SCIENCE CHINA
Information Sciences

• RESEARCH PAPER •

October 2019, Vol. 62 202204:1–202204:16
https://doi.org/10.1007/s11432-018-9732-7

Leader-following consensus of second-order nonlinear multi-agent systems with intermittent position measurements

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Received 9 October 2018/Accepted 12 December 2018/Published online 21 August 2019

Abstract This work studies the leader-following consensus problem of second-order nonlinear multi-agent systems with aperiodically intermittent position measurements. Through the filter-based method, a novel intermittent consensus protocol without velocity measurements is designed for each follower exclusively based on the relative position measurements of neighboring agents. Under the common assumption that only relative position measurements between the neighboring agents are intermittently used, some consensus conditions are derived for second-order leader-following multi-agent systems with inherent delayed nonlinear dynamics. Moreover, for multi-agent systems without inherent delayed nonlinear dynamics, some simpler consensus conditions are presented. Finally, some simulation examples are presented to verify and illustrate the theoretical results.

Keywords leader-following consensus, second-order multi-agent system, delayed nonlinear dynamics, distributed filter, intermittent measurements

Citation Wang F Y, Liu Z X, Chen Z Q. Leader-following consensus of second-order nonlinear multi-agent systems with intermittent position measurements. Sci China Inf Sci, 2019, 62(10): 202204, https://doi.org/10.1007/s11432-018-9732-7

1 Introduction

Over the past few years, distributed collaboration control for networked multi-agent systems has attracted considerable attention from different scientific fields, such as control theory, systems engineering, applied mathematics, and computer science. This topic has wide applications in many areas, such as flocking [1], synchronization [2], formation control [3], and distributed sensor networks [4]. In multi-agent system networks, all autonomous agents can communicate and cooperate with their neighbors. Based on distributed control strategy, all the collaborative agents can complete some large or complex assignments in multi-agent systems, which is the motivation of distributed collaboration control.

One of the most fundamental and important interests to be explored of distributed collaboration control is consensus-seeking. For distributed consensus-seeking, some appropriate distributed control strategies are adopted such that a group of autonomous agents asymptotically reach an agreement on a common value. The consensus-seeking problems have been intensively discussed from different perspectives including leaderless consensus (i.e., consensus without any leader) [5–10], leader-following consensus (i.e., consensus with a leader) [11–15], and containment control (i.e., consensus with multiple leaders) [16–20]. Note that the abovementioned references only address the consensus problems of multi-agent systems under continuous communication. In this case, each agent can communicate and cooperate with its

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neighbors at all times. However, in the real world, owing to some external factors, communication among the autonomous agents is often discontinuous or intermittent. Under periodically intermittent communication, several intermittent control protocols for consensus-seeking of multi-agent systems have been developed [21–24]. Multi-agent consensus problems have also been intensively investigated [25–28] owing to the coexistence of time delays and intermittent communication. In [29], based on aperiodically intermittent control strategy, the second-order consensus problem was addressed for delayed multi-agent systems under aperiodically intermittent communication.

Notably, all of the abovementioned studies on second-order consensus typically require velocity measurements. However, owing to inaccurate velocity measurements, the relative velocity between the neighboring agents is usually unavailable in multi-agent systems. In addition, to save system cost, load weight, and equipment space, the agents may not be equipped with velocity sensors in many actual systems. Thus, velocity measurements are usually unavailable in real systems. For multi-agent systems without relative velocity measurements, many profound theoretical results exist on distributed coordination control [30–33]. According to the filter-based method, the distributed consensus control problem of second-order nonlinear multi-agent systems was studied in [34], in which velocity measurements were unavailable. Note that all the aforementioned results only focused on either intermittent communication or inaccurate velocity measurements. However, these two communication limitations always coexist in real networks. To the best of our knowledge, under intermittent communication, few studies have discussed the dynamic behavior of second-order multi-agent systems with unavailable velocity measurements.

However, a novel intermittent consensus protocol without velocity measurements under aperiodically intermittent communication is proposed herein to address the leader-following consensus problem of second-order nonlinear multi-agent systems. The contributions and novelties of this article can be summarized as follows.

- Compared with most current studies on second-order consensus, in this study, a second-order leader-following consensus problem for delayed nonlinear multi-agent systems without relative velocity measurements is considered under aperiodically intermittent communication, which can describe more realistic distributed coordination control of second-order multi-agent systems.
- To overcome the challenging problem arising from aperiodically intermittent communication and inaccurate velocity measurements, a novel intermittent consensus algorithm without velocity measurements is developed for leader-following consensus based on the distributed filter designed.
- Under aperiodically intermittent communication, in this study, the proposed control protocol only used the relative position measurements among agents for a second-order leader-following consensus. In contrast, current related studies on second-order consensus used both relative position and relative velocity measurements under periodically intermittent communication or continuous communication.

The remaining article is organized as follows. Section 2 states graph theory, problem formulation, and some useful definitions and lemmas. The main theoretical results for the leader-following consensus of second-order nonlinear multi-agent systems with aperiodically intermittent position measurements are derived in Section 3. Section 4 provides two simulation examples to validate the theoretical analysis. Finally, Section 5 presents the study's conclusion.

Notation. Some notations that are used throughout this article are presented below. Let $\mathbb{R}$, $\mathbb{R}^n$, and $\mathbb{R}^{n\times n}$ be the set of integer numbers, the set of n-dimensional real vector space, and the set of $n\times n$ real matrix space, respectively. I_n, $\mathbf{1}_n$, $\mathbf{0}_n$ and $0_{m\times n}$ represent the n-dimensional identity matrix, the n-dimensional column vector with all ones, the n-dimensional column vector with all zeros, and the $m\times n$ matrix with all zeros. $\|\cdot\|$ denotes the Euclidean norm. For a square matrix A, let $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ represent the maximum and minimum eigenvalues of A, respectively.

2 Problem formulation

Suppose that a communication network among n nodes is denoted by a weighted undirected graph $G=(W,E,A)$, where $W=\{w_1,w_2,\dots,w_n\}$ represents the set of nodes, $E\subseteq W\times W$ represents the set of edges, and A represents the adjacency matrix with weighted values. In particular, $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, in which $a_{ij} \geqslant 0$, $i, j = 1, 2, \ldots, n$, denote the adjacency elements of the matrix A. Moreover, $a_{ij} > 0$ if and only if $(w_j, w_i) \in E$. For all $i \in \{1, 2, \ldots, n\}$, $a_{ii} = 0$, implying that self-loops are not allowed. The Laplacian matrix of the graph G is defined by $L = [l_{ij}] \in \mathbb{R}^{n\times n}$, $i, j = 1, 2, \ldots, n$, where $l_{ij} = \sum_{j=1,j\neq i}^{n} a_{ij}$, $i = j$; $l_{ij} = -a_{ij}$, $i \neq j$. In multi-agent systems, node w_i can be regarded as an agent i. An undirected path from w_i to w_j is denoted by $\pi_{i,j} = \{(w_{i1}, w_{i2}), (w_{i2}, w_{i3}), \ldots, (w_{iq-1}, w_{iq})\}$, where $w_{i1} = w_i$, $w_{iq} = w_j$, and $(w_{ip}, w_{ip+1}) \in E \Leftrightarrow (w_{ip+1}, w_{ip}) \in E$, $p \in \{1, 2, \ldots, q-1\}$. For an undirected graph, $(w_j, w_i) \in E \Leftrightarrow (w_i, w_j) \in E$ implies $a_{ij} = a_{ji}$, which means that agent j and agent i can exchange information. An undirected network G is connected if there is an undirected path between any pair of distinct nodes.

Consider a multi-agent network comprising n followers and a leader (labeled 0), which is described by a simple graph $\bar{G} = (\bar{W}, \bar{E}, \bar{A})$. The node set is denoted by $\bar{W} = \{w_0, w_1, w_2, \ldots, w_n\}$, the edge set is denoted by $\bar{E}$ that contains the edge set E, and the directed edges from the leader to the followers, and the weighted adjacency matrix is denoted by $\bar{A} = [a_{ij}] \in \mathbb{R}^{(n+1)\times(n+1)}$. The Laplacian matrix of the graph $\bar{G}$ is defined by $\hat{L}$. Moreover, $\hat{L} = L + \mathrm{diag}\{a_{10}, a_{20}, \ldots, a_{n0}\}$, in which L is the Laplacian matrix of the graph G, $a_{i0} > 0$ if follower i can receive information from the leader, and $a_{i0} = 0$ otherwise, $i = 1, 2, \ldots, n$.

Definition 1 ([11]). The graph $\bar{G}$ is said to be connected if there exists at least one agent in G that can connect to the leader via a directed edge.

Lemma 1 ([11]). If graph $\bar{G}$ is connected, then the symmetric matrix $\hat{L}$ associated with $\bar{G}$ is positive definite.

Consider a leader-following multi-agent system with the following second-order nonlinear dynamic model. Each follower has the following dynamics:

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = f(t, v_i(t), v_i(t-\tau(t))) + u_i(t), \quad i = 1, 2, \ldots, n, \end{cases} \tag{1}$$

where $x_i(t) \in \mathbb{R}^m$, $v_i(t) \in \mathbb{R}^m$, and $u_i(t) \in \mathbb{R}^m$ are the position state, velocity state, and control input of agent i, respectively. $f : \mathbb{R}^m \times \mathbb{R}^m \times [0, +\infty) \to \mathbb{R}^m$ is a uniform continuously differentiable vector-valued function having a time-varying nonlinear property, which describes the inherent dynamics of agent i. In addition, the leader, labeled $i = 0$, has the following dynamics:

$$\begin{cases} \dot{x}_0(t) = v_0(t), \\ \dot{v}_0(t) = f(t, v_0(t), v_0(t-\tau(t))), \end{cases} \tag{2}$$

where $x_0(t) \in \mathbb{R}^m$ and $v_0(t) \in \mathbb{R}^m$ are the position and velocity states of the leader, respectively, and $f(t, v_0(t), v_0(t-\tau(t)))$ is defined as that in (1). $\tau(t) > 0$ represents time-varying delays. Moreover, for the nonlinear function $f(\cdot)$ in systems (1) and (2), the following Lipschitz-type condition is satisfied.

Assumption 1 ([8]). For any $x, y, z, w \in \mathbb{R}^n$, there exist non-negative constants ρ_1 and ρ_2 such that

$$\|f(t, x, w) - f(t, y, z)\| \leqslant \rho_1\|x - y\| + \rho_2\|w - z\|, \quad \forall t \geqslant 0.$$

Definition 2. The second-order leader-following consensus of systems (1) and (2) is said to be achieved if, for any initial conditions $x_i(0)$, $v_i(0)$, $i = 0, 1, \ldots, n$,

$$\begin{cases} \lim\limits_{t\to\infty} \|x_i(t) - x_0(t)\| = 0, \\ \lim\limits_{t\to\infty} \|v_i(t) - v_0(t)\| = 0, \quad i = 1, 2, \ldots, n. \end{cases} \tag{3}$$

In real systems, velocity measurements are usually unavailable. To overcome this challenge, an approximation auxiliary filter vector was introduced for a serial n-link rigid robot manipulator in [35], and a distributed filter was used for the coordination control of multi-agent systems in [34]. Motivated by the studies of [34, 35], a distributed filter is introduced for each follower as follows:

$$\begin{cases} \omega_i(t) = \vartheta_i(t) + b\sum_{j=0}^{n} a_{ij}(x_i(t) - x_j(t)), \\ \dot{\vartheta}_i(t) = -a\omega_i(t), \qquad\qquad i = 1, 2, \ldots, n, \end{cases} \tag{4}$$

where $\omega_i(t) \in \mathbb{R}^m$ is the filter output, $\vartheta_i(t) \in \mathbb{R}^m$ is an auxiliary filter vector, and $a > 0$, $b > 0$ are two constants. For the sake of simplicity, we assume that $m = 1$ in this study. For the case of $m > 1$, all obtained results still hold using the Kronecker product operations.

Let the time span $t \in [0, +\infty)$ be divided into several uniformly bounded and non-overlapping sequences $[t_k, t_{k+1})$, $k = 0, 1, \ldots$, $t_0 = 0$. For any $[t_k, t_{k+1})$, $k = 0, 1, \ldots$, there exists a time point δ_k such that the time interval $[t_k, t_{k+1})$ is divided into $[t_k, \delta_k]$ and (δ_k, t_{k+1}), where $[t_k, \delta_k]$ is said to be the communication time duration and (δ_k, t_{k+1}) is said to be the communication time interruption. In multi-agent networks, over the time periods $[t_k, \delta_k]$, each agent can communicate with their neighbors, whereas over the time periods (δ_k, t_{k+1}) all agents cannot obtain the information of their neighbors. Moreover, $\delta_k - t_k$ denotes the k-th communication width, and $t_{k+1} - \delta_k$ denotes the k-th interruption width. Obviously, the intermittent communication type of the network is aperiodic.

Assumption 2 ([29])**.** Suppose that the following condition is satisfied under aperiodically intermittent communication:

$$\begin{cases} \inf_k \theta_k = \theta, \\ \sup_k T_k = T, \end{cases} \tag{5}$$

where $\theta \in (0, T]$, and $T \in (0, +\infty)$ are two constants, and $\theta_k = \delta_k - t_k$, $T_k = t_{k+1} - t_k$, $k = 0, 1, 2, \ldots$.

Remark 1. Assumption 2 means that the duration of each communication should not be less than θ, and the total time over $[t_k, t_{k+1})$ should not be larger than T; i.e., the duration of each communication interruption should not be larger than $T - \theta$.

The following Lemmas 2–6 are required for the theoretical analysis of the study.

Lemma 2 (Schur complement [36])**.** For a given symmetric matrix

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^{\mathrm{T}} & S_{22} \end{bmatrix},$$

where S_{11} and S_{22} are square matrices. The following statements are equivalent:

(a) $S > 0$;

(b) $S_{11} > 0$, $S_{22} - S_{12}^{\mathrm{T}} S_{11}^{-1} S_{12} > 0$;

(c) $S_{22} > 0$, $S_{11} - S_{12} S_{22}^{-1} S_{12}^{\mathrm{T}} > 0$.

Lemma 3 ([37])**.** For any two real vectors x and y with the same dimensions, the following inequality holds:

$$\pm 2x^{\mathrm{T}} y \leqslant x^{\mathrm{T}} \Phi x + y^{\mathrm{T}} \Phi^{-1} y,$$

where Φ is any positive definite matrix with appropriate dimensions.

Lemma 4 ([38])**.** Suppose that $M \in \mathbb{R}^{n\times n}$ is a positive definite matrix, and $N \in \mathbb{R}^{n\times n}$ is symmetric. Then, for any vector $x \in \mathbb{R}^n$, the following inequality holds:

$$\lambda_{\min}(M^{-1}N) x^{\mathrm{T}} M x \leqslant x^{\mathrm{T}} N x \leqslant \lambda_{\max}(M^{-1}N) x^{\mathrm{T}} M x.$$

Lemma 5 ([26])**.** Suppose that the non-negative function $g(t)$ satisfies the condition, for $t \in [-\tau, +\infty)$,

$$\frac{\mathrm{d}g(t)}{\mathrm{d}t} \leqslant -c_1 g(t) + c_2 g(t - \tau), \quad t \geqslant 0,$$

where constants $c_1 > c_2 > 0$. Then,

$$g(t) \leqslant \bar{g}(0) \exp\{-\varepsilon(t)\}, \quad t \geqslant 0,$$

where $\bar{g}(t) = \sup_{-\tau \leqslant \epsilon \leqslant 0}(g(t+\epsilon))$, and $\varepsilon > 0$ is the unique solution of $\varepsilon - c_1 + c_2 \exp\{\varepsilon\tau\} = 0$.

Lemma 6 ([26])**.** Suppose that the non-negative function $g(t)$ satisfies the condition, for $t \in [-\tau, +\infty)$,

$$\frac{\mathrm{d}g(t)}{\mathrm{d}t} \leqslant h_1 g(t) + h_2 g(t-\tau), \quad t \geqslant 0,$$

where $h_1 > 0$ and $h_2 > 0$ are two constants. Then,

$$g(t) \leqslant |g(0)|_\tau \exp\{(h_1 + h_2)t\}, \quad t \geqslant 0,$$

where $|g(0)|_\tau = \sup_{-\tau \leqslant s \leqslant 0} g(s)$.

3 Main results

Based on only relative position measurements, a novel intermittent consensus protocol is presented below:

$$\begin{cases} u_i(t) = -\gamma \left[\sum\limits_{j=0}^{n} a_{ij}(x_i(t) - x_j(t)) + \omega_i(t) \right], & t \in [t_k, \delta_k], \\ u_i(t) = 0, & t \in (\delta_k, t_{k+1}), \end{cases} \tag{6}$$

where γ is a positive constant to be determined, $i = 1, 2, \ldots, n$.

Remark 2. Under the case where the relative velocity information is unavailable, and the interacting information between neighboring agents is aperiodically intermittent, the control protocol (6) is designed for second-order leader-following consensus. However, in contrast to majority previous studies, under aperiodically intermittent communications, only relative intermittent position measurements can be used for the consensus control protocol (6). Using the filter output $\omega_i(t)$ on the work time $[t_k, \delta_k]$, the relative velocity measurements $\sum_{j=0}^{n} a_{ij}(v_i(t) - v_j(t))$ are replaced by the filtered relative position measurements $\sum_{j=0}^{n} a_{ij}(x_i(t) - x_j(t))$.

Before going forward, in systems (1) and (2), for time-varying delay $\tau(t)$, the following assumption is required for deriving the main results.

Assumption 3. There exists a constant $\tau^* > 0$, such that the time-varying delay satisfies $\tau(t) \leqslant \tau^* < \theta$, where θ is defined in Assumption 2.

Let $\tilde{x}_i(t) = x_i(t) - x_0(t)$ and $\tilde{v}_i(t) = v_i(t) - v_0(t)$, $i = 1, 2, \ldots, n$; then the systems (1) and (2) with protocol (6) can be written in a close-loop form

$$\begin{cases} \dot{\tilde{x}}_i(t) = \tilde{v}_i(t), \\ \dot{\tilde{v}}_i(t) = -\gamma \sum\limits_{j=0}^{n} a_{ij}(x_i(t) - x_j(t)) - \gamma\omega_i(t) + \tilde{f}(t, v_i(t), v_i(t-\tau(t))), & t \in [t_k, \delta_k], \\ \dot{\tilde{v}}_i(t) = \tilde{f}(t, v_i(t), v_i(t-\tau(t))), & t \in (\delta_k, t_{k+1}), \end{cases} \tag{7}$$

where $\tilde{f}(t, v_i(t), v_i(t-\tau(t))) = f(t, v_i(t), v_i(t-\tau(t))) - f(t, v_0(t), v_0(t-\tau(t)))$, $i = 1, 2, \ldots, n$.

Let $\omega(t) = [\omega_1(t), \omega_2(t), \ldots, \omega_n(t)]^{\mathrm{T}}$, $\tilde{x}(t) = [\tilde{x}_1(t), \tilde{x}_2(t), \ldots, \tilde{x}_n(t)]^{\mathrm{T}}$ and $\tilde{v}(t) = [\tilde{v}_1(t), \tilde{v}_2(t), \ldots, \tilde{v}_n(t)]^{\mathrm{T}}$. Then, system (4) can be rewritten as follows:

$$\dot{\omega}(t) = -a\omega(t) + b\hat{L}\tilde{v}(t); \tag{8}$$

and system (7) can be rewritten as

$$\begin{cases} \dot{\tilde{x}}(t) = \tilde{v}(t), \\ \dot{\tilde{v}}(t) = -\gamma\hat{L}\tilde{x}(t) - \gamma\omega(t) + \tilde{f}(t, v(t), v(t-\tau(t))), & t \in [t_k, \delta_k], \\ \dot{\tilde{v}}(t) = \tilde{f}(t, v(t), v(t-\tau(t))), & t \in (\delta_k, t_{k+1}), \end{cases} \tag{9}$$

where $\tilde{f}(t,v(t),v(t-\tau(t)))=f(t,v(t),v(t-\tau(t)))-\mathbf{1}_n\otimes f(t,v_0(t),v_0(t-\tau(t)))$ and $f(t,v(t),v(t-\tau(t)))=[f(t,v_1(t),v_1(t-\tau(t))),f(t,v_2(t),v_2(t-\tau(t))),\ldots,f(t,v_n(t),v_n(t-\tau(t)))]^{\rm T}$.

For convenience, let $z(t)=[\tilde{x}^{\rm T}(t)\hat{L},\tilde{v}^{\rm T}(t),\omega^{\rm T}(t)]^{\rm T}$; then, systems (8) and (9) can be transformed into

$$\begin{cases}\dot{z}(t)=H_1z(t)+F(t,v(t),v(t-\tau(t))), & t\in[t_k,\delta_k],\\ \dot{z}(t)=H_2z(t)+F(t,v(t),v(t-\tau(t))), & t\in(\delta_k,t_{k+1}),\end{cases}\tag{10}$$

where

$$H_1=\begin{bmatrix}0_{n\times n} & \hat{L} & 0_{n\times n}\\ -\gamma I_n & 0_{n\times n} & -\gamma I_n\\ 0_{n\times n} & b\hat{L} & -aI_n\end{bmatrix},\quad H_2=\begin{bmatrix}0_{n\times n} & \hat{L} & 0_{n\times n}\\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n}\\ 0_{n\times n} & b\hat{L} & -aI_n\end{bmatrix},$$

and $F(t,v(t),v(t-\tau(t)))=[\mathbf{0}_n^{\rm T},\tilde{f}^{\rm T}(t,v(t),v(t-\tau(t))),\mathbf{0}_n^{\rm T}]^{\rm T}$.

Theorem 1. Suppose that G is undirected, $\bar{G}$ is connected, and Assumptions 1–3 hold. Then, the control protocol (6) makes the systems (1) and (2) achieve second-order leader-following consensus, if the following condition holds:

$$\begin{cases}\gamma>\max\left\{\dfrac{\lambda_{\max}(\hat{L})}{2},d_1\right\},\\ b>\dfrac{(\gamma+d_2)}{\lambda_{\min}(\hat{L})}+1,\\ a>2\gamma+d_3,\\ \lambda_{\min}(R_1)>\dfrac{2\rho_2\lambda_{\max}(P)}{\lambda_{\min}(P)},\\ \varpi=\mu(\theta-\tau^*)-\upsilon(T-\theta)>0,\end{cases}\tag{11}$$

where μ is the unique positive solution of $\mu-\eta_1+\eta_2\exp\{\mu\tau^*\}=0$, $\eta_1=\frac{\lambda_{\min}(R_1)}{\lambda_{\max}(P)}$, $\eta_2=\frac{2\rho_2}{\lambda_{\min}(P)}$, $\upsilon=\eta_3+\eta_4$, $\eta_3=\lambda_{\max}(P^{-1}R_2)$, $\eta_4=\frac{2\rho_2}{\lambda_{\min}(P)}$, $d_1=\frac{\rho_1+\rho_2}{2}$, $d_2=3\rho_1+\rho_2$, $d_3=\frac{\rho_1+\rho_2}{2}$, and ρ_1 and ρ_2 are defined as that in Assumption 1.

Proof. Consider a Lyapunov function defined as follows:

$$V(t)=\frac{1}{2}z^{\rm T}(t)Pz(t),\tag{12}$$

where

$$P=\begin{bmatrix}\gamma\hat{L}^{-1} & \frac{1}{2}I_n & 0_{n\times n}\\ \frac{1}{2}I_n & I_n & -\frac{1}{2}I_n\\ 0_{n\times n} & -\frac{1}{2}I_n & \frac{1}{2}I_n\end{bmatrix}.$$

According to Lemma 1, $\hat{L}$ is symmetric and positive definite. Through Lemma 2, we have that $P>0$ if and only if

$$\begin{bmatrix}\gamma\hat{L}^{-1} & \frac{1}{2}I_n\\ \frac{1}{2}I_n & I_n\end{bmatrix}-2\begin{bmatrix}0_{n\times n}\\ -\frac{1}{2}I_n\end{bmatrix}\begin{bmatrix}0_{n\times n} & -\frac{1}{2}I_n\end{bmatrix}=\frac{1}{2}\begin{bmatrix}2\gamma\hat{L}^{-1} & I_n\\ I_n & I_n\end{bmatrix}>0.$$

Then, using the expression $\gamma>\frac{\lambda_{\max}(\hat{L})}{2}$ in condition (11), we can easily check that $P>0$ and $V(t)\geqslant 0$. In addition, $V(t)=0$ if and only if $z(t)=\mathbf{0}_{3n}$. Moreover, we obtain

$$V(t)\geqslant\frac{1}{2}\lambda_{\min}(P)z^{\rm T}(t)z(t),\tag{13}$$

$$V(t) \leqslant \frac{1}{2}\lambda_{\max}(P)z^{\mathrm{T}}(t)z(t). \tag{14}$$

When $t \in [t_k, \delta_k]$, $k = 0, 1, 2, \ldots$, considering $\dot{V}(t)$ along the system (10), we have

$$\begin{aligned}
\dot{V}(t) &= z^{\mathrm{T}}(t)P[H_1 z(t) + F(t, v(t), v(t-\tau(t)))] \\
&= -\frac{1}{2}z^{\mathrm{T}}(t)\begin{bmatrix} \gamma I_n & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & (b-1)\hat{L} & \left(\gamma - \frac{a}{2}\right)I_n - \frac{b}{2}\hat{L} \\ 0_{n\times n} & \left(\gamma - \frac{a}{2}\right)I_n - \frac{b}{2}\hat{L} & (a-\gamma)I_n \end{bmatrix} z(t) \\
&\quad + z^{\mathrm{T}}(t)PF(t, v(t), v(t-\tau(t))) \\
&= -\frac{1}{2}\Bigg\{\gamma\tilde{x}^{\mathrm{T}}(t)\hat{L}^2\tilde{x}(t) + (b-1)\tilde{v}^{\mathrm{T}}(t)\hat{L}\tilde{v}(t) + (a-\gamma)\omega^{\mathrm{T}}(t)\omega(t) \\
&\quad + 2\left[\gamma\tilde{v}^{\mathrm{T}}(t)\omega(t) - \frac{a}{2}\tilde{v}^{\mathrm{T}}(t)\omega(t) - \frac{b}{2}\tilde{v}^{\mathrm{T}}(t)\hat{L}\omega(t)\right]\Bigg\} \\
&\quad + \left[\frac{1}{2}\hat{L}\tilde{x}(t) + \tilde{v}(t) - \frac{1}{2}\omega(t)\right]^{\mathrm{T}}\tilde{f}(t, v(t), v(t-\tau(t))).
\end{aligned} \tag{15}$$

Using Lemma 3, we obtain

$$\begin{aligned}
\dot{V}(t) &\leqslant -\frac{1}{2}\Bigg\{\gamma\tilde{x}^{\mathrm{T}}(t)\hat{L}^2\tilde{x}(t) + (b-1)\tilde{v}^{\mathrm{T}}(t)\hat{L}\tilde{v}(t) + (a-\gamma)\omega^{\mathrm{T}}(t)\omega(t) \\
&\quad + \left(-\gamma + \frac{a}{2}\right)\left[\tilde{v}^{\mathrm{T}}(t)\tilde{v}(t) + \omega^{\mathrm{T}}(t)\omega(t)\right] \\
&\quad + \frac{b}{2}\left[\tilde{v}^{\mathrm{T}}(t)\tilde{v}(t) + \omega^{\mathrm{T}}(t)\hat{L}^2\omega(t)\right]\Bigg\} \\
&\quad + \left[\tilde{v}(t) + \frac{1}{2}\hat{L}\tilde{x}(t) - \frac{1}{2}\omega(t)\right]^{\mathrm{T}}\tilde{f}(t, v(t), v(t-\tau(t))) \\
&\leqslant -\frac{1}{2}\{\gamma\tilde{x}^{\mathrm{T}}(t)\hat{L}^2\tilde{x}(t) + \tilde{v}^{\mathrm{T}}(t)[(b-1)\hat{L} - \gamma I_n]\tilde{v}(t) \\
&\quad + (a-2\gamma)\omega^{\mathrm{T}}(t)\omega(t)\} \\
&\quad + \left[\tilde{v}(t) + \frac{1}{2}\hat{L}\tilde{x}(t) - \frac{1}{2}\omega(t)\right]^{\mathrm{T}}\tilde{f}(t, v(t), v(t-\tau(t))) \\
&= -\frac{1}{2}z^{\mathrm{T}}(t)\begin{bmatrix} \gamma I_n & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & (b-1)\hat{L} - \gamma I_n & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & (a-2\gamma)I_n \end{bmatrix} z(t) \\
&\quad + \left[\tilde{v}(t) + \frac{1}{2}\hat{L}\tilde{x}(t) - \frac{1}{2}\omega(t)\right]^{\mathrm{T}}\tilde{f}(t, v(t), v(t-\tau(t))).
\end{aligned} \tag{16}$$

Based on Assumption 1, we can easily show that

$$\begin{aligned}
\frac{1}{2}\tilde{x}^{\mathrm{T}}(t)\hat{L}\tilde{f}(t, v(t), v(t-\tau(t))) &= \frac{1}{2}\tilde{x}^{\mathrm{T}}(t)\hat{L}[f(t, v(t), v(t-\tau(t))) - \mathbf{1}_n \otimes f(t, v_0(t), v_0(t-\tau(t)))] \\
&\leqslant \frac{1}{2}\left\|\hat{L}\tilde{x}(t)\right\| \cdot (\rho_1\|\tilde{v}(t)\| + \rho_2\|\tilde{v}(t-\tau(t))\|) \\
&= \frac{1}{2}\left(\rho_1\left\|\hat{L}\tilde{x}(t)\right\| \cdot \|\tilde{v}(t)\| + \rho_2\left\|\hat{L}\tilde{x}(t)\right\| \cdot \|\tilde{v}(t-\tau(t))\|\right) \\
&\leqslant \frac{1}{2}\left(\frac{\rho_1+\rho_2}{2}\left\|\hat{L}\tilde{x}(t)\right\|^2 + \frac{\rho_1}{2}\|\tilde{v}(t)\|^2 + \frac{\rho_2}{2}\|\tilde{v}(t-\tau(t))\|^2\right),
\end{aligned} \tag{17}$$

$$\begin{aligned}
\tilde{v}^{\mathrm{T}}(t)\tilde{f}(t, v(t), v(t-\tau(t))) &\leqslant \|\tilde{v}(t)\| \cdot (\rho_1\|\tilde{v}(t)\| + \rho_2\|\tilde{v}(t-\tau(t))\|) \\
&\leqslant \left(\rho_1 + \frac{\rho_2}{2}\right)\|\tilde{v}(t)\|^2 + \frac{\rho_2}{2}\|\tilde{v}(t-\tau(t))\|^2,
\end{aligned} \tag{18}$$

$$
\begin{aligned}
-\frac{1}{2}\omega^{\mathrm{T}}(t)\tilde{f}(t,v(t),v(t-\tau(t))) &\leqslant \frac{1}{2}\left\|\omega(t)\right\|\cdot(\rho_1\left\|\tilde{v}(t)\right\|+\rho_2\left\|\tilde{v}(t-\tau(t))\right\|)\\
&\leqslant \frac{1}{2}\left(\frac{\rho_1}{2}\left\|\tilde{v}(t)\right\|^2+\frac{\rho_1+\rho_2}{2}\left\|\omega(t)\right\|^2+\frac{\rho_2}{2}\left\|\tilde{v}(t-\tau(t))\right\|^2\right). \quad (19)
\end{aligned}
$$

Combining (14)–(19), we obtain

$$
\begin{aligned}
\dot{V}(t) &\leqslant -\frac{1}{2}\|z(t)\|^{\mathrm{T}}R_1\left\|z(t)\right\|+\frac{1}{2}\|z(t-\tau(t))\|^{\mathrm{T}}S_1\left\|z(t-\tau(t))\right\|\\
&\leqslant -\frac{1}{2}\lambda_{\min}(R_1)z(t)^{\mathrm{T}}z(t)+\frac{1}{2}\lambda_{\max}(S_1)z(t-\tau(t))^{\mathrm{T}}z(t-\tau(t)), \quad (20)
\end{aligned}
$$

where

$$
R_1=\begin{bmatrix}(\gamma-d_1)I_n & 0_{n\times n} & 0_{n\times n}\\ 0_{n\times n} & (b-1)\hat{L}-(\gamma+d_2)I_n & 0_{n\times n}\\ 0_{n\times n} & 0_{n\times n} & (a-2\gamma-d_3)I_n\end{bmatrix},\quad S_1=\begin{bmatrix}0_{n\times n} & 0_{n\times n} & 0_{n\times n}\\ 0_{n\times n} & 2\rho_2 I_n & 0_{n\times n}\\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n}\end{bmatrix},
$$

and

$$
\begin{cases}
\|z(t)\|=\left[\left\|\hat{L}\tilde{x}(t)\right\|^{\mathrm{T}},\|\tilde{v}(t)\|^{\mathrm{T}},\|\omega(t)\|^{\mathrm{T}}\right]^{\mathrm{T}},\\
\|z(t-\tau(t))\|=\left[\left\|\hat{L}\tilde{x}(t-\tau(t))\right\|^{\mathrm{T}},\|\tilde{v}(t-\tau(t))\|^{\mathrm{T}},\|\omega(t-\tau(t))\|^{\mathrm{T}}\right]^{\mathrm{T}},\\
\|\tilde{x}(t)\|=[\|\tilde{x}_1(t)\|,\ldots,\|\tilde{x}_n(t)\|]^{\mathrm{T}},\\
\|\tilde{v}(t)\|=[\|\tilde{v}_1(t)\|,\ldots,\|\tilde{v}_n(t)\|]^{\mathrm{T}},\\
\|\omega(t)\|=[\|\omega_1(t)\|,\ldots,\|\omega_n(t)\|]^{\mathrm{T}},\\
d_1=d_3=\dfrac{\rho_1+\rho_2}{2},\\
d_2=3\rho_1+\rho_2.
\end{cases}
$$

Moreover, according to condition (11), $R_1>0$.

Inequalities (13), (14), and (20) imply that

$$
\dot{V}(t)\leqslant -\eta_1 V(t)+\eta_2 V(t-\tau(t)), \quad (21)
$$

where $\eta_1=\frac{\lambda_{\min}(R_1)}{\lambda_{\max}(P)}$, $\eta_2=\frac{2\rho_2}{\lambda_{\min}(P)}$.

When $t\in(\delta_k,t_{k+1})$, $k=0,1,2,\ldots$, still considering $\dot{V}(t)$ along the system (10), we have

$$
\begin{aligned}
\dot{V}(t) &= z^{\mathrm{T}}(t)P[H_2 z(t)+F(t,v(t),v(t-\tau(t)))]\\
&= \frac{1}{2}z^{\mathrm{T}}(t)\begin{bmatrix}0_{n\times n} & \gamma I_n & 0_{n\times n}\\ \gamma I_n & (1-b)\hat{L} & \frac{b}{2}\hat{L}+\frac{a}{2}I_n\\ 0_{n\times n} & \frac{b}{2}\hat{L}+\frac{a}{2}I_n & -aI_n\end{bmatrix}z(t)\\
&\quad+\left[\tilde{v}(t)+\frac{1}{2}\hat{L}\tilde{x}(t)-\frac{1}{2}\omega(t)\right]^{\mathrm{T}}\tilde{f}(t,v(t),v(t-\tau(t))). \quad (22)
\end{aligned}
$$

Using a similar analysis method, we obtain

$$
\dot{V}(t)\leqslant \frac{1}{2}\|z(t)\|^{\mathrm{T}}R_2\left\|z(t)\right\|+\frac{1}{2}\|z(t-\tau(t))\|^{\mathrm{T}}S_2\|z(t-\tau(t))\|, \quad (23)
$$

where

$$
R_2=\begin{bmatrix}d_1 I_n & \gamma I_n & 0_{n\times n}\\ \gamma I_n & (1-b)\hat{L}+d_2 I_n & \frac{b}{2}\hat{L}+\frac{a}{2}I_n\\ 0_{n\times n} & \frac{b}{2}\hat{L}+\frac{a}{2}I_n & (d_3-a)I_n\end{bmatrix},\quad S_2=\begin{bmatrix}0_{n\times n} & 0_{n\times n} & 0_{n\times n}\\ 0_{n\times n} & 2\rho_2 I_n & 0_{n\times n}\\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n}\end{bmatrix}.
$$

Based on inequality (23) and Lemma 4, we obtain

$$\dot{V}(t) \leqslant \frac{1}{2}\lambda_{\max}(P^{-1}R_2)z(t)^{\mathrm{T}}Pz(t) + \frac{1}{2}\lambda_{\max}(S_2)z(t-\tau(t))^{\mathrm{T}}z(t-\tau(t)). \tag{24}$$

Using inequalities (13) and (24), we have

$$\dot{V}(t) \leqslant \lambda_{\max}(P^{-1}R_2)V(t) + \frac{\lambda_{\max}(S_2)}{\lambda_{\min}(P)}V(t-\tau(t)) = \eta_3 V(t) + \eta_4 V(t-\tau(t)), \tag{25}$$

where $\eta_3 = \lambda_{\max}(P^{-1}R_2)$, $\eta_4 = \frac{2\rho_2}{\lambda_{\min}(P)}$.

If Assumptions 2, 3 and the condition (11) hold, then we obtain the following results.

For $t \in [0, \delta_0]$, it follows from Lemma 5 that

$$V(t) \leqslant \sup_{-\tau^* \leqslant \epsilon \leqslant 0} V(\epsilon)\exp\{-\mu t\}, \tag{26}$$

where $\mu > 0$ is a solution of $\mu - \eta_1 + \eta_2 \exp\{\mu\tau^*\} = 0$. Moreover, μ is a unique solution. For $t \in (\delta_0, t_1)$, it follows from Lemma 6 that

$$\begin{aligned}
V(t) &\leqslant \sup_{\delta_0-\tau^* \leqslant \epsilon \leqslant \delta_0} V(\epsilon)\exp\{\upsilon(t-\delta_0)\} \\
&\leqslant \sup_{\delta_0-\tau^* \leqslant \epsilon \leqslant \delta_0} \left\{ \sup_{-\tau^* \leqslant s \leqslant 0} V(s)\exp\{-\mu\epsilon\} \right\} \exp\{\upsilon(t-\delta_0)\} \\
&\leqslant \sup_{-\tau^* \leqslant \epsilon \leqslant 0} V(\epsilon)\exp\{-\mu(\delta_0-\tau^*) + \upsilon(t-\delta_0)\} \\
&= \sup_{-\tau^* \leqslant \epsilon \leqslant 0} V(\epsilon)\exp\{-\mu(\delta_0-t_0-\tau^*) + \upsilon(t-\delta_0)\} \\
&\leqslant \sup_{-\tau^* \leqslant \epsilon \leqslant 0} V(\epsilon)\exp\{-\mu(\theta-\tau^*) + \upsilon(t-\delta_0)\},
\end{aligned} \tag{27}$$

where $\upsilon = \eta_3 + \eta_4$.

For $t \in [t_1, \delta_1]$, we obtain

$$\begin{aligned}
V(t) &\leqslant \sup_{t_1-\tau^* \leqslant \epsilon \leqslant t_1} V(\epsilon)\exp\{-\mu(t-t_1)\} \\
&\leqslant \sup_{t_1-\tau^* \leqslant \epsilon \leqslant t_1} \left\{ \sup_{-\tau^* \leqslant s \leqslant 0} V(s)\exp\{-\mu(\theta-\tau^*) + \upsilon(\epsilon-\delta_0)\} \right\} \times \exp\{-\mu(t-t_1)\} \\
&\leqslant \sup_{-\tau^* \leqslant \epsilon \leqslant 0} V(\epsilon)\exp\{-\mu(\theta-\tau^*) + \upsilon(t_1-\delta_0) - \mu(t-t_1)\} \\
&= \sup_{-\tau^* \leqslant \epsilon \leqslant 0} V(\epsilon)\exp\{-\mu(\theta-\tau^*) + \upsilon[(t_1-t_0)-(\delta_0-t_0)] - \mu(t-t_1)\} \\
&\leqslant \sup_{-\tau^* \leqslant \epsilon \leqslant 0} V(\epsilon)\exp\{-\mu(\theta-\tau^*) + \upsilon(T-\theta) - \mu(t-t_1)\} \\
&= \sup_{-\tau^* \leqslant \epsilon \leqslant 0} V(\epsilon)\exp\{-\varpi - \mu(t-t_1)\},
\end{aligned} \tag{28}$$

where $\varpi = \mu(\theta-\tau^*) - \upsilon(T-\theta) > 0$. For $t \in (\delta_1, t_2)$, we obtain

$$\begin{aligned}
V(t) &\leqslant \sup_{\delta_1-\tau^* \leqslant \epsilon \leqslant \delta_1} V(\epsilon)\exp\{\upsilon(t-\delta_1)\} \\
&\leqslant \sup_{\delta_1-\tau^* \leqslant \epsilon \leqslant \delta_1} \left\{ \sup_{-\tau^* \leqslant s \leqslant 0} V(s)\exp\{-\varpi - \mu(\epsilon-t_1)\} \right\} \times \exp\{\upsilon(t-\delta_1)\} \\
&\leqslant \sup_{-\tau^* \leqslant \epsilon \leqslant 0} V(\epsilon)\exp\{-\varpi - \mu(\delta_1-\tau^*-t_1) + \upsilon(t-\delta_1)\} \\
&\leqslant \sup_{-\tau^* \leqslant \epsilon \leqslant 0} V(\epsilon)\exp\{-\varpi - \mu(\theta-\tau^*) + \upsilon(t-\delta_1)\}.
\end{aligned} \tag{29}$$

For $t \in [t_2, \delta_2]$, we have

$$\begin{aligned} V(t) &\leqslant \sup_{t_2-\tau^*\leqslant\epsilon\leqslant t_2} V(\epsilon)\exp\{-\mu(t-t_2)\} \\ &\leqslant \sup_{t_2-\tau^*\leqslant\epsilon\leqslant t_2} \left\{ \sup_{-\tau^*\leqslant s\leqslant 0} V(s)\exp\{-\varpi-\mu(\theta-\tau^*)+\upsilon(\epsilon-\delta_1)\} \right\} \times \exp\{-\mu(t-t_2)\} \\ &\leqslant \sup_{-\tau^*\leqslant\epsilon\leqslant 0} V(\epsilon)\exp\{-\varpi-\mu(\theta-\tau^*)+\upsilon(t_2-\delta_1)-\mu(t-t_2)\} \\ &= \sup_{-\tau^*\leqslant\epsilon\leqslant 0} V(\epsilon)\exp\{-\varpi-\mu(\theta-\tau^*)+\upsilon[(t_2-t_1)-(\delta_1-t_1)]-\mu(t-t_2)\} \\ &\leqslant \sup_{-\tau^*\leqslant\epsilon\leqslant 0} V(\epsilon)\exp\{-\varpi-\mu(\theta-\tau^*)+\upsilon(T-\theta)-\mu(t-t_2)\} \\ &= \sup_{-\tau^*\leqslant\epsilon\leqslant 0} V(\epsilon)\exp\{-2\varpi-\mu(t-t_2)\}. \end{aligned} \tag{30}$$

For $t \in (\delta_2, t_3)$, we obtain the following expressions:

$$\begin{aligned} V(t) &\leqslant \sup_{\delta_2-\tau^*\leqslant\epsilon\leqslant\delta_2} V(\epsilon)\exp\{\upsilon(t-\delta_2)\} \\ &\leqslant \sup_{\delta_2-\tau^*\leqslant\epsilon\leqslant\delta_2} \left\{ \sup_{-\tau^*\leqslant s\leqslant 0} V(s)\exp\{-2\varpi-\mu(\epsilon-t_2)\} \right\} \times \exp\{\upsilon(t-\delta_2)\} \\ &\leqslant \sup_{-\tau^*\leqslant\epsilon\leqslant 0} V(\epsilon)\exp\{-2\varpi-\mu(\delta_2-\tau^*-t_2)+\upsilon(t-\delta_2)\} \\ &\leqslant \sup_{-\tau^*\leqslant\epsilon\leqslant 0} V(\epsilon)\exp\{-2\varpi-\mu(\theta-\tau^*)+\upsilon(t-\delta_2)\}. \end{aligned} \tag{31}$$

By repeating the abovementioned procedure, there exists a natural number $s \geqslant 0$ such that $t_s \leqslant t < t_{s+1}$, for arbitrary $t > 0$. For $t \in [t_s, \delta_s]$, we obtain

$$V(t) \leqslant \sup_{-\tau^*\leqslant\epsilon\leqslant 0} V(\epsilon)\exp\{-s\varpi-\mu(t-t_s)\} \leqslant \sup_{-\tau^*\leqslant\epsilon\leqslant 0} V(\epsilon)\exp\{-s\varpi\}. \tag{32}$$

For $t \in (\delta_s, t_{s+1})$, we obtain

$$\begin{aligned} V(t) &\leqslant \sup_{-\tau^*\leqslant\epsilon\leqslant 0} V(\epsilon)\exp\{-s\varpi-\mu(\theta-\tau^*)+\upsilon(t-\delta_s)\} \\ &\leqslant \sup_{-\tau^*\leqslant\epsilon\leqslant 0} V(\epsilon)\exp\{-s\varpi-\mu(\theta-\tau^*)+\upsilon(t_{s+1}-\delta_s)\} \\ &= \sup_{-\tau^*\leqslant\epsilon\leqslant 0} V(\epsilon)\exp\{-s\varpi-\mu(\theta-\tau^*)+\upsilon[(t_{s+1}-t_s)-(\delta_s-t_s)]\} \\ &\leqslant \sup_{-\tau^*\leqslant\epsilon\leqslant 0} V(\epsilon)\exp\{-s\varpi-\mu(\theta-\tau^*)+\upsilon(T-\theta)\} \\ &= \sup_{-\tau^*\leqslant\epsilon\leqslant 0} V(\epsilon)\exp\{-(s+1)\varpi\} \\ &\leqslant \sup_{-\tau^*\leqslant\epsilon\leqslant 0} V(\epsilon)\exp\{-s\varpi\}. \end{aligned} \tag{33}$$

In conclusion, for arbitrary $t > 0$, it follows that

$$\begin{aligned} V(t) &\leqslant \sup_{-\tau^*\leqslant\epsilon\leqslant 0} V(\epsilon)\exp\{-s\varpi\} \\ &\leqslant \sup_{-\tau^*\leqslant\epsilon\leqslant 0} V(\epsilon)\exp\left\{\left(-\frac{t}{T}+1\right)\varpi\right\} \\ &= K_0\exp\{-K_1 t\}, \end{aligned} \tag{34}$$

where $K_0 = \sup_{-\tau^*\leqslant\epsilon\leqslant 0} V(\epsilon)\exp\{\varpi\}$ and $K_1 = \varpi/T$, which indicates that $z(t) = \mathbf{0}_{3n}$ is globally exponentially stable. As $t \to \infty$, it follows that $\hat{L}\tilde{x}(t) \to \mathbf{0}_n$ and $\tilde{v}(t) \to \mathbf{0}_n$. In addition, because $\hat{L}$ is symmetric and positive definite, $\tilde{x}(t) \to \mathbf{0}_n$ as $t \to \infty$. Therefore, the conclusions of $\lim_{t\to\infty} \|x_i(t) - x_0(t)\| = 0$ and $\lim_{t\to\infty} \|v_i(t) - v_0(t)\| = 0$ hold, for $\forall i = 1, 2, \ldots, n$. This completes the proof.

Next, we consider the special case where $f(t, v_i(t), v_i(t - \tau(t))) \equiv 0$, $i = 1, 2, \ldots, n$. In this case, systems (1) and (2) are reduced to double integrator systems as follows:

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = u_i(t), \end{cases} \quad i = 1, 2, \ldots, n, \tag{35}$$

$$\begin{cases} \dot{x}_0(t) = v_0(t), \\ \dot{v}_0(t) = 0. \end{cases} \tag{36}$$

By applying the control protocol (6) for system (35), we obtain

$$\begin{cases} \dot{\tilde{x}}_i(t) = \tilde{v}_i(t), \\ \dot{\tilde{v}}_i(t) = -\gamma \sum\limits_{j=0}^{n} a_{ij}(x_i(t) - x_j(t)) - \gamma\omega_i(t), & t \in [t_k, \delta_k], \\ \dot{\tilde{v}}_i(t) = 0, & t \in (\delta_k, t_{k+1}), \end{cases} \tag{37}$$

where $\tilde{x}_i(t) = x_i(t) - x_0(t)$ and $\tilde{v}_i(t) = v_i(t) - v_0(t)$, $i = 1, 2, \ldots, n$. Moreover, based on system (37), we have

$$\begin{cases} \dot{\tilde{x}}(t) = \tilde{v}(t), \\ \dot{\tilde{v}}(t) = -\gamma\hat{L}\tilde{x}(t) - \gamma\omega(t), & t \in [t_k, \delta_k], \\ \dot{\tilde{v}}(t) = 0, & t \in (\delta_k, t_{k+1}). \end{cases} \tag{38}$$

Next, we define $z(t) = [\tilde{x}^{\mathrm{T}}(t)\hat{L}, \tilde{v}^{\mathrm{T}}(t), \omega^{\mathrm{T}}(t)]^{\mathrm{T}}$ in the following. Hence, according to systems (8) and (38), we easily obtain the following expressions:

$$\begin{cases} \dot{z}(t) = \Gamma_1 z(t), & t \in [t_k, \delta_k], \\ \dot{z}(t) = \Gamma_2 z(t), & t \in (\delta_k, t_{k+1}), \end{cases} \tag{39}$$

where

$$\Gamma_1 = \begin{bmatrix} 0_{n\times n} & \hat{L} & 0_{n\times n} \\ -\gamma I_n & 0_n & -\gamma I_n \\ 0_{n\times n} & b\hat{L} & -aI_n \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} 0_{n\times n} & \hat{L} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & b\hat{L} & -aI_n \end{bmatrix}.$$

Theorem 2. Suppose that G is undirected, $\bar{G}$ is connected, and Assumption 2 holds. Then, the control protocol (6) makes the systems (35) and (36) achieve second-order leader-following consensus if the following condition holds:

$$\begin{cases} \gamma > \dfrac{\lambda_{\max}(\hat{L})}{2}, \\ b > \dfrac{\gamma}{\lambda_{\min}(\hat{L})} + 1, \\ a > 2\gamma, \\ \eta_3\theta - \eta_4(T - \theta) > 0, \end{cases} \tag{40}$$

where $\eta_3 = \frac{\lambda_{\min}(\Xi_1)}{\lambda_{\max}(P)}$, $\eta_4 = \lambda_{\max}(P^{-1}\Xi_2)$.

Proof. Consider the same Lyapunov function $V(t)$ as that defined in the proof of Theorem 1 in Section 3. Thus, if $\gamma > \frac{\lambda_{\max}(\hat{L})}{2}$ in condition (40) holds, then $V(t)$ is still an effective Lyapunov function for the system (39).

When $t \in [t_k, \delta_k]$, $k = 0, 1, 2, \ldots$, calculating $\dot{V}(t)$ along the system (39), we obtain

$$\begin{aligned}\dot{V}(t) =& \frac{1}{2} z^{\mathrm{T}}(t)[PH_1 + H_1^{\mathrm{T}} P] z(t) \\ =& -\frac{1}{2}\Bigg\{\gamma \tilde{x}^{\mathrm{T}}(t)\hat{L}^2\tilde{x}(t) + (b-1)\tilde{v}^{\mathrm{T}}(t)\hat{L}\tilde{v}(t) \\ &+ (a-\gamma)\omega^{\mathrm{T}}(t)\omega(t) + 2\Bigg[\gamma\tilde{v}^{\mathrm{T}}(t)\omega(t) \\ &- \frac{a}{2}\tilde{v}^{\mathrm{T}}(t)\omega(t) - \frac{b}{2}\tilde{v}^{\mathrm{T}}(t)\hat{L}\omega(t)\Bigg]\Bigg\}. \end{aligned} \tag{41}$$

Using Lemma 3, it follows from (41) that

$$\begin{aligned}\dot{V}(t) \leqslant& -\frac{1}{2}\Bigg\{\gamma\tilde{x}^{\mathrm{T}}(t)\hat{L}^2\tilde{x}(t) + (b-1)\tilde{v}^{\mathrm{T}}(t)\hat{L}\tilde{v}(t) + (a-\gamma)\omega^{\mathrm{T}}(t)\omega(t) \\ &+ \left(-\gamma + \frac{a}{2}\right)[\tilde{v}^{\mathrm{T}}(t)\tilde{v}(t) + \omega^{\mathrm{T}}(t)\omega(t)] \\ &+ \frac{b}{2}[\tilde{v}^{\mathrm{T}}(t)\tilde{v}(t) + \omega^{\mathrm{T}}(t)\hat{L}^2\omega(t)]\Bigg\} \\ \leqslant& -\frac{1}{2}\{\gamma\tilde{x}^{\mathrm{T}}(t)\hat{L}^2\tilde{x}(t) + (b-1)\tilde{v}^{\mathrm{T}}(t)\hat{L}\tilde{v}(t) \\ &+ (a-\gamma)\omega^{\mathrm{T}}(t)\omega(t) - \gamma[\tilde{v}^{\mathrm{T}}(t)\tilde{v}(t) + \omega^{\mathrm{T}}(t)\omega(t)]\} \\ =& -\frac{1}{2} z^{\mathrm{T}}(t)\Xi_1 z(t). \end{aligned} \tag{42}$$

Here,

$$\Xi_1 = \begin{bmatrix} \gamma I_n & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & (b-1)\hat{L} - \gamma I_n & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & (a-2\gamma)I_n \end{bmatrix} > 0.$$

Then, it follows from inequalities (14) and (42) that

$$\dot{V}(t) \leqslant -\frac{1}{2}\lambda_{\min}(\Xi_1) z^{\mathrm{T}}(t)z(t) \leqslant -\eta_3 V(t), \tag{43}$$

where $\eta_3 = \frac{\lambda_{\min}(\Xi_1)}{\lambda_{\max}(P)}$.

When $t \in (\delta_k, t_{k+1})$, $k = 0, 1, 2, \ldots$, calculating $\dot{V}(t)$ along the system (39), we obtain

$$\dot{V}(t) = \frac{1}{2} z^{\mathrm{T}}(t)[PH_2 + H_2^{\mathrm{T}} P] z(t) = \frac{1}{2} z^{\mathrm{T}}(t)\Xi_2 z(t), \tag{44}$$

where

$$\Xi_2 = \begin{bmatrix} 0_{n\times n} & \gamma I_n & 0_{n\times n} \\ \gamma I_n & (1-b)\hat{L} & \frac{b}{2}\hat{L} + \frac{a}{2} I_n \\ 0_{n\times n} & \frac{b}{2}\hat{L} + \frac{a}{2} I_n & -a I_n \end{bmatrix}.$$

Using Lemma 4 and (44), we obtain

$$\dot{V}(t) \leqslant \frac{1}{2}\lambda_{\max}(P^{-1}\Xi_2) z^{\mathrm{T}}(t)Pz(t) = \lambda_{\max}(P^{-1}\Xi_2)V(t) = \eta_4 V(t), \tag{45}$$

where $\eta_4 = \lambda_{\max}(P^{-1}\Xi_2)$.

For $t \in [t_0, \delta_0]$, it follows from (43) that

$$V(t) \leqslant V(t_0)\exp\{-\eta_3(t - t_0)\}. \tag{46}$$

For $t \in [\delta_0, t_1]$, it follows from (45) that

$$V(t) \leqslant V(\delta_0)\exp\{\eta_4(t-\delta_0)\}. \tag{47}$$

Then, according to (46) and (47), it follows that

$$\begin{aligned} V(t_1) &\leqslant V(\delta_0)\exp\{\eta_4(t_1-\delta_0)\} \\ &\leqslant V(t_0)\exp\{-\eta_3(\delta_0-t_0)+\eta_4(t_1-\delta_0)\} \\ &= V(t_0)\exp\{-\eta_3(\delta_0-t_0)+\eta_4[(t_1-t_0)-(\delta_0-t_0)]\} \\ &= V(t_0)\exp\{-\eta_3\theta_0+\eta_4(T_0-\theta_0)\} \\ &= V(0)\exp\{-\Delta_0\}, \end{aligned} \tag{48}$$

where $\Delta_0 = \eta_3\theta_0 - \eta_4(T_0-\theta_0)$ and $t_0 = 0$. Then, using condition (40), we have $\Delta_0 > 0$. By recursion, for any positive integer k, we have

$$V(t_{k+1}) \leqslant V(0)\exp\left\{-\sum_{j=0}^{k}\Delta_j\right\}, \tag{49}$$

where $\Delta_j = \eta_3\theta_j - \eta_4(T_j-\theta_j) > 0$, $j = 0, 1, 2, \ldots, k$.

Furthermore, for all $t > 0$, there exists a natural number s^* such that $t_{s^*+1} \leqslant t < t_{s^*+2}$. For $t \in [t_{s^*+1}, \delta_{s^*+1}]$, we have

$$V(t) \leqslant V(t_{s^*+1})\exp\{-\eta_3(t-t_{s^*+1})\} \leqslant V(0)\exp\left\{-\sum_{j=0}^{s^*}\Delta_j\right\}. \tag{50}$$

For $t \in (\delta_{s^*+1}, t_{s^*+2})$, we have

$$\begin{aligned} V(t) &\leqslant V(\delta_{s^*+1})\exp\{\eta_4(t-\delta_{s^*+1})\} \\ &\leqslant V(t_{s^*+1})\exp\{-\eta_3(\delta_{s^*+1}-t_{s^*+1})+\eta_4(t_{s^*+2}-\delta_{s^*+1})\} \\ &\leqslant V(0)\exp\left\{-\sum_{j=0}^{s^*}\Delta_j - \eta_1\theta_{s^*+1} + \eta_4(T_{s^*+1}-\theta_{s^*+1})\right\} \\ &= V(0)\exp\left\{-\sum_{j=0}^{s^*+1}\Delta_j\right\} \\ &\leqslant V(0)\exp\left\{-\sum_{j=0}^{s^*}\Delta_j\right\}. \end{aligned} \tag{51}$$

Thus, for arbitrary $t > 0$, it follows that

$$\begin{aligned} V(t) &\leqslant V(0)\exp\left\{-\sum_{j=0}^{s^*}\Delta_j\right\} \\ &\leqslant V(0)\exp\{-s^*\Delta_{\min}\} \\ &\leqslant V(0)\exp\left\{\left(-\frac{t}{T}+1\right)\Delta_{\min}\right\} \\ &= K_0\exp\{-K_1 t\}, \end{aligned} \tag{52}$$

where $K_0 = V(0)\exp\{\Delta_{\min}\}$, $K_1 = \Delta_{\min}/T$, $\Delta_{\min} = \eta_3\theta - \eta_4(T-\theta)$; this implies that $z(t) = \mathbf{0}_{3n}$, as $t \to \infty$. Then, using the control protocol (6) with a filter (4), the second-order leader-following consensus for systems (35) and (36) is achieved. This completes the proof.

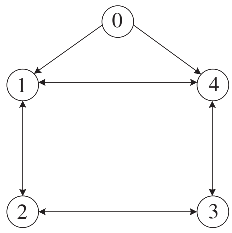

Figure 1 Communication topology $\bar{G}$.

Remark 3. In [34], the authors investigated the consensus problem of second-order multi-agent systems without velocity measurements under continuous communication. However, under aperiodically intermittent communication, only few studies exist that consider the consensus problems of multi-agent systems with immeasurable velocity. To the best of our knowledge, this study is the first one to investigate the leader-following consensus problem of second-order nonlinear multi-agent systems with aperiodically intermittent position measurements. In addition, herein, a new type of intermittent consensus control protocol combined with a distributed filter is designed using the relative intermittent position measurements between neighboring agents, which is different from [34].

4 Simulation examples

Consider a multi-agent communication network with four followers and one leader with the topology graph $\bar{G}$ as shown in Figure 1. Suppose that the connection weighted values of the communication topology are all equal to 1 in this study. Then, based on the topology graph $\bar{G}$, we have $\lambda_{\max}(\hat{L}) = 4.618$ and $\lambda_{\min}(\hat{L}) = 0.382$.

Example 1. Consider the second-order nonlinear dynamic model (1) and (2). Suppose the multi-agent network interaction graph among the agents is selected as that shown in Figure 1. Let $f(t, v_i(t), v_i(t - \tau(t))) = 0.001 \sin v_i(t) + 0.001 \cos v_i(t - \tau(t))$, where $\tau(t) = 0.1|\sin t| \leqslant 0.1$, $x_i(t) \in \mathbb{R}$, $v_i(t) \in \mathbb{R}$, and $i \in \{0, 1, 2, 3, 4\}$. Considering Assumption 1, we know that $\rho_1 = 0.001$ and $\rho_2 = 0.001$. The initial position and velocity of the followers are selected as $x_i(0) = 0.05i$, $v_i(0) = 0.01i$, $i = 1, 2, 3, 4$. The initial values of the leader are selected as $x_0(0) = 0$, $v_0(0) = 0.02$. Using condition (11), we can consider that $\gamma = 3$, $a = 8$, and $b = 12$. According to Theorem 1, we obtain $\lambda_{\min}(R_1) = 1.198 > \frac{\pi\lambda_{\max}(P)}{\lambda_{\min}(P)} = 0.374$, $\mu = 0.1039$, and $\upsilon = 2.0031$. Then, we can choose T and θ such that $\varpi = \mu(\theta - \tau^*) - \upsilon(T - \theta) > 0$. Figure 2(a) shows the differences in the position states between the four followers and the leader, and Figure 2(b) shows the differences in the velocity states between the four followers and the leader. These differences imply that second-order leader-following consensus can be achieved for nonlinear multi-agent systems with time-varying delay and aperiodically intermittent position measurements.

Example 2. Here, we again use the communication topology shown in Figure 1. The second-order nonlinear dynamic model is reduced to the double integrator model similar to systems (35) and (36); i.e., $f(t, v_i(t), v_i(t - \tau(t))) \equiv 0$. The initial states of the followers and the leader are selected identical to those in Example 1. Similarly, we take $\gamma = 4$, $a = 10$, and $b = 14$, which satisfies the condition (40). Based on Theorem 2, we obtain $\eta_3 = 0.1224$ and $\eta_4 = 2.4977$. Then, we can choose T and θ, such that $\eta_3\theta - \eta_4(T - \theta) > 0$. Figure 3 shows the simulation results. This shows that leader-following consensus can be achieved for multi-agent systems with double-integrator dynamics and aperiodically intermittent position measurements.

5 Conclusion

In this study, the leader-following consensus problem for a second-order nonlinear multi-agent system with time-varying delay and aperiodically intermittent position measurements is investigated. Different from majority of the current studies, the intermittent communication type used in this study can be aperiodic,

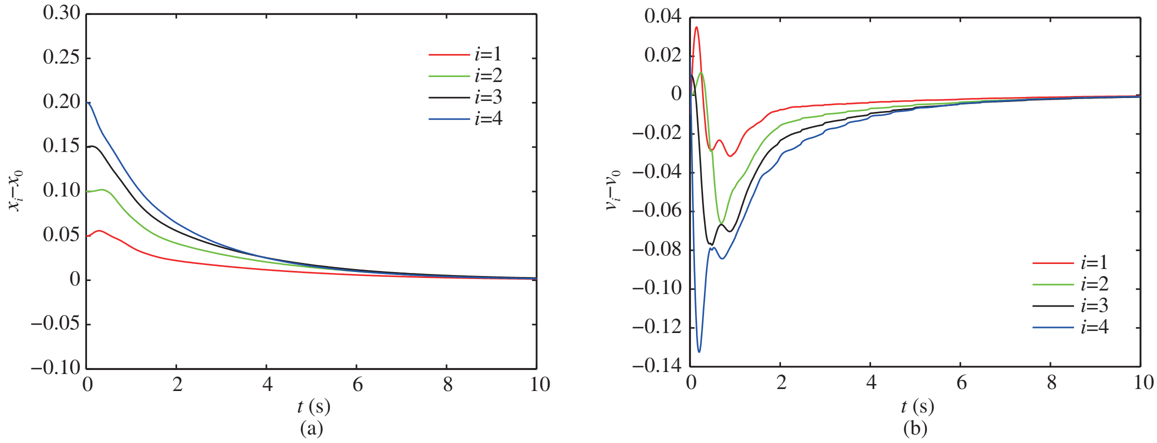

Figure 2 (Color online) Errors of the states between the followers and the leader in Example 1. (a) Position; (b) velocity.

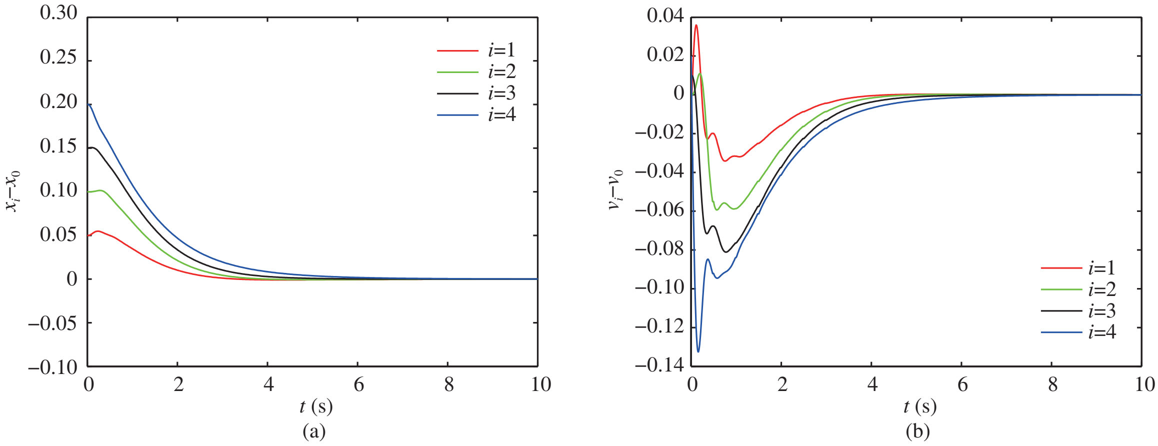

Figure 3 (Color online) Errors of the states between the followers and the leader in Example 2. (a) Position; (b) velocity.

and only the intermittent position measurements among the agents can be used. Under aperiodically intermittent communication, a novel consensus control protocol combined with a distributed filter is designed to guarantee that all followers can track the leader using the relative position measurements between the neighboring agents. With the help of the intermittent control method and Lyapunov function technology, some consensus conditions are obtained for second-order leader-following multi-agent systems under aperiodically intermittent communications. As communication noises are usually inevitable, in the future, we will focus on investigating the intermittent mean square consensus tracking problem for second-order nonlinear delayed multi-agent systems without velocity measurements.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 61573200, 61573199).

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