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Necessary and sufficient conditions for the dynamic output feedback stabilization of fractional-order systems with order $0 < \alpha < 1$

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Dear editor,

Fractional calculus is the generalization of integer order calculus that enables us to obtain a considerably accurate description of several real world systems because of its memory property and practical applications. The fractional-order systems (FOS) have attracted significant attention in recent years and have yielded several valuable results, especially with respect to stability and stabilization [1–4]. The authors of [1–3] have investigated the robust stability and stabilization for FOS with order $0 < \alpha < 1$ using linear matrix inequalities (LMIs) and have obtained the state feedback stabilization conditions in terms of LMIs.

In practical applications, the output feedback can be extensively applied for system control because not all the system state variables are available for feedback and because the output feedback considers only the output signal that can be detected as the feedback signal. Further, the static output feedback, dynamic output feedback, and observer-based stabilization for FOS have been investigated in [5–8]. However, the conditions of dynamic output feedback stabilization are sufficient until now and they result in various limitations in case of practical applications. Therefore, the dynamic output feedback stabilization of FOS still needs to be investigated, both in terms of theory and practical applications.

In this study, we focus on the dynamic output feedback stabilization of FOS with order $0 < \alpha < 1$. Further, a necessary and sufficient condition is presented for the design of the dynamic output feedback controller using LMIs, and we provide a numerical example to illustrate our results. In addition, we use the definition of the Caputo fractional derivative.

Definition 1 ([9])**.** The Caputo derivative of order α for a function $f(t)$ can be defined as follows:

$$D^\alpha f(t) = \frac{1}{\Gamma(N-\alpha)} \int_0^t \frac{f^{(N)}(s)}{(t-s)^{\alpha-N+1}} \mathrm{d}s, \quad (1)$$

where N is a positive integer that satisfies $N-1 < \alpha \leqslant N$; further, $\Gamma(\cdot)$ denotes the Gamma function that can be defined as $\Gamma(\tau) = \int_0^\infty \mathrm{e}^{-t} t^{\tau-1} \mathrm{d}t$.

Notation. X^{-1} and X^{T} denote the inverse and transpose of X, respectively. $A > 0$ ($A < 0$) indicates that a symmetric matrix A is positive definite (negative definite). The notation $\mathrm{sym}\{T\}$ denotes $T + T^{\mathrm{T}}$. $\mathbb{P}_\alpha(X,Y) = \{\sin(\frac{\alpha\pi}{2})X + \cos(\frac{\alpha\pi}{2})Y, \left[\begin{smallmatrix} X & Y \\ -Y & X \end{smallmatrix}\right] > 0\}$ with $\alpha \in (0,1)$ and X, Y denoting the real square matrices is the matrix set that will be used in the sequel. The matrices, if

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not explicitly stated, are assumed to exhibit compatible dimensions.

Problem formulation. Consider the FOS that is described by

$$D^{\alpha}x(t) = Ax(t) + Bu(t), \tag{2}$$

$$y(t) = Cx(t) + Du(t), \tag{3}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^q$ denote the state, the control input, and the measurement, respectively. A, B, C and D are the known real constant matrices with appropriate dimensions, where $0 < \alpha < 1$.

Our objective is to estimate a dynamic output feedback controller.

$$D^{\alpha}\bar{x}(t) = A_K\bar{x}(t) + B_K y(t), \tag{4}$$

$$u(t) = C_K\bar{x}(t), \tag{5}$$

where $\bar{x}(t) \in \mathbb{R}^n$ is the controller state and A_K, B_K and C_K are the matrices that are to be determined, by ensuring that the following closed-loop system is stable:

$$D^{\alpha}x_c(t) = A_c x_c(t), \tag{6}$$

where

$$x_c(t)=\begin{bmatrix} x(t) \\ \bar{x}(t) \end{bmatrix},\ A_c=\begin{bmatrix} A & BC_K \\ B_K C & A_K + B_K D C_K \end{bmatrix}. \tag{7}$$

It is renowned (refer to [2, 3]) that the system (6) is asymptotically stable if and only if there exists a matrix, $P_c \in P_\alpha(X_c, Y_c)$, i.e., $P_c = \sin(\frac{\alpha\pi}{2})X_c + \cos(\frac{\alpha\pi}{2})Y_c$, $\left[\begin{smallmatrix} X_c & Y_c \\ -Y_c & X_c \end{smallmatrix}\right] > 0$, such that $\mathrm{sym}\{A_c P_c\} < 0$.

Main results. The following lemma plays an important role in the sequel.

Lemma 1 ([4])**.** (1) If $P \in \mathbb{P}_\alpha(X, Y)$, P is invertible and $P^{-1} \in \mathbb{P}_\alpha(X, Y)$.

(2) If $P \in \mathbb{P}_\alpha(X, Y)$, M is a real matrix; further, $\det(M) \neq 0$ and $M^{\mathrm{T}}PM \in \mathbb{P}_\alpha(X, Y)$.

Theorem 1. There exists a dynamic output feedback controller of the form (4) and (5) such that the closed-loop system (6) is stable if and only if there exist matrices $P_1 \in \mathbb{P}_\alpha(X_1, Y_1), P_2 \in \mathbb{P}_\alpha(X_2, Y_2), \Phi$ and Ψ such that

$$\mathrm{sym}\{AP_1 + B\Phi\} < 0, \tag{8}$$

$$\mathrm{sym}\{P_2^{\mathrm{T}}A + \Psi C\} < 0. \tag{9}$$

Furthermore, if Eqs. (8) and (9) hold, there exist matrices P_1, P_2, Φ and Ψ such that Eqs. (8) and (9) hold and $P_1 - P_2^{-1} \in \mathbb{P}_\alpha(X_{12}, Y_{12})$; further, the desired stabilizing dynamic output feedback controller in (4) and (5) can be computed using the following parameters:

$$A_K = (P_2^{-\mathrm{T}}A^{\mathrm{T}} + AP_1 + B\Phi + P_2^{-\mathrm{T}}\Psi C P_1)(P_1 - P_2^{-1})^{-1} + P_2^{-1}\Psi D\Phi(P_1 - P_2^{-1})^{-1}, \tag{10}$$

$$B_K = P_2^{-\mathrm{T}}\Psi, \tag{11}$$

$$C_K = \Phi(P_2^{-1} - P_1)^{-1}. \tag{12}$$

First, to proof the necessity, we assume that there exists a dynamic output feedback controller in (4) and (5) such that the closed-loop system (6) is stable. Further, there exists a matrix $P_c \in P_\alpha(X_c, Y_c)$ such that

$$\mathrm{sym}\{A_c P_c\} < 0. \tag{13}$$

Let $P_c = \left[\begin{smallmatrix} P_{c1} & P_{c2} \\ P_{c3} & P_{c4} \end{smallmatrix}\right] = \sin(\frac{\alpha\pi}{2})\left[\begin{smallmatrix} X_{c1} & X_{c2} \\ X_{c3} & X_{c4} \end{smallmatrix}\right] + \cos(\frac{\alpha\pi}{2})\left[\begin{smallmatrix} Y_{c1} & Y_{c2} \\ Y_{c3} & Y_{c4} \end{smallmatrix}\right]$, where the partition is compatible with A_c in (6). Using $\left[\begin{smallmatrix} X_c & Y_c \\ -Y_c & X_c \end{smallmatrix}\right] > 0$, we can obtain $\left[\begin{smallmatrix} X_{c1} & Y_{c1} \\ -Y_{c1} & X_{c1} \end{smallmatrix}\right] > 0$ and $\left[\begin{smallmatrix} X_{c4} & Y_{c4} \\ -Y_{c4} & X_{c4} \end{smallmatrix}\right] > 0$, which indicates that $P_{c1}, P_{c4} \in P_\alpha(X, Y)$. By Lemma 1, P_{c4} is invertible. Let $U = \left[\begin{smallmatrix} I & 0 \\ -P_{c4}^{-1}P_{c3} & I \end{smallmatrix}\right]$. By left- and right-multiplying P_c with U^{T} and U, we obtain $U^{\mathrm{T}}P_cU = \sin(\frac{\alpha\pi}{2})\left[\begin{smallmatrix} \hat{X}_{c1} & \hat{X}_{c2} \\ \hat{X}_{c3} & \hat{X}_{c4} \end{smallmatrix}\right] + \cos(\frac{\alpha\pi}{2})\left[\begin{smallmatrix} \hat{Y}_{c1} & \hat{Y}_{c2} \\ \hat{Y}_{c3} & \hat{Y}_{c4} \end{smallmatrix}\right]$. By Lemma 1, $\left[\begin{smallmatrix} \hat{X}_{c1} & \hat{Y}_{c1} \\ -\hat{Y}_{c1} & \hat{X}_{c1} \end{smallmatrix}\right] > 0$. The 1–1 block of $U^{\mathrm{T}}P_cU$ is $P_{c1} - P_{c2}P_{c4}^{-1}P_{c3}$ and is invertible.

Based on the 1–1 block of (13), we can obtain

$$\mathrm{sym}\{AP_{c1} + BC_K P_{c3}\} < 0, \tag{14}$$

which can be used to yield (8) by setting $P_1 = P_{c1}$, $\Phi = C_K P_{c3}$. By left- and right-multiplying (13) with U^{T} and U, and by setting $P_2 = (P_{c1} - P_{c2}P_{c4}^{-1}P_{c3})^{-1}$, we can obtain the 1–1 block in the following manner:

$$\mathrm{sym}\{AP_2^{-1} - P_{c3}^{\mathrm{T}}P_{c4}^{-\mathrm{T}}B_K C P_2^{-1}\} < 0. \tag{15}$$

By left- and right-multiplying (15) with P_2^{T} and P_2, respectively, and by setting $\Psi = -P_2^{\mathrm{T}}P_{c3}^{\mathrm{T}}P_{c4}^{-\mathrm{T}}B_K$, we can obtain (9).

In the proof of sufficiency, there exist matrices $P_1 \in \mathbb{P}_\alpha(X_1, Y_1), P_2 \in \mathbb{P}_\alpha(X_2, Y_2), \Phi$ and Ψ such that Eqs. (8) and (9) hold. Without the loss of generality, we can assume that $P_1 - P_2^{-1} \in \mathbb{P}_\alpha(X_{12}, Y_{12})$, i.e, $P_1 - P_2^{-1} = \sin(\frac{\alpha\pi}{2})(X_1 - \bar{X}_2) + \cos(\frac{\alpha\pi}{2})(Y_1 - \bar{Y}_2)$, $\left[\begin{smallmatrix} X_1 - \bar{X}_2 & Y_1 - \bar{Y}_2 \\ \bar{Y}_2 - Y_1 & X_1 - \bar{X}_2 \end{smallmatrix}\right] > 0$. Otherwise, if $\left[\begin{smallmatrix} X_1 - \bar{X}_2 & Y_1 - \bar{Y}_2 \\ \bar{Y}_2 - Y_1 & X_1 - \bar{X}_2 \end{smallmatrix}\right] > 0$ is not set, there exist $\theta \in \mathbb{R}$ and $\theta > 0$, when P_1 is replaced by θP_1, Φ is replaced by $\theta\Phi$, Eq. (8) still holds, and $\left[\begin{smallmatrix} \theta X_1 - \bar{X}_2 & \theta Y_1 - \bar{Y}_2 \\ -\theta Y_1 + \bar{Y}_2 & \theta X_1 - \bar{X}_2 \end{smallmatrix}\right] > 0$. Using the dynamic output feedback controller with parameters (10)–(12), we can obtain the closed-loop system as

$$D^{\alpha}x_c(t) = \bar{A}_c x_c(t), \tag{16}$$

where $x_c(t)$ is given in (7), and $\bar{A}_c = \left[\begin{smallmatrix} A & B\Phi(P_2^{-1} - P_1)^{-1} \\ P_2^{-1}\Psi C & \Delta \end{smallmatrix}\right]$, where $\Delta = (P_2^{-\mathrm{T}}A^{\mathrm{T}} +$

$AP_1+B\Phi+P_2^{-\mathrm{T}}\Psi CP_1)(P_1-P_2^{-1})^{-1}$. Further, we set $\bar{P}_c=\left[\begin{smallmatrix} P_1 & P_2^{-1}-P_1 \\ P_2^{-1}-P_1 & P_1-P_2^{-1}\end{smallmatrix}\right]=\sin(\frac{\alpha\pi}{2})\bar{X}_c+\cos(\frac{\alpha\pi}{2})\bar{Y}_c$, where $\bar{X}_c=\left[\begin{smallmatrix} X_1 & \bar{X}_2-X_1 \\ \bar{X}_2-X_1 & X_1-\bar{X}_2\end{smallmatrix}\right]$, $\bar{Y}_c=\left[\begin{smallmatrix} Y_1 & \bar{Y}_2-Y_1 \\ \bar{Y}_2-Y_1 & Y_1-\bar{Y}_2\end{smallmatrix}\right]$. Let

$$V=\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}.$$

We perform left- and right-multiplication of $\left[\begin{smallmatrix} \bar{X}_c & \bar{Y}_c \\ -\bar{Y}_c & \bar{X}_c\end{smallmatrix}\right]$ with V^{T} and V, respectively. By $\left[\begin{smallmatrix} \bar{X}_2 & \bar{Y}_2 \\ -\bar{Y}_2 & \bar{X}_2\end{smallmatrix}\right]>0$ and $\left[\begin{smallmatrix} X_1-\bar{X}_2 & Y_1-\bar{Y}_2 \\ \bar{Y}_2-Y_1 & X_1-\bar{X}_2\end{smallmatrix}\right]>0$, we can obtain $\left[\begin{smallmatrix} \bar{X}_c & \bar{Y}_c \\ -\bar{Y}_c & \bar{X}_c\end{smallmatrix}\right]>0$. Hence, $\bar{P}_c\in\mathbb{P}_\alpha(\bar{X}_c,\bar{Y}_c)$.

With $\mathrm{sym}\{AP_1+B\Phi\}<0$ and left- and right-multiplication of $\mathrm{sym}\{P_2^{\mathrm{T}}A+\Psi C\}<0$ with $P_2^{-\mathrm{T}}$ and P_2^{-1}, based on Schur complement, we can obtain

$$\mathrm{sym}\{\bar{A}_c\bar{P}_c\}=\begin{bmatrix} \Omega_1 & -\Omega_1 \\ -\Omega_1 & \Omega_2+\Omega_1 \end{bmatrix}<0, \qquad (17)$$

where $\Omega_1=\mathrm{sym}\{AP_1+B\Phi\}$, $\Omega_2=P_2^{-\mathrm{T}}\mathrm{sym}\{P_2^{\mathrm{T}}A+\Psi C\}P_2^{-1}$.

The aforementioned fact implies that the closed-loop system (16) is stable, and this completes the proof.

Numerical example. Consider the system (2) and (3) with parameters $\alpha=0.8$, and

$$A=\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, B=\begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix}, C=\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} D=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

There is no feasible solution for this system using the LMIs, as presented in [6]. However, using Theorem 1, the desired dynamic output feedback controller in (4) and (5) can be designed using the following parameters by solving the LMIs (8) and (9):

$$A_K=\begin{bmatrix} 8.9028 & -20.5694 \\ -1.0233 & -13.6787 \end{bmatrix},$$
$$B_K=\begin{bmatrix} 5.5003 & -0.7629 \\ 0.3430 & -2.8687 \end{bmatrix},$$
$$C_K=10^3\times\begin{bmatrix} -1.8558 & 4.4151 \\ -1.8606 & 4.4282 \end{bmatrix}.$$

Therefore, using the dynamic output feedback controller, the closed-loop system (6) is observed to be stable.

Conclusion. In this study, we investigated the dynamic output feedback stabilization of FOS with order $0<\alpha<1$ and proposed a necessary and sufficient condition for designing the dynamic output feedback controller. The developed result is observed to be more general and useful than that of some of the existing studies, in which only the sufficient conditions were presented. Further, the effectiveness of the conditions can be verified using a numerical example.

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