

Hence, the assertions (28) and (29) follow from Corollary 1.

4 Examples

In this section, we will use three examples to illustrate our theorems. Although these examples are scalar highly nonlinear hybrid NSDDEs with constant delays, they are fully covered our theorems.

Example 1. Consider the following scalar hybrid NSDDE:

$$\mathrm{d}[X(t)-\Lambda(X(t-\tau_1),r(t),t)]=f(X(t),X(t-\tau_1),X(t-\tau_2),r(t),t)\mathrm{d}t+g(X(t-\tau_2),r(t),t)\mathrm{d}B(t), \quad (30)$$

where $r(t)$ is a Markov chain on the state space $S=\{1,2\}$ with its generator

$$\Gamma=\begin{pmatrix}-1 & 1\\ 2 & -2\end{pmatrix},$$

and the coefficients are defined as follows:

$$f(x,y_1,y_2,1,t)=y_1+y_2^3-6x-6x^3, \quad f(x,y_1,y_2,2,t)=y_1+y_2^3-3x-4x^3,$$

$$g(y_2,1,t)=g(y_2,2,t)=0.5y_2^2, \quad \Lambda(y_1,1,t)=\Lambda(y_1,2,t)=0.1y_1.$$

In mode 1, the system is described by the NSDDE

$$\mathrm{d}[X(t)-0.1X(t-\tau_1)]=[X(t-\tau_1)+X^3(t-\tau_2)-6X(t)-6X^3(t)]\mathrm{d}t+0.5X^2(t-\tau_2)\mathrm{d}B(t),$$

while in mode 2 it is described by

$$\mathrm{d}[X(t)-0.1X(t-\tau_1)]=[X(t-\tau_1)+X^3(t-\tau_2)-3X(t)-4X^3(t)]\mathrm{d}t+0.5X^2(t-\tau_2)\mathrm{d}B(t).$$

It is easy to see that the two modes have the same structure but different parameters, which means that the system only experiences abrupt changes in its parameters.

Before applying our theorem, we set $\tau_1=2$ and $\tau=\tau_2=3$. Let

$$V(x,i,t)=\begin{cases} x^2+x^4, & \text{if } i=1,\\ 2x^2+2x^4, & \text{if } i=2.\end{cases}$$

Then, we obtain

$$\begin{aligned} LV(x-\Lambda(y_1,1,t),y_1,y_2,1,t) &= (2(x-0.1y_1)+4(x-0.1y_1)^3)(y_1+y_2^3-6x-6x^3) \\ &\quad +(0.25+1.25(x-0.1y_1)^2)y_2^4+(x-0.1y_1)^2+(x-0.1y_1)^4, \end{aligned}$$

and

$$\begin{aligned} LV(x-\Lambda(y_1,2,t),y_1,y_2,2,t) &= (4(x-0.1y_1)+8(x-0.1y_1)^3)(y_1+y_2^3-3x-4x^3) \\ &\quad +(0.5+3(x-0.1y_1)^2)y_2^4-2(x-0.1y_1)^2-2(x-0.1y_1)^4. \end{aligned}$$

Applying the inequalities $(a+b)^p \leqslant (1+\rho)^{p-1}(a^p+\rho^{1-p}b^p)$, and setting $\rho=\kappa/(1-\kappa)=1/9$, and $a^\beta b^{1-\beta} \leqslant \beta a+(1-\beta)b$, we obtain

$$\begin{aligned}4(x-0.1y_1)^3(y_1+y_2^3) &\leqslant 4.94|x|^3|y_1|+0.4y_1^4+4.94|x|^3|y_2|^3+0.4|y_1|^3|y_2|^3\\ &\leqslant 3.72x^4+1.64y_1^4+2.47x^6+0.2y_1^6+2.67y_2^6,\end{aligned}$$

and

$$\begin{aligned}&(x-0.1y_1)^2 \leqslant 1.1x^2+0.11y_1^2, \quad (x-0.1y_1)^4 \leqslant 1.38x^4+0.1y_1^4,\\ &(0.25+1.5(x-0.1y_1)^2)y_2^4 \leqslant 0.56x^6+0.25y_2^4+0.05y_1^6+1.44y_2^6.\end{aligned}$$

By the well-known Young inequality, we have

$$2(x-0.1y_1)(y_1+y_2^3-6x-6x^3) \leqslant -10.4x^2+1.4y_1^2-10.6x^4+0.35y_1^4+1.65y_2^4,$$

$$-24x(x-0.1y_1)^3 \leqslant -18.594x^4+1.818y_1^4, \quad -24x^3(x-0.1y_1)^3 \leqslant -17.988x^6+1.212y_1^6.$$

Hence,

$$\begin{aligned}&LV(x-\Lambda(y_1,1,t),y_1,y_2,1,t)\\ &\leqslant -14.958x^6-24.094x^4-9.3x^2+1.462y_1^6+3.908y_1^4+1.51y_1^2+3.89y_2^6+1.9y_2^4.\end{aligned}$$

Similarly,

$$\begin{aligned}&LV(x-\Lambda(y_1,2,t),y_1,y_2,2,t)\\ &\leqslant -17.924x^6-26.348x^4-11.2x^2+2.116y_1^6+5.8y_1^4+2.38y_1^2+7.78y_2^6+3.8y_2^4.\end{aligned}$$

Therefore,

$$\begin{aligned}LV(x-\Lambda(y_1,i,t),y_1,y_2,i,t) &\leqslant -14.958x^6-24.094x^4-9.3x^2+2.116y_1^6+5.68y_1^4+2.38y_1^2+7.78y_2^6+3.8y_2^4\\ &\leqslant -4.6(3x^6+5x^4+2x^2)+1.2(3y_1^6+5y_1^4+2y_1^2)+2.6(3y_2^6+5y_2^4+2y_2^2).\end{aligned}$$

Letting $U_1(x,t)=x^2+x^4$ and $U_2(x,t)=2x^2+5x^4+3x^6$, we have $U_1(x,t) \leqslant V(x,i,t) \leqslant U_2(x,t)$. Thus, we have $a_1=0$, $a_2=4.6$, $c_1=1.2$, $c_2=2.6$. By Theorem 2 and Corollary 1, we conclude that the hybrid system (30) is fourth moment exponential stable. This result is also supported by the simulation analysis carried out (see Figure 1).

Example 2. Let $r(t)$ be a Markov chain on the state space $S=\{1,2\}$ with generator

$$\Gamma=\begin{pmatrix}-1 & 1\\ 10 & -10\end{pmatrix}.$$

Consider the following scalar hybrid NSDDE:

$$\mathrm{d}[X(t)-\Lambda(X(t-\tau_1),r(t),t)]=f(X(t),X(t-\tau_1),r(t),t)\mathrm{d}t+g(X(t-\tau_2),r(t),t)\mathrm{d}B(t), \tag{31}$$

where $\tau_1=1$, $\tau_2=2$,

$$f(x,y_1,1,t)=y_1-4x-4x^3, \quad f(x,y_1,2,t)=0.5x, \quad g(y_2,1,t)=y_2^2,$$

$$g(y_2,2,t)=0.5y_2, \quad \Lambda(y_1,1,t)=\Lambda(y_1,2,t)=0.1y_1.$$

It is clear that the system has different structures in different modes. In mode 1, both f and g are highly nonlinear functions, while in mode 2, f and g are linear functions. This shows that the system experiences abrupt changes in its structure. Next, we will show that our theorem can be applied to systems which have different structures in different modes.

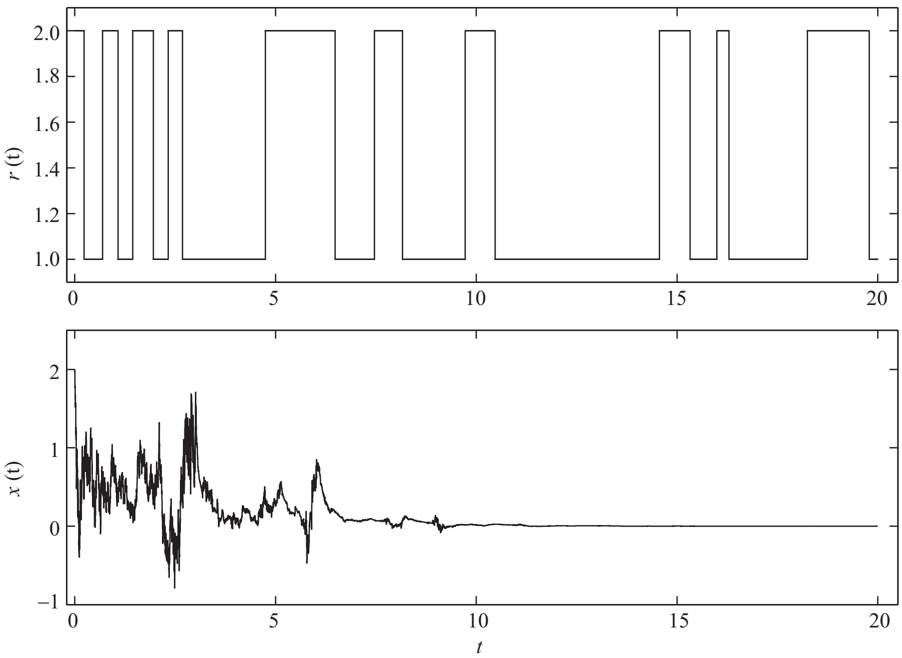

Figure 1 Computer simulation of the stochastic trajectories of the Markov chain and the NSDDE (30).

Let

$$V(x,i,t) = \begin{cases} x^2, & \text{if } i = 1, \\ 2x^2 + x^4, & \text{if } i = 2. \end{cases}$$

It is easy to show that

$$LV(x - \Lambda(y_1, 1, t), y_1, y_2, 1, t) \leqslant -5.4x^2 - 6x^4 + 1.3y_1^2 + 0.3y_1^4 + y_2^4,$$

and

$$LV(x - \Lambda(y_1, 2, t), y_1, y_2, 2, t) \leqslant -6.9x^2 - 3x^4 + y_1^2 + 1.4y_1^4 + 0.5y_2^2 + y_2^4.$$

Hence, we have

$$\begin{aligned} LV(x - \Lambda(y_1, i, t), y_1, y_2, i, t) &\leqslant -5.4x^2 - 3x^4 + 1.3y_1^2 + 1.4y_1^4 + 0.5y_2^2 + y_2^4 \\ &\leqslant -2.7(2x^2 + x^4) + 1.4(2y_1^2 + y_1^4) + (2y_2^2 + y_2^4). \end{aligned}$$

If we set $U_1(x,t) = x^2$ and $U_2(x,t) = 2x^2 + x^4$, then we have $U_1(x,t) \leqslant V(x,i,t) \leqslant U_2(x,t)$, $a_1 = 0$, $a_2 = 2.7$, $c_1 = 1.4$ and $c_2 = 1$. By Theorem 2 and Corollary 1, we conclude that the NSDDE (31) is almost sure exponential stable. The sample paths of the Markov chain and the solution of the NSDDE (31) are plotted in Figure 2.

Example 3. In this example, we use the method of M-matrix to analysis the stability of NSDDEs. Let $r(t)$ be a Markov chain on the state space $S = \{1, 2\}$ with its generator as

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 10 & -10 \end{pmatrix}.$$

Consider the following scalar hybrid NSDDE:

$$\mathrm{d}[X(t) - \Lambda(X(t-\tau_1), r(t), t)] = f(X(t), X(t-\tau_1), X(t-\tau_2), r(t), t)\mathrm{d}t + g(X(t-\tau_2), r(t), t)\mathrm{d}B(t), \quad (32)$$

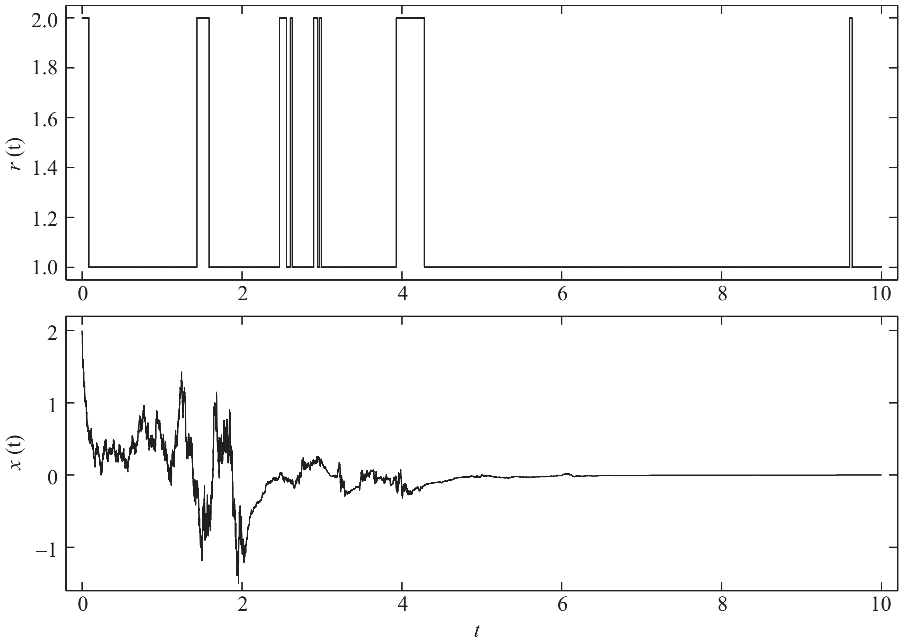

Figure 2 Computer simulation of the sample paths of the Markov chain and the NSDDE (31).

on $t \geqslant 0$, where

$$\tau_1 = 1, \quad \tau_2 = 2, \quad f(x, y_1, y_2, 1, t) = y_1 + y_2^3 - 6x - 6x^3, \quad f(x, y_1, y_2, 2, t) = 0.2y_1 + \frac{1}{2}y_2^3 - 4x^3,$$

$$g(y_2, 1, t) = y_2, \quad g(y_2, 2, t) = 0.5y_2, \quad \Lambda(y_1, 1, t) = \Lambda(y_1, 2, t) = 0.1y_1.$$

We estimate as follows:

$$\begin{aligned}
&(x - \Lambda(y_1, 1, t))^{\mathrm{T}} f(x, y_1, y_2, 1, t) + \frac{1}{2}|g(x, y_2, 1, t)|^2 \\
&= (x - 0.1y_1)(y_1 + y_2^3 - 6x - 6x^3) + \frac{1}{2}y_2^2 \\
&\leqslant -5.2x^2 + 0.7y_1^2 + 0.5y_2^2 - 5.35x^4 + 0.225y_1^4 + 0.825y_2^4,
\end{aligned}$$

and

$$\begin{aligned}
&(x - \Lambda(y_1, 2, t))^{\mathrm{T}} f(x, y_1, y_2, 2, t) + \frac{1}{2}|g(x, y_2, 2, t)|^2 \\
&= (x - 0.1y_1)\left(0.2y_1 + \frac{1}{2}y_2^3 - 4x^3\right) + \frac{1}{2}y_2^2 \\
&\leqslant 0.1x^2 + 0.08y_1^2 + 0.125y_2^2 - 3.575x^4 + 0.1125y_1^4 + 0.4125y_2^4.
\end{aligned}$$

Thus, we have

$$\beta_{11} = -5.2,\ \beta_{21} = 0.1,\ \beta_{121} = 0.7,\ \beta_{122} = 0.5,\ \beta_{221} = 0.08,\ \beta_{222} = 0.125,$$

$$\beta_{13} = 5.35,\ \beta_{23} = 3.45,\ \beta_{141} = 0.225,\ \beta_{142} = 0.825,\ \beta_{241} = 0.1125,\ \beta_{242} = 0.4125.$$

By Assumption 5, we derive that

$$\mathcal{A} = \begin{pmatrix} 11.3 & -1.1 \\ -11 & 8.8 \end{pmatrix}, \quad \mathcal{A}^{-1} = \begin{pmatrix} 0.1008 & 0.0126 \\ 0.1259 & 0.1294 \end{pmatrix}, \quad \lambda_1 = 0.1234, \quad \lambda_2 = 0.2553.$$

Thus, by Theorem 3 we have the following computed:

$$b_1 = 0.1234,\ b_2 = 0.2553,\ \theta_{21} = 0.432,\ \theta_{22} = 0.1234,\ \theta_3 = 1.3203,$$

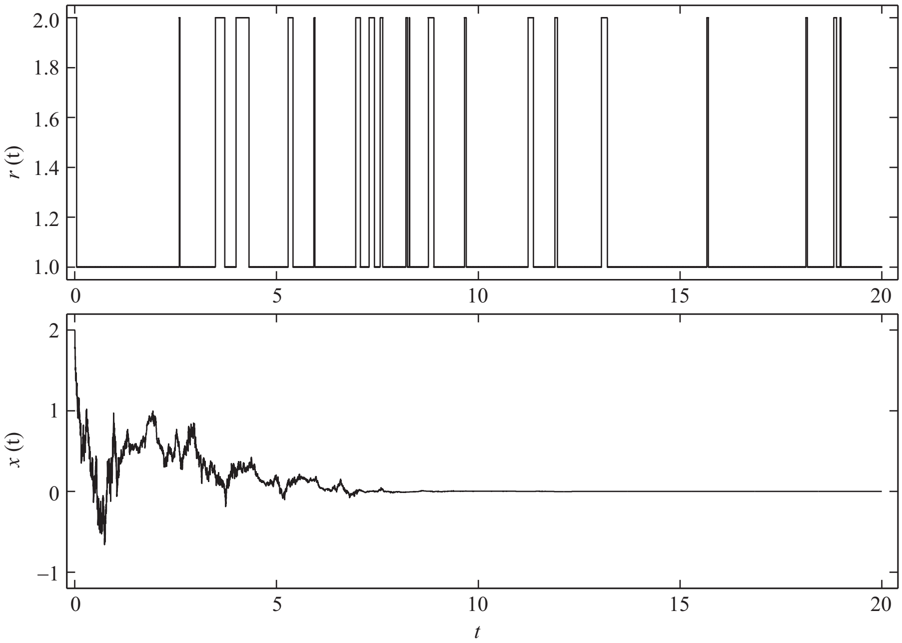

Figure 3 Computer simulation of the sample paths of the Markov chain and the NSDDE (32).

$$\theta_{41} = 0.0556,\ \theta_{42} = 0.2107,\ c_1 = 0.432,\ c_2 = 0.1596.$$

Hence, inequality (27) holds. Consequently, by Theorem 3, we conclude that the NSDDE (32) is stable, although the second subsystem is unstable. Figure 3 illustrates the sample paths of the Markov chain and the NSDDE (32).

5 Conclusion

In this study, we investigated boundedness and stability of highly nonlinear hybrid NSDDEs with multiple delays without assuming linear growth condition. We also introduced a general Lyapunov function to over come the difficulties in handling nonlinear growth conditions. Furthermore, the method of M-matrix was utilized to analyze mean square exponential stability of NSDDEs. Our results can be applied to a larger class of hybrid NSDDEs which may have different structures or parameters in different modes. We have also presented three examples to demonstrate the applicability of our theorems.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 71571001, 71873002), Natural Science Foundation of Universities of Anhui Province (Grant No. KJ2018A0119), and Promoting Plan of Higher Education of Anhui Province (Grant No. TSKJ2016B11).

References

1 Song J, Niu Y, Zou Y Y. Asynchronous sliding mode control of Markovian jump systems with time-varying delays and partly accessible mode detection probabilities. Automatica, 2018, 93: 33–41
2 Fei W Y, Hu L J, Mao X R, et al. Generalized criteria on delay dependent stability of highly nonlinear hybrid stochastic systems. Int J Robust Nonlinear Control, 2019, 29: 1201–1215
3 Wang B, Zhu Q. Stability analysis of semi-Markov switched stochastic systems. Automatica, 2018, 94: 72–80
4 Fei C, Shen M X, Fei W Y, et al. Stability of highly nonlinear hybrid stochastic integro-differential delay equations. Nonlinear Anal-Hybrid Syst, 2019, 31: 180–199
5 Mao X R, Yuan C G. Stochastic Differential Equations with Markovian Switching. London: Imperial College Press, 2006
6 Song G F, Lu Z Y, Zheng B C, et al. Almost sure stabilization of hybrid systems by feedback control based on discrete-time observations of mode and state. Sci China Inf Sci, 2018, 61: 070213
7 Yan Z G, Song Y X, Park J H. Finite-time stability and stabilization for stochastic markov jump systems with mode-dependent time delays. ISA Trans, 2017, 68: 141–149
8 Mao X R. Stochastic Differential Equations and Applications. 2nd ed. Chichester: Horwood Publishing, 2007

9 Cao Y Y, Lam J, Hu L. Delay-dependent stochastic stability and H_∞ analysis for time-delay systems with Markovian jumping parameters. J Franklin Inst, 2003, 340: 423–434
10 Wu Y, Liu M, Wu X, et al. Input-to-state stability analysis for stochastic delayed systems with markovian switching. IEEE Access, 2017, 5: 23663–23671
11 Wu X T, Tang Y, Zhang W B. Stability analysis of stochastic delayed systems with an application to multi-agent systems. IEEE Trans Autom Control, 2016, 61: 4143–4149
12 Hu L J, Mao X R, Shen Y. Stability and boundedness of nonlinear hybrid stochastic differential delay equations. Syst Control Lett, 2013, 62: 178–187
13 Hu L J, Mao X R, Zhang L G. Robust stability and boundedness of nonlinear hybrid stochastic differential delay equations. IEEE Trans Autom Control, 2013, 58: 2319–2332
14 Fei W Y, Hu L J, Mao X R, et al. Delay dependent stability of highly nonlinear hybrid stochastic systems. Automatica, 2017, 82: 165–170
15 Fei W Y, Hu L J, Mao X R, et al. Structured robust stability and boundedness of nonlinear hybrid delay systems. SIAM J Control Optim, 2018, 56: 2662–2689
16 Obradović M, Milošević M. Stability of a class of neutral stochastic differential equations with unbounded delay and Markovian switching and the Euler-Maruyama method. J Comput Appl Math, 2017, 309: 244–266
17 Chen H B, Shi P, Lim C L, et al. Exponential stability for neutral stochastic Markov systems with time-varying delay and its applications. IEEE Trans Cybern, 2016, 46: 1350–1362
18 Mao W H, Deng F Q, Wan A H. Robust H_2/H_∞ global linearization filter design for nonlinear stochastic time-varying delay systems. Sci China Inf Sci, 2016, 59: 032204
19 Mo H Y, Li M L, Deng F Q, et al. Exponential stability of the Euler-Maruyama method for neutral stochastic functional differential equations with jumps. Sci China Inf Sci, 2018, 61: 070214
20 Deng F Q, Mao W H, Wan A H. A novel result on stability analysis for uncertain neutral stochastic time-varying delay systems. Appl Math Comput, 2013, 221: 132–143
21 Shen M X, Fei W Y, Mao X R, et al. Exponential stability of highly nonlinear neutral pantograph stochastic differential equations. Asian J Control, 2018. doi: 10.1002/asjc.1903
22 Wu F K, Hu S G, Huang C M. Robustness of general decay stability of nonlinear neutral stochastic functional differential equations with infinite delay. Syst Control Lett, 2010, 59: 195–202
23 Shen M X, Fei W Y, Mao X R, et al. Stability of highly nonlinear neutral stochastic differential delay equations. Syst Control Lett, 2018, 115: 1–8
24 Luo Q, Mao X R, Shen Y. New criteria on exponential stability of neutral stochastic differential delay equations. Syst Control Lett, 2006, 55: 826–834
25 Song Y, Shen Y. New criteria on asymptotic behavior of neutral stochastic functional differential equations. Automatica, 2013, 49: 626–632
26 Kolmanovskii V, Koroleva N, Maizenberg T, et al. Neutral stochastic differential delay equations with Markovian switching. Stoch Anal Appl, 2003, 21: 819–847
27 Mao X R, Shen Y, Yuan C G. Almost surely asymptotic stability of neutral stochastic differential delay equations with Markovian switching. Stoch Proc Appl, 2008, 118: 1385–1406
28 Li M L, Deng F Q. Almost sure stability with general decay rate of neutral stochastic delayed hybrid systems with Lévy noise. Nonlinear Anal-Hybrid Syst, 2017, 24: 171–185
29 Zhao N, Zhang X, Xue Y, et al. Necessary conditions for exponential stability of linear neutral type systems with multiple time delays. J Franklin Inst, 2018, 355: 458–473
30 Wang J J, Hu P, Chen H B. Delay-dependent exponential stability for neutral stochastic system with multiple time-varying delays. IET Control Theory Appl, 2014, 8: 2092–2101
31 Park J H. A new delay-dependent criterion for neutral systems with multiple delays. J Comput Appl Math, 2001, 136: 177–184
32 Zhang H G, Dong M, Wang Y C, et al. Stochastic stability analysis of neutral-type impulsive neural networks with mixed time-varying delays and Markovian jumping. Neurocomputing, 2010, 73: 2689–2695
33 Lu S, Ge W. Existence of positive periodic solutions for neutral logarithmic population model with multiple delays. J Comput Appl Math, 2004, 166: 371–383