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Halanay-type inequality with delayed impulses and its applications

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Abstract In this study, some properties of a novel Halanay-type inequality that simultaneously contains impulses and delayed impulses are investigated. Two concepts with respect to average impulsive gain are proposed to describe hybrid impulsive strength and hybrid delayed impulsive strength. Then, using the obtained results, two stability criteria are derived for the linear systems with impulses and delayed impulses. It is found that the stability of impulsive systems is robust with respect to delayed impulses of which the magnitude strength is relatively small. Whereas, if the impulse strength is small, the time-delayed impulses can also promote the stability of unstable systems. Two numerical examples are employed to illustrate the efficiency of our results.

Keywords Halanay-type inequality, hybrid impulses, average impulsive gain, delayed impulses, linear impulsive systems

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1 Introduction

When studying dynamic systems in nature and in the real world, we often find that transient disturbance of a system state or a sudden change of states at a given moment will considerably affect the dynamic behavior. Moreover, the duration of these rapid changes or mutations is very short compared to the entire motion process. This transient disturbance or sudden change of the state value is called an impulsive phenomenon. It is well-known that, the stability problems for impulsive dynamical systems (DSs) have received a significant amount of attention for their application in many areas such as image processing [1], optimization problems [2], biology [3], and harmonic oscillation generation [4]. In fact, there are many stability phenomena in artificial systems and in nature, such as fireflies in the forest [5], routing messages in the internet [6], and ecosystems management [7], and so on [8–13]. Many excellent achievements have been obtained with respect to both theoretical analysis and applications of stability for DSs.

Several differential inequalities, such as the Halanay inequality [14], Hilger-type impulsive differential inequality [15], and Lieb-Thirring-type inequality [16, 17], have received a significant amount of attention as powerful tools for studying impulsive dynamic behaviors of differential systems [18–21]. These inequalities have proven to be effective in the investigation of stability problems for differential DSs. One of these, the Halanay inequality, is presented:

$$u'(t) \leqslant \alpha_1 u(t) + \alpha_2 [u(t)]_\tau, \quad -\alpha_1 > \alpha_2 > 0,$$

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and the generalized Halanay inequality is described by

$$u'(t) \leqslant \alpha_1(t)u(t) + \alpha_2(t)[u(t)]_{\tau(t)},$$

where $\tau(t) < t$, $-\alpha_1(t) > \alpha_2(t) > 0$ and $\lim_{t\to\infty} \tau(t) = \infty$. A new generalized Halanay-type inequality without the restriction of $\lim_{t\to\infty} a_1(t) = 0$ (uniform positiveness condition) was proposed in [22] to study the global generalized exponential stability (ES) of nonlinear nonautonomous time-delayed systems. Song et al. [23] introduced discrete Halanay-type inequalities and applied the results to establish the μ-stable criteria for discrete delayed neural networks. He et al. [24] extended the Halanay inequality to the case of fractional-order impulsive differential inequality using an integral inequality.

As a type of hybrid character, time-delayed impulses commonly arise in many different fields. In general these may cause instability, chaos, or other undesirable performance in dynamical systems. The analysis of dynamical behaviors of DSs with delayed impulses, including stability and asymptotic stability has attracted substantial attention. Many significant achievements have been reported [25–29]. Recently, impulsive differential inequalities with delays were established to study the stability problems related to delayed DSs with impulsive effects. The greatest advantage of the impulsive delay differential inequality-based method is that the derivatives of the discrete delays do not have any constraint. It was reported in [30] that the global ES of impulsive DSs with delays could be accomplished via an impulsive differential inequality with time-varying delays. By building a novel extended Halanay-type differential inequality with delayed impulses, the ES of recurrent neural networks with delays via impulsive protocol was analyzed [31]. Then, Yang et al. [32] extended this inequality to impulsive differential inequalities with delays, where the time-varying delays in impulsive items are multiple. Using this result, exponential synchronization problems of TS fuzzy complex networks [32] and complex-valued complex networks with stochastic perturbations [33] were studied. It is natural and necessary to study the effects of delayed impulses on differential systems.

Note that the above literature only discussed robustness with respect to delayed impulses, where the delayed impulses were regarded as a type of instantaneous perturbation or some destabilizing source. However, not all delayed impulses destabilize systems. Li et al. [34] found that time-delayed impulses may contribute to the stabilization of delayed systems by restricting the impulse interval and impulsive gain. Subsequently, Yang et al. [35] used this result to design a distributed delayed impulsive controller to investigate the exponential synchronization for nonlinear complex dynamical systems. However, few work considered dynamical systems with hybrid impulses and hybrid delayed impulses, including simultaneous stabilizing and destabilizing cases. A natural question can be proposed: Under what circumstances would both normal impulses and delayed impulses enhance stability?

Motivated by the above arguments, this paper focuses on establishing a novel Halanay-type inequality with simultaneous impulses and delayed impulses, and then employs the results of the Halanay-type inequality to study the exponential stability problems of linear DSs with impulsive effects. The main contributions of the current results are as follows. First, this paper presents an investigation of the impact of the relationship of the average delayed impulsive gain and average impulsive gain on the stability of impulsive systems. Second, to depict the hybrid impulsive gain and hybrid delayed impulsive gain, two novel concepts concerning the average impulsive gain are proposed. Finally, we apply the obtained results of the Halanay-type inequality to study ES problems for linear impulsive DSs. The remainder of this article is organized as follows. Section 2 formulates the model description and two important definitions. In Section 3, we describe our study of the dynamical behaviors of the Halanay-type inequality. The applications of the obtained results to linear impulsive systems are presented in Section 4. Section 5 presents two examples to validate our theoretical results.

Notation. The symbol D^+ represents the upper right-hand Dini derivative, whereas $\mathbb{R}^{n\times m}$, $\mathbb{R}^n$, and $\mathbb{R}^+$ denote the set of $n \times m$ real matrices, the n-dimensional Euclidean space, and the set of positive real numbers, respectively. Set $\mathrm{PC}(U,V) = \{u : U \to V$ is continuous everywhere except at a finite number of point t, at which right hand limit $u(t^+)$ and left hand limit $u(t^-)$ exist, and $u(t^+) = u(t)\}$. I denotes the identity matrix with suitable order, and superscript T represents the transpose of a vector or a matrix.

$|\cdot|$ represents the Euclidean norm. The minimum and maximum eigenvalues of a symmetric matrix P are represented as $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$, respectively.

2 Model formulation and some preliminaries

The following differential inequality with hybrid impulsive effects and delayed impulsive effects are considered:

$$\begin{cases} D^{+}u(t) \leqslant pu(t), \ t \neq t_k, \\ u(t_k) \leqslant b_k u(t_k^{-}) + d_k[u(t_k)^{-}]_{\tau}, \end{cases} \tag{1}$$

where $u \in \mathrm{PC}(\mathbb{R}, \mathbb{R}^{+})$, $p \in \mathbb{R}$, $b_k > 0$, $d_k > 0$, $\tau > 0$, and the impulsive sequence $\zeta = \{t_1, t_2, t_3, \ldots\}$. The impulsive sequence satisfies $\lim_{k\to\infty} t_k = \infty$ and $t_{k+1} - t_k > \tau$.

Remark 1. In the above inequality, p is not restricted to being negative, which relaxes the restrictions in the existing Halanay-type inequalities. However, different time-dependent parameters b_k and d_k are adopted to illustrate the impulsive strength and delayed impulsive strength, which is more general and has a broader range of applications.

In previous studies, the range of an impulsive interval was restricted to study the impulsive effects on the stability of DSs. In 2010, the concept known as the average impulsive interval (AII) was proposed [36], which can be used to adjust the impulsive frequency. This technique has attracted a significant amount of attention and has been used in many types of impulsive systems. Then, Wang et al. [37] extended this concept to a form of limited as following, which is more general.

Definition 1 ([36, 37])**.** The limit form of AII T_α of the impulsive sequence ζ is defined as follows:

$$T_\alpha = \lim_{t\to\infty} \frac{t - t_0}{N_\zeta(t, t_0)},$$

where $N_\zeta(t, t_0)$ denotes the number of impulses in the impulsive sequence ζ for the interval (t_0, t).

In general, there are two types of impulses for a DS, stabilizing impulses and destabilizing impulses. In [37], the concept of average impulsive gain was first proposed to delineate the hybrid impulsive gain, which is a powerful method for investigating impulsive systems with two types of impulses. Motivated by [37], for the Halanay-type inequality (1), the following definition is proposed to illustrate the hybrid impulsive gain and hybrid delayed impulsive gain.

Definition 2. The average impulsive gain μ_b and the average delayed impulsive gain μ_d of impulsive sequence ζ in inequality (1) are respectively defined as follows:

$$\mu_b = \lim_{t\to\infty} \frac{|b_1| + |b_2| + \cdots + |b_{N_\zeta(t,t_0)}|}{N_\zeta(t, t_0)}, \tag{2}$$

and

$$\mu_d = \lim_{t\to\infty} \frac{|d_1| + |d_2| + \cdots + |d_{N_\zeta(t,t_0)}|}{N_\zeta(t, t_0)}. \tag{3}$$

3 Main results

Some results concerning the Halanay-type inequality (1) are presented in this section.

Theorem 1. For the impulsive inequality (1), if there exists $M > 0$ such that

$$b_k \geqslant M d_k \mathrm{e}^{-p\tau}, \tag{4}$$

then it follows that, for any solution $u(t)$ of inequality (1),

$$u(t) \leqslant \left(1+\frac{1}{M}\right)^{N_\zeta(t,t_0)} \prod_{i=1}^{N_\zeta(t,t_0)} b_i \mathrm{e}^{p(t-t_0)} u_0, \tag{5}$$

where $u_0 = \sup_{t_0-\tau \leqslant s \leqslant t_0} u(s)$.

Proof. When $t \in [t_0, t_1)$,

$$u(t) \leqslant u_0 \mathrm{e}^{p(t-t_0)},$$

and

$$u(t_1) \leqslant b_1 u(t_1^-) + d_1[u(t_1^-)]_{\tau^-} \leqslant [b_1 \mathrm{e}^{p(t_1-t_0)} + d_1 \mathrm{e}^{p(t_1-\tau-t_0)}]u_0.$$

When $t \in [t_1, t_2)$, we have

$$u(t) \leqslant \mathrm{e}^{p(t-t_1)} u(t_1) \leqslant [b_1 \mathrm{e}^{p(t-t_0)} + d_1 \mathrm{e}^{p(t-\tau-t_0)}]u_0,$$

and

$$\begin{aligned} u(t_2) &\leqslant b_2 u(t_2^-) + d_2[u(t_2^-)]_{\tau^-} \\ &\leqslant [b_1 b_2 \mathrm{e}^{p(t_2-t_0)} + b_2 d_1 \mathrm{e}^{p(t_2-\tau-t_0)} + b_1 d_2 \mathrm{e}^{p(t_2-\tau-t_0)} + d_1 d_2 \mathrm{e}^{p(t_2-2\tau-t_0)}]u_0 \\ &\leqslant \left(1+\frac{1}{M}\right)^2 b_1 b_2 \mathrm{e}^{p(t_2-t_0)} u_0. \end{aligned}$$

Suppose that Eq. (5) holds for $N(t,t_0) = k$. We will prove that Eq. (5) holds if $N(t,t_0) = k+1$. In fact,

$$u(t_{k+1}^-) \leqslant \left(1+\frac{1}{M}\right)^k \prod_{i=1}^{k} b_i \mathrm{e}^{p(t_{k+1}-t_0)} u_0,$$

and from (4),

$$\begin{aligned} u(t_{k+1}) &\leqslant b_{k+1} u(t_{k+1}^-) + d_{k+1}[u(t_{k+1}^-)]_\tau \\ &\leqslant \left(1+\frac{1}{M}\right)^k \prod_{i=1}^{k+1} b_i \mathrm{e}^{p(t_{k+1}-t_0)} u_0 + d_{k+1}\left(1+\frac{1}{M}\right)^k \prod_{i=1}^{k} b_i \mathrm{e}^{p(t_{k+1}-\tau-t_0)} u_0 \\ &\leqslant \left(1+\frac{1}{M}\right)^k \prod_{i=1}^{k+1} b_i \mathrm{e}^{p(t_{k+1}-t_0)} u_0 + \frac{1}{M}\left(1+\frac{1}{M}\right)^k \prod_{i=1}^{k+1} b_i \mathrm{e}^{p(t_{k+1}-t_0)} u_0 \\ &\leqslant \left(1+\frac{1}{M}\right)^{k+1} \prod_{i=1}^{k+1} b_i \mathrm{e}^{p(t_{k+1}-t_0)} u_0. \end{aligned}$$

For $t \in [t_{k+1}, t_{k+2})$, we have

$$u(t) \leqslant \mathrm{e}^{p(t-t_{k+1})} u(t_{k+1}) \leqslant \left(1+\frac{1}{M}\right)^{k+1} \prod_{i=1}^{k+1} b_i \mathrm{e}^{p(t-t_0)} u_0.$$

From the above discussion and using mathematical induction,

$$u(t) \leqslant \left(1+\frac{1}{M}\right)^{N_\zeta(t,t_0)} u_0 \mathrm{e}^{p(t-t_0)} \prod_{i=1}^{N_\zeta(t,t_0)} b_i.$$

Hence, the proof is complete.

Corollary 1. Suppose that Eq. (4) holds. For any solution $u(t)$ of inequality (1), there exists a sufficient large $T > 0$ such that the following inequality holds:

$$u(t) \leqslant \mathrm{e}^{\eta_1(t-t_0)}u_0, \quad t \geqslant T,$$

where $\eta_1 > \eta = \frac{\ln \frac{M}{M+1}\mu_b}{T_\alpha} + p$, T_α is the average impulsive interval and μ_b is defined as (2).

Proof. From the mean value inequality, we have

$$\begin{aligned} u(t) &\leqslant \prod_{i=1}^{N_\zeta(t,t_0)} \left(\frac{M+1}{M}b_i\right)\mathrm{e}^{p(t-t_0)}u_0 \\ &\leqslant \left(\frac{M+1}{M}\right)^{N_\zeta(t,t_0)} \left(\frac{|b_1|+|b_2|+\cdots+|b_{N_\zeta(t,t_0)}|}{N_\zeta(t,t_0)}\right)^{N_\zeta(t,t_0)} \mathrm{e}^{p(t-t_0)}u_0 \\ &= \left(\frac{M+1}{M}\cdot\frac{|b_1|+|b_2|+\cdots+|b_{N_\zeta(t,t_0)}|}{N_\zeta(t,t_0)}\right)^{N_\zeta(t,t_0)} \mathrm{e}^{p(t-t_0)}u_0 \\ &= \mathrm{e}^{N_\zeta(t,t_0)\ln(\frac{\frac{M}{M+1}(|b_1|+|b_2|+\cdots+|b_{N_\zeta(t,t_0)}|)}{N_\zeta(t,t_0)})}\mathrm{e}^{p(t-t_0)}u_0 \\ &= \mathrm{e}^{\frac{A}{B}(t-t_0)}\mathrm{e}^{p(t-t_0)}u_0, \end{aligned}$$

where $A = \ln(\frac{\frac{M+1}{M}(|b_1|+|b_2|+\cdots+|b_{N_\zeta(t,t_0)}|)}{N_\zeta(t,t_0)})$ and $B = \frac{t-t_0}{N_\zeta(t,t_0)}$. For any η_1 satisfying $\eta_1 > \eta = \frac{\ln\frac{M+1}{M}\mu_b}{T_\alpha} + p$, there exists $T > 0$ such that when $t \geqslant T$, we have

$$u(t) \leqslant \mathrm{e}^{(\frac{\ln\frac{M+1}{M}\mu_b}{T_\alpha}+\eta_1-\eta+p)(t-t_0)}u_0 = \mathrm{e}^{\eta_1(t-t_0)}u_0.$$

Theorem 2. Consider inequality (1); then any solution $u(t)$ of inequality (1) satisfies

$$u(t) \leqslant \left[\left(1+\frac{1}{\widetilde{M}}\right)\mathrm{e}^{-p\tau}\right]^{N_\zeta(t,t_0)} \prod_{i=1}^{N_\zeta(t,t_0)} d_i\mathrm{e}^{p(t-t_0)}u_0, \tag{6}$$

if there exists a positive constant $\widetilde{M}$ such that

$$d_k \geqslant \widetilde{M}b_k\mathrm{e}^{p\tau}, \tag{7}$$

where $u_0 = \sup_{t_0-\tau\leqslant s\leqslant t_0} u(s)$.

Proof. When $t \in [t_0, t_1)$, $u(t) \leqslant \mathrm{e}^{p(t-t_0)}u_0$ and

$$u(t_1) \leqslant b_1u(t_1^-) + d_1[u(t_1^-)]_{\tau^-} \leqslant [b_1\mathrm{e}^{p(t_1-t_0)} + d_1\mathrm{e}^{p(t_1-\tau-t_0)}]u_0.$$

For $t \in [t_1, t_2)$, from (1) and (7),

$$u(t) \leqslant \mathrm{e}^{p(t-t_1)}u(t_1) \leqslant [b_1\mathrm{e}^{p(t-t_0)} + d_1\mathrm{e}^{p(t-\tau-t_0)}]u_0,$$

and

$$\begin{aligned} u(t_2) &\leqslant b_2u(t_2^-) + d_2[u(t_2^-)]_\tau \\ &\leqslant [b_1b_2\mathrm{e}^{p(t_2-t_0)} + b_2d_1\mathrm{e}^{p(t_2-\tau-t_0)} + b_1d_2\mathrm{e}^{p(t_2-\tau-t_0)} + d_1d_2\mathrm{e}^{p(t_2-2\tau-t_0)}]u_0 \\ &\leqslant \left(1+\frac{1}{\widetilde{M}}\right)^2 d_1d_2\mathrm{e}^{p(t_2-2\tau-t_0)}u_0. \end{aligned}$$

Assume that Eq. (6) holds for $N(t,t_0) = k$. Next, analyze the condition of $N(t,t_0) = k+1$. From (7) we have

$$u(t_{k+1}^-) \leqslant \left[\left(1+\frac{1}{\widetilde{M}}\right)\mathrm{e}^{-p\tau}\right]^k \prod_{i=1}^{k} d_i\mathrm{e}^{p(t_{k+1}-t_0)}u_0,$$

and

$$\begin{aligned}u(t_{k+1}) &\leqslant b_{k+1}u(t_{k+1}^-)+d_{k+1}[u(t_{k+1}^-)]_\tau\\&\leqslant b_{k+1}\left[\left(1+\frac{1}{\widetilde{\widetilde{M}}}\right)\mathrm{e}^{-p\tau}\right]^k\prod_{i=1}^{k}d_i\mathrm{e}^{p(t_{k+1}-t_0)}u_0+\left[\left(1+\frac{1}{\widetilde{\widetilde{M}}}\right)\mathrm{e}^{-p\tau}\right]^k\prod_{i=1}^{k+1}d_i\mathrm{e}^{p(t_{k+1}-\tau-t_0)}u_0\\&\leqslant \frac{1}{\widetilde{\widetilde{M}}}\mathrm{e}^{-p\tau}\left[\left(1+\frac{1}{\widetilde{\widetilde{M}}}\right)\mathrm{e}^{-p\tau}\right]^k\prod_{i=1}^{k+1}d_i\mathrm{e}^{p(t_{k+1}-t_0)}u_0+\left[\left(1+\frac{1}{\widetilde{\widetilde{M}}}\right)\mathrm{e}^{-p\tau}\right]^k\prod_{i=1}^{k+1}d_i\mathrm{e}^{p(t_{k+1}-\tau-t_0)}u_0\\&\leqslant \left[\left(1+\frac{1}{\widetilde{\widetilde{M}}}\right)\mathrm{e}^{-p\tau}\right]^{k+1}\prod_{i=1}^{k+1}d_i\mathrm{e}^{p(t_{k+1}-t_0)}u_0.\end{aligned}$$

When $t\in[t_{k+1},t_{k+2})$,

$$u(t)\leqslant \mathrm{e}^{p(t-t_{k+1})}u(t_{k+1})\leqslant\left(1+\frac{1}{\widetilde{\widetilde{M}}}\right)^{k+1}\prod_{i=1}^{k+1}b_i\mathrm{e}^{p(t-t_0)}u_0.$$

Using mathematical induction, we obtain

$$u(t)\leqslant\left[\left(1+\frac{1}{\widetilde{\widetilde{M}}}\right)\mathrm{e}^{-p\tau}\right]^{N_\zeta(t,t_0)}u_0\prod_{i=1}^{N_\zeta(t,t_0)}d_i\mathrm{e}^{p(t-t_0)}.$$

Hence, the proof is complete.

Corollary 2. Suppose that Eq. (7) holds. Then, for any solution $u(t)$ of inequality (1), there exists $T>0$ such that

$$u(t)\leqslant \mathrm{e}^{\eta_1(t-t_0)}u_0,\quad t\geqslant T,$$

where $\eta_1>\eta=\frac{\ln\frac{\widetilde{\widetilde{M}}+1}{\widetilde{\widetilde{M}}}\mathrm{e}^{-p\tau}\mu_d}{T_\alpha}+p$, T_α is the average impulsive interval and μ_d is defined as (3).

Proof. Applying the mean value inequality yields

$$\begin{aligned}u(t)&\leqslant\prod_{i=1}^{N_\zeta(t,t_0)}\left(\frac{\widetilde{\widetilde{M}}+1}{\widetilde{\widetilde{M}}}\mathrm{e}^{-p\tau}d_i\right)\mathrm{e}^{p(t-t_0)}u_0\\&\leqslant\left(\frac{\widetilde{\widetilde{M}}+1}{\widetilde{\widetilde{M}}}\mathrm{e}^{-p\tau}\right)^{N_\zeta(t,t_0)}\left(\frac{|d_1|+|d_2|+\cdots+|d_{N_\zeta(t,t_0)}|}{N_\zeta(t,t_0)}\right)^{N_\zeta(t,t_0)}\mathrm{e}^{p(t-t_0)}u_0\\&=\mathrm{e}^{N_\zeta(t,t_0)\ln(\frac{\frac{\widetilde{\widetilde{M}}+1}{\widetilde{\widetilde{M}}}\mathrm{e}^{-p\tau}(|d_1|+|d_2|+\cdots+|d_{N_\zeta(t,t_0)}|)}{N_\zeta(t,t_0)})}\mathrm{e}^{p(t-t_0)}u_0\\&=\mathrm{e}^{\frac{\widetilde{A}}{\widetilde{B}}(t-t_0)}\mathrm{e}^{p(t-t_0)}u_0,\end{aligned}$$

where $\widetilde{A}=\ln(\frac{\frac{\widetilde{\widetilde{M}}+1}{\widetilde{\widetilde{M}}}\mathrm{e}^{-p\tau}(|d_1|+|d_2|+\cdots+|d_{N_\zeta(t,t_0)}|)}{N_\zeta(t,t_0)})$ and $\widetilde{B}=\frac{t-t_0}{N_\zeta(t,t_0)}$. For any η_1 satisfying $\eta_1>\eta=\frac{\ln\frac{\widetilde{\widetilde{M}}+1}{\widetilde{\widetilde{M}}}\mathrm{e}^{-p\tau}\mu_d}{T_\alpha}+p$, there exists $T>0$ such that when $t\geqslant T$ we have

$$u(t)\leqslant \mathrm{e}^{(\frac{\ln\frac{\widetilde{\widetilde{M}}+1}{\widetilde{\widetilde{M}}}\mathrm{e}^{-p\tau}\mu_d}{T_\alpha}+\eta_1-\eta+p)(t-t_0)}u_0=\mathrm{e}^{\eta_1(t-t_0)}u_0.$$

Remark 2. In the real world, the effects of impulses and delayed impulses on the stability of DSs can be very complex. Impulses can not only promote the stability of DSs but also suppress the stability. Previous study [34] showed that the delayed impulses could destabilize the DS. For example, to guarantee stability, the delayed impulsive gain in [33] should be kept as small as possible. Although Li et al. [34] proved that the delayed impulses may facilitate stability, the DS in this case only contains delayed impulses without delay-free impulses. The systems considered in the current study incorporate both hybrid impulses and hybrid delayed impulses simultaneously to stabilize the system. Comparison of their parameters, the influences of impulses and delayed impulses on the stability, will be discussed later.

Remark 3. Zhang et al. [33] noted that the delayed impulsive gain should be as small as possible, which is consistent with Theorem 1 and Corollary 1. Moreover, our results presented the explicit relationship between impulsive parameters and delayed impulsive parameters. On the other hand, considering the stabilizing delayed impulses, Theorem 2 and Corollary 2 concerned the dynamical behavior of impulsive inequality with stabilizing delayed impulses.

4 Applications

Using the results obtained in Section 3, this section studies the stability problems of a linear time-invariant system

$$\begin{cases} \dot{x}(t) = Ax(t), \quad t \neq t_k, \\ x(t_k) = C_k x(t_k^-) + D_k x(t_k^- - \tau_k), \\ x(t) = \varphi(t), \quad t \in [t_0 - \tau, t_0], \end{cases} \tag{8}$$

where $x(t) \in \mathbb{R}^n$ is the state variable, A, C_k, $D_k \in \mathbb{R}^{n\times n}$ are constant matrices, $0 < \tau_k < \tau$, and $\varphi(t) \in C([t_0 - \tau, t_0], \mathbb{R}^n)$ is the initial function. The impulsive sequence $\zeta = \{t_1, t_2, t_3, \ldots\}$ is the same as that in impulsive differential inequality (1).

Theorem 3. System (8) is globally exponentially stable if there exist constants $\beta \in \mathbb{R}$, $b_k > 0$, $d_k > 0$, and a positive-definite matrix P such that (2), (4), and the following LMIs hold:

$$PA + A^{\mathrm{T}}P - \beta P < 0, \tag{9}$$

$$\begin{pmatrix} C_k^{\mathrm{T}} P C_k - b_k P & C_k^{\mathrm{T}} P D_k \\ D_k^{\mathrm{T}} P C_k & D_k^{\mathrm{T}} P D_k - d_k P \end{pmatrix} < 0, \tag{10}$$

where

$$0 > \eta_1 > \eta = \frac{\ln \frac{M}{M+1}\mu_b}{T_\alpha} + \beta.$$

Proof. Consider the function $V(t) = x^{\mathrm{T}}(t)Px(t)$. We have

$$D^+V(t) = x^{\mathrm{T}}(t)PAx(t) + x^{\mathrm{T}}(t)A^{\mathrm{T}}Px(t).$$

From (9), we have

$$D^+V(t) \leqslant \beta V(t).$$

Furthermore, substituting (10) into (8) yields

$$\begin{aligned} V(t_k) &= x^{\mathrm{T}}(t_k^-)C_k^{\mathrm{T}}PC_k x(t_k^-) + x^{\mathrm{T}}(t_k^-)C_k^{\mathrm{T}}PD_k x(t_k^- - \tau_k) \\ &\quad + x^{\mathrm{T}}(t_k^- - \tau_k)D_k^{\mathrm{T}}PC_k x(t_k^-) + x^{\mathrm{T}}(t_k^- - \tau_k)D_k^{\mathrm{T}}PD_k x(t_k^- - \tau_k) \\ &\leqslant b_k x^{\mathrm{T}}(t_k^-)Px(t_k^-) + d_k x^{\mathrm{T}}(t_k^- - \tau_k)Px(t_k^- - \tau_k) \\ &= b_k V(t_k^-) + d_k V(t_k^- - \tau_k). \end{aligned}$$

Then, from Corollary 1 we can deduce that

$$V(t) \leqslant V_0 \mathrm{e}^{\eta_1(t-t_0)},$$

where $V_0 = \sup_{t_0-\tau \leqslant s \leqslant t_0} V(s)$. Then

$$\|x(t)\|^2 \leqslant \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\|\varphi\|^2 \mathrm{e}^{\eta(t-t_0)}. \tag{11}$$

Hence, one can conclude that the ES of system (8) is obtained from (11).

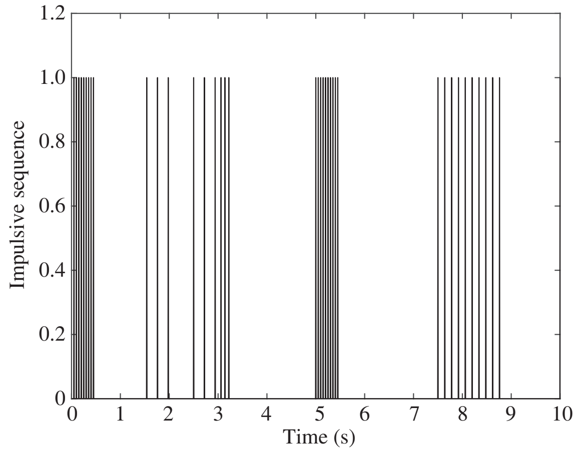

Figure 1 Impulsive sequence with average impulsive interval $T_\alpha = 0.25$. This sequence is repeated.

Similarly, one can deduce the following result by Corollary 2.

Theorem 4. System (8) is globally exponentially stable if there exist constants $\beta_1 \in \mathbb{R}$, $b_k > 0$, $d_k > 0$, and a positive-definite matrix $P \in \mathbb{R}^{n\times n}$ such that (3), (7), and the following LMIs hold:

$$PA + A^{\mathrm{T}}P - \beta_1 P < 0, \tag{12}$$

$$\begin{pmatrix} C_k^{\mathrm{T}}PC_k - b_kP & C_k^{\mathrm{T}}PD_k \\ D_k^{\mathrm{T}}PC_k & D_k^{\mathrm{T}}PD_k - d_kP \end{pmatrix} < 0, \tag{13}$$

where

$$0 > \eta_1 > \eta = \frac{\ln \frac{M}{M+1}\mathrm{e}^{-\beta_1\tau}\mu_d}{T_\alpha} + p. \tag{14}$$

Remark 4. The above two theorems show two different delayed impulses. Theorem 3 states that the ES of impulsive DS (8) is robust with respect to delayed impulses, while the impulses without delays contribute to stabilization. Theorem 4 shows that the stability of an unstable system can be achieved under delayed impulsive control, which illustrates that delayed impulses can promote the stability of DSs.

5 Numerical simulations

In this section, we validate our theoretical results with two examples.

Example 1. Consider the following DS:

$$\begin{cases} \dot{x}(t) = Ax(t), \quad t \neq t_k, \\ x(t_k) = C_kx(t_k^-) + D_kx(t_k^- - \tau_k), \end{cases} \tag{15}$$

where $A = \left(\begin{smallmatrix} 1 & 1 \\ -2 & 0.7 \end{smallmatrix}\right)$. It follows from $\mathrm{Re}(\lambda_A) > 0$ that this linear system is unstable without impulses. According to Theorem 3, we will design some parameters to stabilize the system. Let the time decay $\tau_k = \tau = 0.3$, $T_\alpha = 0.25$, $M = 2$, and $\alpha = 0.3$. The impulsive sequence is shown in Figure 1. The matrices C_k can be chosen from the set $\{0.2I, 0.4I, 0.3I\}$ and $D_k \in \{0.3I, 0.2I, 0.2I\}$ with equal probability. Let impulsive parameters $b_k \in \{0.3, 0.5, 0.4\}$ and delayed impulsive parameters $d_k \in \{0.16, 0.27, 0.21\}$; then $\mu_b = 0.4$ and $\mu_d = 0.213$. Using the LMI toolbox of MATLAB, we can find

$$P = \begin{pmatrix} 26.0614 & 0.7118 \\ 0.7118 & 25.672 \end{pmatrix},$$

which satisfies (9) and (10). From Theorem 3, we can deduce that the system (15) is stable. The corresponding trajectory of $x(t)$ is depicted in Figure 2.

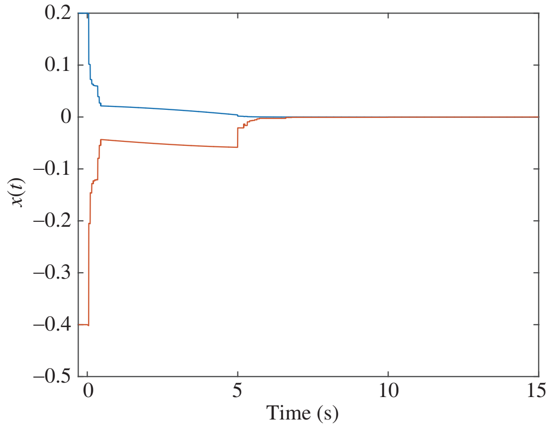

Figure 2 (Color online) Trajectory of system (15) in Example 1.

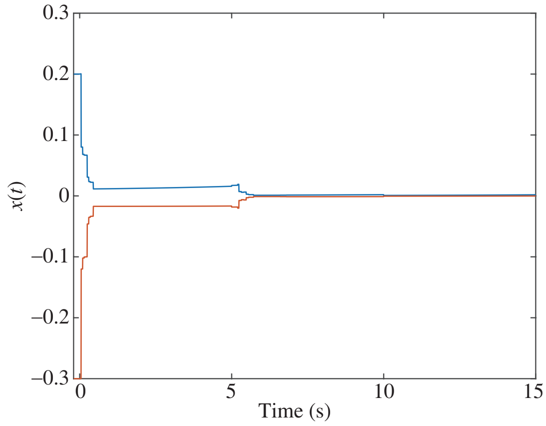

Figure 3 (Color online) Trajectory of system (15) in Example 2.

Example 2. This example considers linear system (8) with $A = \left(\begin{smallmatrix} 0.2 & 0.1 \\ 0.06 & 0.04 \end{smallmatrix}\right)$. By simple calculation, A is not Hurwize; hence, the system is unstable without impulses. Let $C_k \in \{0.3I, 0.1I, 0.15I\}$ and $D_k \in \{0.8I, 0.4I, 0.3I\}$. The impulsive sequence is the same as in Example 1. Let the impulsive delays $\tau_k = \tau = 0.2$ and $\widetilde{M} = 2$. The impulsive parameters are chosen as $b_k \in \{0.18, 0.14, 0.23\}$ and delayed impulsive parameters are chosen as $d_k \in \{0.4, 0.3, 0.35\}$. Then, we can obtain $\mu_b = 0.183$, $\mu_d = 0.35$, and $p = 0.3$, which satisfy (7) and (14). Using the LMI toolbox of MATLAB, we can find

$$P = \begin{pmatrix} 0.3903 & 0.0947 \\ 0.0947 & 0.2723 \end{pmatrix},$$

satisfying (12) and (13). From Theorem 4, we can see that this system is stable under delayed impulses, and Figure 3 shows the trajectory of $x(t)$.

6 Conclusion and further work

In this study, a novel Halanay-type inequality with hybrid impulses and hybrid delayed impulses was investigated. Two concepts were proposed to describe the average impulsive and delayed impulsive gains. To study the effects of impulses and delayed impulses on the stability, the impulsive parameters and the delayed impulsive parameters were compared. Stabilizing impulses and stabilizing delayed impulses were analyzed to study the stability problem for impulsive DSs. Using the obtained results concerning Halanay-type inequality, we derived a few sufficient conditions for the ES of linear DSs with hybrid impulsive effects. In addition, although the stability problem for DSs with hybrid impulses was considered in this study, many issues should be considered in further work. For example, it is interesting to consider the stability of delayed systems with hybrid impulsive effects and hybrid delayed impulsive effects.

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