

Optimal control with irregular performance

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Received 23 August 2018/Accepted 30 September 2018/Published online 2 August 2019

Abstract In this paper, we solve the long-standing fundamental problem of irregular linear-quadratic (LQ) optimal control, which has received significant attention since the 1960s. We derive the optimal controllers via the key technique of finding the analytical solutions to two different forward and backward differential equations (FBDEs). We give a complete solution to the finite-horizon irregular LQ control problem using a new ‘two-layer optimization’ approach. We also obtain the necessary and sufficient condition for the existence of optimal and stabilizing solutions in the infinite-horizon case in terms of solutions to two Riccati equations and the stabilization of one specific system. For the first time, we explore the essential differences between irregular and standard LQ control, making a fundamental contribution to classical LQ control theory. We show that irregular LQ control is totally different from regular control as the irregular controller must guarantee the terminal state constraint of $P_1(T)x(T) = 0$.

Keywords irregular, LQ control, Riccati equation, stabilization

Citation Zhang H S, Xu J J. Optimal control with irregular performance. *Sci China Inf Sci*, 2019, 62(9): 192203, <https://doi.org/10.1007/s11432-018-9685-8>

1 Introduction

Because of the widespread use in modern engineering, linear-quadratic (LQ) optimal control has become one of the most fundamental problems in the field and has received much attention in recent years [1–18]. Since 1950s, researchers have studied classical LQ control problems where the controller’s weight matrix in the cost function is positive definite [19, 20]. Two main approaches have been developed to handle such problems: the maximum principle [21] and dynamic programming [22]. When the controller’s weight matrix is positive definite, a unique optimal solution exists, and can be given in terms of the standard Riccati equation, which is clearly regular in this case.

When the weight matrix is positive semi-definite (and possibly zero), the LQ problem, called singular control, has remained a significant challenge since the 1960s. Here, there are three main approaches. The first approach is ‘transformation in state space’ (see [23–25] and references therein for details). The problem with a zero weight matrix R was studied by Ho [1], who showed that the problem is solvable if the initial values are of the form $x_2(0) = C_{21}(0)x_1(0)$. Otherwise, an initial impulse control must be applied. It should also be noted that only sufficient conditions have been given for the general case $R \geq 0$. The second approach is the higher-order maximum principle [26–30]. Here, we note that a singular control cannot be found with this approach if the higher derivatives vanish [26]. Furthermore, particular initial values must usually be chosen to ensure the problem is solvable. The third approach applies a perturbation [3, 5]. In [5], open-loop solutions for irregular LQ problems were obtained by applying a minimize convergence sequence while closed-loop controllers were derived based on the regularity of the

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Riccati equation. In summary, although singular control has been studied for more than 50 years, some fundamental problems still remain to be solved, especially irregular LQ control, which is the essence of singular control.

In this paper, we consider the solvability of irregular LQ control problems with both finite and infinite horizons. One key issue in LQ control problems is to find an optimal controller that minimizes the cost function while also stabilizing the original system. For traditional LQ control problems, there is always a unique positive-definite solution to the standard algebraic Riccati equation under stabilizability and observability assumptions. However, these conditions are too strict for infinite-horizon irregular LQ control problems to be solvable. Here, we first derive the necessary and sufficient condition for solving finite-horizon irregular LQ control problems. Based on a new “two-layer optimization” approach, we show that the controller entries of irregular-LQ controller can be derived from two equilibrium conditions in two different layers, unlike in classical regular LQ control where all these entries are obtained from an equilibrium condition in one layer, based on a regular Riccati equation. Then, we give the necessary and sufficient condition for the existence of optimal and stabilizing solutions to infinite-horizon irregular LQ control problems. Specially, we derive the optimal controller in terms of two layers, where the first layer is based on the solutions to two Riccati equations and the second one is designed based on the stabilization of one specific system.

The remainder of this paper is organized as follows. Section 2 introduces the problems studied in this paper. Sections 3 and 4 present the solutions to the finite-horizon and infinite-horizon irregular control problems, respectively. Finally, Section 5 gives some concluding remarks, and Appendixes A–E provide additional proofs.

Throughout this paper, we use the following notations. First, \mathbb{R}^n denotes the family of n -dimensional vectors, x' represents the transpose of x , and $\|x\|^2 = x'x$. Stating that a symmetric matrix $M > 0$ (≥ 0) means it is strictly positive definite (positive semi-definite), and $\text{Range}(M)$ represents its range. Finally, M^\dagger is the Moore-Penrose inverse [31] of the matrix M if it satisfies $MM^\dagger M = M$, $M^\dagger MM^\dagger = M^\dagger$, $(MM^\dagger)' = MM^\dagger$, and $(M^\dagger M)' = M^\dagger M$.

2 Problem formulation

Consider the linear system governed by the following differential equation:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0, \quad (1)$$

where $x \in \mathbb{R}^n$ is the system's state, $u \in \mathbb{R}^m$ is its control input, and x_0 is its initial value. The matrices A and B are constant and have compatible dimensions. The cost function is given by

$$J_T(t_0, x_0; u) = \int_{t_0}^T [x'(t)Qx(t) + u'(t)Ru(t)]dt + x'(T)Hx(T), \quad (2)$$

where $Q \geq 0$, $R \geq 0$, and $H \geq 0$ are symmetric matrices with compatible dimensions. Here, we consider two problems, namely finite-horizon and infinite-horizon optimal control. The finite-horizon optimal control problem can be stated as follows.

Problem 1. For any given initial pair (t_0, x_0) , find a $u^*(t)$ such that

$$J_T^*(t_0, x_0; u^*) = \min_{u(\cdot)} J_T(t_0, x_0; u).$$

Remark 1. Unlike with standard LQ control, the weight matrix R in the cost function (2) is singular, which makes the problem significantly more complex, as mentioned in Section 1. We will show that the difficulty of singular control problems is caused by the so-called irregularity, that is, the fact that the controller cannot be derived from the equilibrium condition. In other words, irregular control problems cannot be solved by simply completing the square. To make this clearer, we present the following example.

Consider the system

$$\dot{x}(t) = [1 \ 1]u(t), \quad x(t_0) = x_0,$$

and the cost function

$$J_T(t_0, x_0; u) = \int_{t_0}^T u'(t) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} u(t) dt + x'(T)x(T). \tag{3}$$

The solution to the Riccati equation $0 = \dot{P}(t) - P^2(t)$ with $P(T) = 1$ is $P(t) = -\frac{1}{t-T-1}$. We then find that $\text{Range}(B'P(t)) \not\subseteq \text{Range}(R)$, implying that we cannot obtain $u(t)$ from the equation $Ru(t) + B'P(t)x(t) = 0$ for any $x(t)$.

In addition, taking the derivative of $x'(t)P(t)x(t)$ yields

$$\frac{d}{dt}[x'(t)P(t)x(t)] = 2u'(t)B'P(t)x(t) + x'(t)P^2(t)x(t).$$

Then, by integrating this from t_0 to T , the cost function (3) can be rewritten as

$$\begin{aligned} J_T(t_0, x_0; u) &= x'(t_0)P(t_0)x(t_0) + \int_{t_0}^T (u'(t)Ru(t) + 2u'(t)B'P(t)x(t) + x'(t)P^2(t)x(t)) dt \\ &= x'(t_0)P(t_0)x(t_0) + \int_{t_0}^T \left[(u(t) + R^\dagger B'P(t)x(t))' R (u(t) + R^\dagger B'P(t)x(t)) Ru(t) \right. \\ &\quad \left. + 2u'(t)(I - RR^\dagger)B'P(t)x(t) \right] dt. \end{aligned}$$

From this cost function and the fact that $\text{Range}(B'P(t)) \not\subseteq \text{Range}(R)$, we can see that the optimal controller cannot be obtained by completing the square.

Given this, we cannot apply any of the classical approaches to this irregular optimal control problem. However, it is nonetheless solvable; in fact, one optimal solution is given by

$$u(t) = - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{x(t_0)}{T - t_0} \quad \text{or} \quad u(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{x(t)}{t - T},$$

and the corresponding optimal cost is 0.

In some researches, there exists some work on LQ control in the $R \geq 0$ case, even for stochastic control with indefinite R (see [32] and references therein). However, the problems they studied were assumed to be regular, and the solvability of the irregular LQ problem remains a challenge. One contribution of this paper is that we present the complete solution to the irregular LQ problem.

Remark 2. Irregular LQ control has broad applications in numerous areas, such as engineering systems, economic models, and the production of natural resources [6, 33, 34]. For example, a fundamental space navigation problem is to control rockets so as to minimize the propellant expenditure based on a mathematical model that includes bounds on the rocket's thrust, which is a typical irregular LQ problem [6].

In the infinite-horizon case, we are only interested in stabilizing controllers; thus, we introduce the admissible control set

$$\mathcal{U} = \left\{ u(t), t \in [0, \infty] \mid \int_0^\infty \|u(t)\|^2 dt < \infty, u(t) \text{ is stabilizing control} \right\}.$$

The infinite-horizon problem is described in detail below.

Problem 2. Find a control $u(t) \in \mathcal{U}$ which minimizes the cost function $J(x_0; u)$ and stabilizes the system (1) where $J(x_0; u)$ is defined by

$$J(x_0; u) = \int_0^\infty [x'(t)Qx(t) + u'(t)Ru(t)] dt. \tag{4}$$

We now discuss this irregularity in detail. It is well known [18] that Problem 1 is solvable if and only if the following forward and backward differential equations (FBDEs) are solvable:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0, \tag{5}$$

$$\dot{p}(t) = -[A'p(t) + Qx(t)], \quad p(T) = Hx(T), \tag{6}$$

$$0 = Ru(t) + B'p(t). \tag{7}$$

This implies that the key to deriving optimal solutions is the solvability of the FBDEs (5)–(7). To this end, we introduce the following Riccati equation:

$$0 = \dot{P}(t) + A'P(t) + P(t)A + Q - P(t)BR^\dagger B'P(t), \tag{8}$$

where the terminal condition is given by $P(T) = H$ and R^\dagger represents the Moore-Penrose inverse of R . Throughout this paper, we assume that the Riccati equation (8) is solvable. The solvability condition has been given in Corollary 2.10 of Chapter 6 in [35].

In the $R > 0$ case, if we replace $p(t)$ with $P(t)x(t)$, the equilibrium condition (7) becomes

$$0 = Ru(t) + B'P(t)x(t). \tag{9}$$

Thus, when

$$\text{Range}(B'P(t)) \subseteq \text{Range}(R), \tag{10}$$

the linear equation (9) is solvable for any $x(t) \in \mathbb{R}^n$, which implies the optimization problem is solvable as well. Conversely, when

$$\text{Range}(B'P(t)) \not\subseteq \text{Range}(R), \tag{11}$$

the linear equation (9) is unsolvable. This case of (11) is termed as irregular control. As shown in the previous example (Remark 1), an optimal control cannot be obtained from the equilibrium condition or by reformulating the cost function by completing the square. However, the irregular case may still be solvable. In the next section, we give a complete solution to the irregular LQ control problem.

In summary, as discussed above, the irregularity of the problem implies that we cannot simply complete the square for the cost function (2).

3 Solution to Problem 1

In this section, we study the finite-horizon optimal control problem, and obtain results for both the regular and irregular cases. First, we give the optimal solution in the regular case.

Theorem 1. In the regular case (10), the optimal solution to Problem 1 is given by

$$u(t) = -R^\dagger B'P(t)x(t) + (I - R^\dagger R)z(t), \tag{12}$$

where $z(t)$ is an arbitrary vector of compatible dimension. In this case, the optimal cost is

$$J_T^*(t_0, x_0; u^*) = x_0' P(t_0) x_0. \tag{13}$$

Proof. This result can be obtained directly from [32].

Now, we turn to the irregular case, namely $\text{Range}(B'P(t)) \not\subseteq \text{Range}(R)$. First, we transform the task of solving Problem 1 into that of solving new FBDEs. Before presenting the result, we introduce some notations for future convenience. Without loss of generality, let $\text{rank}(R) = m_0 < m$, so $\text{rank}(I - R^\dagger R) = m - m_0 > 0$. There is also an elementary row transformation matrix T_0 such that

$$T_0(I - R^\dagger R) = \begin{bmatrix} 0 \\ \Upsilon_{T_0} \end{bmatrix}, \tag{14}$$

where $\Upsilon_{T_0} \in \mathbb{R}^{[m-m_0] \times m}$ is of full row rank. In addition, let

$$\begin{aligned} A_0(t) &= A - BR^\dagger B' P(t), \quad D_0 = -BR^\dagger B', \quad \begin{bmatrix} * & G_0 \end{bmatrix} = T_0^{-1}, \\ \begin{bmatrix} * & C'_0(t) \end{bmatrix} &= P(t)B(I - R^\dagger R)T_0^{-1}, \quad \begin{bmatrix} * & B_0 \end{bmatrix} = B(I - R^\dagger R)T_0^{-1}, \end{aligned}$$

and define

$$0 = \dot{P}_1(t) + P_1(t)A_0(t) + A'_0(t)P_1(t) + P_1(t)D_0P_1(t), \tag{15}$$

where the terminal value $P_1(T)$ is yet to be determined. Next, Lemma 1 states that solving Problem 1 is equivalent to solving new FBDEs.

Lemma 1. Given condition (11), Problem 1 is solvable if and only if there exists a $u_1(t) \in \mathbb{R}^{m-m_0}$ such that

$$0 = C_0(t)x(t) + B'_0\Theta(t), \Theta(T) = 0, \tag{16}$$

where $u_1(t)$, $x(t)$, and $\Theta(t)$ satisfy the FBDEs

$$\dot{x}(t) = A_0(t)x(t) + D_0\Theta(t) + B_0u_1(t), \tag{17}$$

$$\dot{\Theta}(t) = -(A'_0(t)\Theta(t) + C'_0(t)u_1(t)). \tag{18}$$

Proof. The proof is given in Appendix A.

We are now in a position to give the solvability condition for Problem 1.

Theorem 2. Given condition (11), Problem 1 is solvable if and only if there exists a $P_1(t)$ in (15) with terminal value $P_1(T)$ such that

$$0 = C_0(t) + B'_0P_1(t), \quad t_0 \leq t \leq T, \tag{19}$$

and a $u_1(t)$ that achieves

$$P_1(T)x(T) = 0, \tag{20}$$

where $x(t)$ obeys

$$\dot{x}(t) = (A_0(t) + D_0P_1(t))x(t) + B_0u_1(t) \tag{21}$$

and has initial value $x(t_0) = x_0$. In this case, the optimal controller $u(t)$ is given by

$$u(t) = -R^\dagger B' (P(t) + P_1(t))x(t) + G_0u_1(t), \tag{22}$$

and the optimal cost is given by

$$J_T^*(t_0, x_0; u^*) = x'_0 (P(t_0) + P_1(t_0)) x_0.$$

Proof. The proof is given in Appendix B.

Corollary 1. The following statements hold.

- (1) Under the conditions of Theorem 1, $P(t) \geq 0$.
- (2) Under the conditions of Theorem 2, $P(t) + P_1(t) \geq 0$.

Proof. Because the matrices Q , R , and H are positive semi-definite, the cost function (2) is non-negative. Thus, the optimal costs obtained in Theorems 1 and 2 are non-negative. Considered together with the arbitrariness of the initial time and value, these results follow, completing the proof.

Next, we formulate the open-loop and closed-loop solutions.

Theorem 3. Given condition (11), Problem 1 is open-loop solvable if and only if there exists a $P_1(t)$ in (15) such that Eq. (19) holds and

$$\text{Range}[P_1(t_0)] \subseteq \text{Range}(G_1[t_0, T]), \tag{23}$$

where the Gramian matrix $G_1[t_0, T]$ is defined by

$$G_1[t_0, T] = \int_{t_0}^T P_2(t_0, s)C_0'(s)C_0(s)P_2'(t_0, s)ds, \tag{24}$$

and $P_2(t, s)$ satisfies $\dot{P}_2(t, s) = -A_0'(t)P_2(t, s)$, with $P_2(t, t) = I$. In this case, the open-loop solution is given by (22), while $u_1(t)$ is given by

$$u_1(t) = C_0(t)P_2'(t_0, t)G_1^\dagger[t_0, T]P_1(t_0)x_0. \tag{25}$$

Proof. The proof is given in Appendix C.

Next, we consider the closed-loop solution. To this end, let $P_1(t) = \begin{bmatrix} P_{11}^1(t) & P_{12}^1(t) \\ P_{12}^2(t) & P_{22}^1(t) \end{bmatrix}$. Assume that $P_1(t)$ is singular. Then there exists a $\mathcal{T}_1(t)$ such that $\mathcal{T}_1'(t)P_1(t)\mathcal{T}_1(t) = \begin{bmatrix} \hat{P}(t) & 0 \\ 0 & 0 \end{bmatrix}$, where $\hat{P}(t)$ is invertible. Let $\dot{\mathcal{T}}_1'(t)\mathcal{T}_1(t) = \begin{bmatrix} \hat{A}_1(t) \\ \hat{A}_2(t) \end{bmatrix}$, $\mathcal{T}_1'(t)B_0\mathcal{T}_1(t) = \begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix}$, and $\mathcal{T}_1'(t)(A_0(t) + D_0P_1(t))\mathcal{T}_1(t) = \begin{bmatrix} \hat{A}_1(t) \\ \hat{A}_2(t) \end{bmatrix}$.

Theorem 4. Given condition (11), assuming that Eq. (19) holds and that there exists a $K(t)$ such that

$$\tilde{\mathcal{T}}_1(t) + \hat{A}_1(t) + B_1(t)\mathcal{T}_1'(t)K(t)\mathcal{T}_1(t) = \begin{bmatrix} I & \\ \frac{I}{t-T} & 0 \end{bmatrix}, \tag{26}$$

then Problem 1 is closed-loop solvable. The closed-loop solution is given by (22) with $u_1(t) = K(t)x(t)$, where $K(t)$ satisfies (26).

Proof. The proof is given in Appendix D.

4 Solution to Problem 2

In this section, we focus on the infinite-horizon optimal control problem, giving necessary and sufficient conditions for the solvability of this optimization problem in both the regular and irregular cases. In the discussion below, we make use of the following algebraic Riccati equation:

$$0 = A'P + PA + Q - PBR^\dagger B'P. \tag{27}$$

4.1 Regular case

First, we consider the case where $\text{Range}(B'P(t)) \subseteq \text{Range}(R)$.

Theorem 5. Given the condition (10), there exists an optimal and stabilizing solution to Problem 2 if and only if the following conditions hold.

- (1) There exists a solution P to (27) such that $P \geq 0$.
- (2) The system $(A - BR^\dagger B'P, B(I - R^\dagger R))$ is stabilizable.

In this case, the optimal and stabilizing solution is given by

$$u(t) = -R^\dagger B'Px(t) + (I - R^\dagger R)z(t), \tag{28}$$

$$z(t) = Kx(t), \tag{29}$$

where K must be chosen such that the matrix

$$A - BR^\dagger B'P + B(I - R^\dagger R)K \tag{30}$$

is stable.

Proof. The proof can be found in [17].

Remark 3. Without loss of generality, we can assume the existence of an orthogonal matrix T such that $T'RT = \begin{bmatrix} R_1 & 0 \\ 0 & 0 \end{bmatrix}$, with $R_1 > 0$. If we let $\bar{u}(t) = T'u(t)$ and $BT = [B_1 \ B_2]$, the system (1) reduces to

$$\dot{x}(t) = Ax(t) + \begin{bmatrix} B_1 & B_2 \end{bmatrix} \bar{u}(t).$$

The Riccati equation (27) then becomes

$$0 = A'P + PA + Q - PB_1R_1^{-1}B_1'P. \tag{31}$$

By applying the standard results in [10, 15], we have that when (A, Q) is observable and (A, B_1) is stabilizable, the algebraic Riccati equation (31) admits a unique positive-definite solution P . An iterative algorithm for solving (31) is given in Corollary 1 of [36]. In addition, the matrix $A - B_1R_1^{-1}B_1'P$ is stable. In this case, K in (29) can be chosen to be the zero matrix, leading to the matrix in (30) being exactly $A - BR^\dagger B'P = A - B_1R_1^{-1}B_1'P$, which is stable. Thus, if (A, Q) is observable and (A, B_1) is stabilizable, then there exists an optimal and stabilizing solution that minimizes the cost function (4), according to Theorem 5. Accordingly, these conditions are sufficient for such a solution to exist. However, the reverse is not true. For example, consider the system $\dot{x}(t) = [1 \ 1] \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$ and the cost function

$$J = \int_0^\infty \begin{bmatrix} u_1'(t) & u_2'(t) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} dt.$$

According to Theorem 5, the optimal controller is $u(t) = \begin{bmatrix} 0 \\ u_2(t) \end{bmatrix}$, where $u_2(t)$ can be chosen as $u_2(t) = -x(t)$, stabilizing the state $x(t)$, and the optimal cost function is $J^* = 0$. However, in this case we find that $A = 0$, $Q = 0$, and $B_1 = 1$, indicating that (A, Q) is unobservable. Thus, it is not necessary for (A, Q) to be observable and (A, B_1) to be stabilizable.

4.2 Irregular case

Next, we consider the case where $\text{Range}(B'P(t)) \not\subseteq \text{Range}(R)$. Here, we define the algebraic Riccati equation corresponding to (15) as

$$0 = P_1A_0 + A_0'P_1 + P_1D_0P_1, \tag{32}$$

where $A_0 = A - BR^\dagger B'P$. In addition, let $[* \ C_0'] = PB(I - R^\dagger R)T_0^{-1}$.

Theorem 6. Given condition (11), there exists an optimal and stabilizing solution to Problem 2 if and only if the following conditions hold.

- (1) There exist solutions P and P_1 to (27) and (32), respectively, that satisfy $P + P_1 \geq 0$.
- (2) P and P_1 satisfy

$$C_0 + B_0'P_1 = 0. \tag{33}$$

- (3) The system $(A_0 + D_0P_1, B_0)$ is stabilizable, i.e., there exists a controller $u_1(t)$ such that the following system is stable:

$$\dot{x}(t) = (A_0 + D_0P_1)x(t) + B_0u_1(t). \tag{34}$$

In this case, the optimal controller is given by

$$u(t) = [-R^\dagger B'(P + P_1) + G_0K]x(t), \tag{35}$$

where K must be chosen such that $A_0 + D_0P_1 + B_0K$ is stable, and the optimal cost is

$$J^* = x_0'(P + P_1)x_0.$$

Proof. The proof is given in Appendix E.

5 Conclusion

In this paper, we have solved the long-standing fundamental problem of optimal LQ control with irregular performance. We have obtained optimal solutions to this problem for both the finite and infinite-horizon cases. In particular, for the finite-horizon case, we have been able to obtain a complete solution by applying a ‘two-layer optimization’ approach. For the infinite-horizon case, we have given a necessary and sufficient condition for the existence of an optimal and stabilizing solution in terms of solutions to two Riccati equations and the stabilization of one specific system. These results have possible applications in areas such as irregular measurement feedback control, robust control, H_∞ control, and stochastic control.

We would also like to emphasize that the proposed technique used in this paper, namely the solving of an associated FBDEs, is a very general strategy, with which we have successfully solved other complicated control problems such as stochastic LQ control with time delay [15, 18], Stackelberg game control [37], mean field stochastic control [38], stabilization of NCSs with simultaneous transmission delay and packet dropout [39].

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 61633014, 61573221, 61873332) and the Qilu Youth Scholar Discipline Construction Funding from Shandong University.

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Appendix A Proof of Lemma 1

Proof of necessity. Based on the discussion of (9)–(11), we can see that $p(t) \neq P(t)x(t)$ under the condition (11), where $P(t)$ is the solution to (8). We therefore define a new variable $\Theta(t)$ as

$$p(t) = P(t)x(t) + \Theta(t), \quad (\text{A1})$$

where it is clear that $\Theta(T) = 0$. Next, we aim to derive the new FBDEs (16)–(18) under the solvability of Problem 1.

First, we take the derivative of (A1), obtaining

$$\dot{p}(t) = \dot{P}(t)x(t) + P(t)[Ax(t) + Bu(t)] + \dot{\Theta}(t). \quad (\text{A2})$$

From (6) and (A1), we then find that

$$\dot{p}(t) = -[A'P(t)x(t) + A'\Theta(t) + Qx(t)]. \quad (\text{A3})$$

By comparing (A2) and (A3), we obtain

$$0 = \dot{P}(t)x(t) + P(t)Ax(t) + P(t)Bu(t) + \dot{\Theta}(t) + A'P(t)x(t) + A'\Theta(t) + Qx(t). \quad (\text{A4})$$

Second, we aim to find the controller $u(t)$ and the new equilibrium condition (16). By using (A1), we can formulate the equilibrium condition (7) as

$$0 = Ru(t) + B'p(t) = Ru(t) + B'P(t)x(t) + B'\Theta(t). \quad (\text{A5})$$

Taken together with (11), this can also be written as

$$u(t) = -R^\dagger (B'P(t)x(t) + B'\Theta(t)) + (I - R^\dagger R)z(t), \quad (\text{A6})$$

where $z(t)$ is an arbitrary vector with compatible dimension such that the following equality holds:

$$0 = (I - RR^\dagger) (B'P(t)x(t) + B'\Theta(t)). \quad (\text{A7})$$

Let

$$T_0(I - RR^\dagger)z(t) = \begin{bmatrix} 0 \\ u_1(t) \end{bmatrix}, \quad (\text{A8})$$

where $u_1(t) = \Upsilon_{T_0}z(t) \in \mathbb{R}^{m-m_0(t)}$. Now, we can rewrite (A7) as (16). Note that

$$\begin{aligned} I - RR^\dagger &= (I - RR^\dagger)(I - RR^\dagger) \\ &= (I - RR^\dagger)T_0'(T_0^{-1})'(I - RR^\dagger) \\ &= \begin{bmatrix} 0 & \Upsilon_{T_0}' \end{bmatrix} (T_0^{-1})'(I - RR^\dagger), \end{aligned} \quad (\text{A9})$$

where we have used (14) to derive the last equality. By using the definitions below (14), we can rewrite (A7) as

$$0 = \Upsilon_{T_0}' [C_0(t)x(t) + B_0'\Theta(t)]. \quad (\text{A10})$$

Note that Υ'_{T_0} is of full column rank, and thus Eq. (A10) can be directly rewritten as (16).

Third, we derive the dynamics of $\Theta(t)$. Substituting (A6) into (A4) and using (8) yields

$$\begin{aligned} 0 &= \dot{P}(t)x(t) + P(t)Ax(t) + A'P(t)x(t) + A'\Theta(t) + Qx(t) + \dot{\Theta}(t) \\ &\quad - P(t)BR^\dagger (B'P(t)x(t) + B'\Theta(t)) + P(t)B(I - R^\dagger R)z(t) \\ &= \dot{\Theta}(t) + (A' - P(t)BR^\dagger B')\Theta(t) + P(t)B(I - R^\dagger R)z(t). \end{aligned} \tag{A11}$$

As $(I - R^\dagger R)^2 = I - R^\dagger R$, we find that

$$\begin{aligned} P(t)B(I - R^\dagger R)z(t) &= P(t)B(I - R^\dagger R)T_0^{-1}T_0(I - R^\dagger R)z(t) \\ &= P(t)B(I - R^\dagger R)T_0^{-1} \begin{bmatrix} 0 \\ u_1(t) \end{bmatrix} \\ &= \begin{bmatrix} * & C'_0(t) \end{bmatrix} \begin{bmatrix} 0 \\ u_1(t) \end{bmatrix} = C'_0(t)u_1(t). \end{aligned} \tag{A12}$$

Thus, from (A11) we obtain $\dot{\Theta}(t) = -[A'_0(t)\Theta(t) + C'_0(t)u_1(t)]$, which implies that the dynamics of $\Theta(t)$ is given by (18).

Finally, we derive the dynamics equation (17). By substituting (A6) into (5) and combining this with the fact that $B(I - R^\dagger R)z(t) = B_0u_1(t)$, which can be obtained in a similar way to (A12), we can derive the state dynamics (17).

Proof of sufficiency. Now, we show Problem 1 is solvable if there exists a $u_1(t)$ that enables us to achieve (16). In fact, if Eq. (16) is true then Eqs. (A6) and (A7) can be jointly rewritten as (A5). Further, by reversing the process for (A1)–(A5), we can easily verify that $p(t) = P(t)x(t) + \Theta(t)$, where $x(t)$ and $\Theta(t)$ satisfy (16)–(18), solving (5)–(7). Thus, Problem 1 is solvable, completing the proof.

Appendix B Proof of Theorem 2

Proof of sufficiency. Based on Lemma 1, it is sufficient to verify that $(\Theta(t), x(t)) = (P_1(t)x(t), x(t))$ is the solution to the FBDEs (16)–(18). Taking the derivative of $P_1(t)x(t)$ yields

$$\begin{aligned} \frac{d[P_1(t)x(t)]}{dt} &= \dot{P}_1(t)x(t) + P_1(t)[A_0(t) + D_0P_1(t)]x(t) + P_1(t)B_0u_1(t) \\ &= -A'_0(t)P_1(t)x(t) + P_1(t)B_0u_1(t) \\ &= -A'_0(t)P_1(t)x(t) - C'_0(t)u_1(t), \end{aligned} \tag{B1}$$

where we have used (15) and (19) to derive the last equality. In addition, again using (19), we have

$$C_0(t)x(t) + B'_0P_1(t)x(t) = 0. \tag{B2}$$

By comparing (16)–(18) with (B2), (B1), and (21), we can see that Eqs. (16)–(18) are solvable with $\Theta(t) = P_1(t)x(t)$ if $P_1(T)x(T) = 0$. Thus, based on Lemma 1, Problem 1 is solvable.

Proof of necessity. This proof is divided into two parts. First, we consider the case where the optimal solution is of closed-loop form, namely $u_1(t) = K_1(t)x(t)$. Based on Lemma 1, Eqs. (16)–(18) are solvable if Problem 1 is solvable. By substituting $u_1(t) = K_1(t)x(t)$ into (17) and (18), we obtain

$$\begin{aligned} \dot{x}(t) &= A_0(t)x(t) + D_0\Theta(t) + B_0K_1(t)x(t), \\ \dot{\Theta}(t) &= -(A'_0(t)\Theta(t) + C'_0(t)K_1(t)x(t)). \end{aligned}$$

Solving the above FBDEs gives us $\Theta(t) = \bar{P}(t)x(t)$, where $\bar{P}(t)$ satisfies

$$0 = \dot{\bar{P}}(t) + \bar{P}(t)A_0(t) + \bar{P}(t)D_0\bar{P}(t) + A'_0(t)\bar{P}(t) + (\bar{P}(t)B_0 + C'_0(t))K_1(t). \tag{B3}$$

In addition, substituting $\Theta(t) = \bar{P}(t)x(t)$ into (16) yields

$$0 = C_0(t) + B'_0\bar{P}(t). \tag{B4}$$

Thus, we can reformulate (B3) as

$$0 = \dot{\bar{P}}(t) + \bar{P}(t)A_0(t) + \bar{P}(t)D_0\bar{P}(t) + A'_0(t)\bar{P}(t).$$

Comparing this with (15), we find that $\bar{P}(t) = P_1(t)$. Thus, Eq. (19) follows from (B4) and Eq. (20) follows from $\Theta(T) = 0$ and $\Theta(T) = P_1(T)x(T)$.

Second, the case where the controller $u_1(t)$ is of open-loop form can be solved similarly to the closed-loop case. This completes the proof.

Appendix C Proof of Theorem 3

Proof of sufficiency. Under the condition (19), it is sufficient to verify that $P_1(T)x(T) = 0$, given Theorem 2. To do this, we first state a formula relating $P_1(t)x(t)$ to the control $u_1(t)$ in terms of its dynamics. Similar to (B1), the dynamics of $P_1(t)x(t)$ is given by

$$\frac{d[P_1(t)x(t)]}{dt} = -A'_0(t)P_1(t)x(t) - C'_0(t)u_1(t).$$

Solving this differential equation yields

$$P_1(t)x(t) = \int_t^T P_2(t, s)C'_0(s)u_1(s)ds + P_2(t, T)C, \tag{C1}$$

where $C = P_1(T)x(T)$.

Next, we aim to prove that $C = 0$ under the controller $u_1(t)$ defined in (25). If Eq. (23) holds, then for any x_0 , there exists a ζ such that $P_1(t_0)x_0 = G_1[t_0, T]\zeta$, where $\zeta = G_1^\dagger[t_0, T]P_1(t_0)x_0$. We can now rewrite $u_1(t)$ in (25) as $u_1(t) = C_0(t)P'_2(t_0, t)\zeta$. By substituting $u_1(t)$ into (C1), we obtain

$$P_1(t_0)x_0 = \left[\int_{t_0}^T P_2(t_0, s)C'_0(s)C_0(s)P'_2(t_0, s)ds \right] \zeta + P_2(t_0, T)C = P_1(t_0)x_0 + P_2(t_0, T)C.$$

As $P_2(t_0, T)$ is invertible, we have $C = 0$, implying that $P_1(T)x(T) = 0$. This completes the proof of sufficiency based on Theorem 2.

Proof of necessity. If the control problem is solvable, it follows from Theorem 2 that there exists a $P_1(t)$ such that Eq. (19) holds. We now prove that Eq. (23) does indeed hold. Otherwise, we would have that $\text{Range}[P_1(t_0)] \not\subseteq \text{Range}(G_1[t_0, T])$, meaning that a non-zero vector ρ would exist such that $\rho'P_1(t_0)\rho \neq 0, \rho'G_1[t_0, T]\rho = 0$. Then, we would obtain

$$0 = \rho'G_1[t_0, T]\rho = \rho' \left[\int_{t_0}^T P_2(t_0, s)C'_0(s)C_0(s)P'_2(t_0, s)ds \right] \rho = \int_{t_0}^T \|C_0(s)P'_2(t_0, s)\rho\|^2 ds,$$

implying that $C_0(s)P'_2(t_0, s)\rho = 0$. Thus, we would have $\rho' \int_{t_0}^T P_2(t_0, s)C'_0(s)u_1(s)ds = 0$. Let $x_0 = \rho$. From $\Theta(t_0) = P_1(t_0)x_0$, we would then have $\Theta(t_0) = P_1(t_0)\rho$. Combining this with $\Theta(t) = \int_t^T P_2(t, s)C'_0(s)u_1(s)ds$ gives

$$\rho'P_1(t_0)\rho = \rho' \left[\int_t^T P_2(t, s)C'_0(s)u_1(s)ds \right] \rho = 0.$$

This is a contradiction, so Eq. (23) must hold, completing the proof.

Appendix D Proof of Theorem 4

Let

$$y(t) = \mathcal{T}'_1(t)x(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}. \tag{D1}$$

Then, using (17) and the feedback controller $u_1(t) = K(t)x(t)$, we have

$$\begin{aligned} \dot{y}(t) &= \dot{\mathcal{T}}'_1(t)x(t) + \mathcal{T}'_1(t)(A_0(t)x(t) + D_0\Theta(t) + B_0K(t)x(t)) \\ &= [\dot{\mathcal{T}}'_1(t)\mathcal{T}_1(t) + \mathcal{T}'_1(t)(A_0(t) + D_0P_1(t) + B_0K(t))\mathcal{T}_1(t)]\mathcal{T}'_1(t)x(t) \\ &= \left(\begin{bmatrix} \hat{T}_1(t) \\ \hat{T}_2(t) \end{bmatrix} + \begin{bmatrix} \hat{A}_1(t) \\ \hat{A}_2(t) \end{bmatrix} + \begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix} \mathcal{T}'_1(t)K(t)\mathcal{T}_1(t) \right) y(t) \\ &= \begin{bmatrix} \hat{T}_1(t) + \hat{A}_1(t) + B_1(t)\mathcal{T}'_1(t)K(t)\mathcal{T}_1(t) \\ \hat{T}_2(t) + \hat{A}_2(t) + B_2(t)\mathcal{T}'_1(t)K(t)\mathcal{T}_1(t) \end{bmatrix} y(t). \end{aligned}$$

By applying (26), we obtain

$$\begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix} = \begin{bmatrix} \frac{I}{t-T} & 0 \\ * & * \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}.$$

This implies that $\dot{y}_1(t) = \frac{I}{t-T}y_1(t)$. Then, solving this equation gives us $y_1(t) = \frac{T-t}{T-t_0}y_1(t_0)$, further implying that

$$y_1(T) = 0. \tag{D2}$$

As

$$0 = \mathcal{T}'_1(T)P_1(T)\mathcal{T}_1(T)\mathcal{T}'_1(T)x(T) = \begin{bmatrix} \hat{P}(T) & 0 \\ 0 & 0 \end{bmatrix} y(T) = \hat{P}(T)y_1(T), \tag{D3}$$

we can combine this with the invertibility of $\hat{P}(t)$ to obtain $y_1(T) = 0$. We also have (D1) and $y(T) = \mathcal{T}'_1(T)x(T) = \begin{bmatrix} 0 \\ y_2(T) \end{bmatrix}$, so that

$$\begin{aligned} P_1(T)x(T) &= P_1(T)\mathcal{T}_1(T) \begin{bmatrix} 0 \\ y_2(T) \end{bmatrix} = \mathcal{T}_1(T)\mathcal{T}'_1(T)P_1(T)\mathcal{T}_1(T) \begin{bmatrix} 0 \\ y_2(T) \end{bmatrix} \\ &= \mathcal{T}_1(T) \begin{bmatrix} \hat{P}(T) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ y_2(T) \end{bmatrix} = 0. \end{aligned}$$

This completes the proof.

Appendix E Proof of Theorem 6

Proof of sufficiency. As solutions P and P_1 exist to (27) and (32) that satisfy $P + P_1 \geq 0$, we will show that the cost function is bounded below by $x'_0(P + P_1)x_0$. By taking the derivative of $x'(t)(P + P_1)x(t)$, we obtain

$$\begin{aligned} \frac{d}{dt} (x'(t)(P + P_1)x(t)) &= (Ax(t) + Bu(t))'(P + P_1)x(t) + x'(t)(P + P_1)(Ax(t) + Bu(t)) \\ &= -x'(t)Qx(t) + x'(t)(P + P_1)BR^\dagger B'(P + P_1)x(t) + u'(t)B'(P + P_1)x(t) \\ &\quad + x'(t)(P + P_1)Bu(t), \end{aligned}$$

where we have used (27) and (32) to derive the last equality. By integrating this from 0 to T , we can further obtain

$$\begin{aligned} &\int_0^T (x'(t)Qx(t) + u'(t)Ru(t)) dt \\ &= x'(0)(P + P_1)x(0) - x'(T)(P + P_1)x(T) + \int_0^T \left(u'(t)Ru(t) + x'(t)(P + P_1)BR^\dagger B'(P + P_1)x(t) \right. \\ &\quad \left. + u'(t)B'(P + P_1)x(t) + x'(t)(P + P_1)Bu(t) \right) dt \\ &= x'(0)(P + P_1)x(0) - x'(T)(P + P_1)x(T) + \int_0^T \left[\left(u(t) + R^\dagger B'(P + P_1)x(t) \right)' R(u(t) \right. \\ &\quad \left. + R^\dagger B'(P + P_1)x(t)) + u'(t)(I - RR^\dagger)B'(P + P_1)x(t) + x'(t)(P + P_1)B(I - R^\dagger R)u(t) \right] dt \\ &= x'(0)(P + P_1)x(0) - x'(T)(P + P_1)x(T) + \int_0^T \left(u(t) + R^\dagger B'(P + P_1)x(t) \right)' R(u(t) \\ &\quad + R^\dagger B'(P + P_1)x(t)) dt, \end{aligned}$$

where we have used $(I - RR^\dagger)B'(P + P_1) = 0$ to derive the last equality, obtained from (33). As $u(t) \in \mathcal{U}$, we thus have $\lim_{T \rightarrow \infty} x'(T)(P + P_1)x(T) = 0$. This implies that

$$\begin{aligned} J(x_0; u) &= \lim_{T \rightarrow \infty} \int_0^T (x'(t)Qx(t) + u'(t)Ru(t)) dt \\ &= x'(0)(P + P_1)x(0) + \lim_{T \rightarrow \infty} \int_0^T \left(u(t) + R^\dagger B'(P + P_1)x(t) \right)' R \left(u(t) + R^\dagger B'(P + P_1)x(t) \right) dt. \end{aligned} \tag{E1}$$

Because $R \geq 0$, we obtain $J(x_0; u) \geq x'(0)(P + P_1)x(0)$.

Next, we show that the controller (35) is stabilizing. Substituting (35) into (1) yields

$$\begin{aligned} \dot{x}(t) &= Ax(t) - BR^\dagger B'(P + P_1)x(t) + BG_0Kx(t) \\ &= (A_0 + D_0P_1)x(t) + BG_0Kx(t) \\ &= (A_0 + D_0P_1)x(t) + BT_0^{-1} \begin{bmatrix} 0 \\ K \end{bmatrix} x(t). \end{aligned} \tag{E2}$$

Because Υ_{T_0} is of full row rank, there exists a K_1 such that $\Upsilon_{T_0}K_1 = K$. From (14), we find that

$$T_0(I - R^\dagger R)K_1 = \begin{bmatrix} 0 \\ \Upsilon_{T_0} \end{bmatrix} K_1 = \begin{bmatrix} 0 \\ K \end{bmatrix}. \tag{E3}$$

By substituting the above equation into (E2), we then have

$$\begin{aligned} \dot{x}(t) &= (A_0 + D_0P_1)x(t) + BT_0^{-1}T_0(I - R^\dagger R)K_1x(t) \\ &= (A_0 + D_0P_1)x(t) + B(I - R^\dagger R)T_0^{-1}T_0(I - R^\dagger R)K_1x(t) \\ &= (A_0 + D_0P_1)x(t) + B_0\Upsilon_{T_0}K_1x(t) \\ &= (A_0 + D_0P_1)x(t) + B_0Kx(t). \end{aligned}$$

K was chosen to satisfy that $A_0 + D_0P_1 + B_0K$ is stable, so the above system is stable, and hence the controller (35) is stabilizing.

Finally, we substitute the stabilizing controller (35) into the cost function (E1) to verify that Eq. (35) is an optimal controller as desired. In fact, with this controller, Eq. (E1) becomes

$$\begin{aligned} J(x_0; u) &= x'(0)(P + P_1)x(0) + \lim_{T \rightarrow \infty} \int_0^T \left([-R^\dagger B'(P + P_1) + G_0K]x(t) + R^\dagger B'(P + P_1)x(t) \right)' \\ &\quad \times R \left([-R^\dagger B'(P + P_1) + G_0K]x(t) + R^\dagger B'(P + P_1)x(t) \right) dt \\ &= x'(0)(P + P_1)x(0) + \lim_{T \rightarrow \infty} \int_0^T x'(t)K'G'_0RG_0Kx(t)dt. \end{aligned} \tag{E4}$$

By again using (E3), it follows that $G_0K = T_0^{-1} \begin{bmatrix} 0 \\ K \end{bmatrix} = (I - R^\dagger R)K_1$. This implies that $RG_0K = R(I - R^\dagger R)K_1 = 0$. Thus, with the controller (35), the cost function (E4) reduces to

$$J(x_0; u) = x'(0)(P + P_1)x(0).$$

This shows that Eq. (35) is an optimal controller, and the optimal cost is $J^*(x_0; u) = x_0'(P + P_1)x_0$.

Proof of necessity. Here, we derive the three conditions given in the theorem. First, we discuss the results for the finite-horizon optimization problem. Considering the asymptotic behavior of the solutions to the Riccati differential equations enables us to obtain the first and second conditions. Then, by applying the maximum principle, we find that the stabilizability condition is as stated by the third condition. The detailed proof is given below.

First, based on Theorem 2, there exists a $P^T(t)$ in (8) and a $P_1^T(t)$ in (15) with terminal values of $P^T(T) = 0$ and $P_1^T(T)$ such that Eq. (19) holds, and there also exists a $u_1(t)$ that achieves (20), where $x(t)$ obeys (21) with initial value $x(0) = x_0$. In this case, the optimal cost is given by

$$J_T^*(x_0; u) = x_0' \hat{P}^T(0)x_0, \tag{E5}$$

where $\hat{P}^T(t) = P^T(t) + P_1^T(t)$. Given that $Q \geq 0$ and $R \geq 0$, we have $J_T(x_0; u) \geq 0$. Accordingly, for $T_1 \leq T_2$, we obtain $J_{T_1}(x_0; u) \leq J_{T_2}(x_0; u)$. Together with (E5) and the arbitrariness of x_0 , we thus find that

$$\hat{P}^{T_1}(0) \leq \hat{P}^{T_2}(0). \tag{E6}$$

In addition, consider the cost function

$$J_T^t(x_0; u) = \int_t^T [x'(t)Qx(t) + u'(t)Ru(t)]dt.$$

By applying a similar argument to that for Theorem 2, the optimal cost yielded by minimizing $J_T^t(x_0; u)$ subject to (1) is given by

$$J_T^t(x_0; u) = x'(t)\hat{P}^T(t)x(t).$$

For $t_1 \leq t_2$, we have that

$$J_{T_1}^{t_1}(x_0; u) \geq J_{T_2}^{t_2}(x_0; u),$$

which implies that

$$\hat{P}^T(t_1) \geq \hat{P}^T(t_2). \tag{E7}$$

Combining (E6) and (E7), we see that $\hat{P}^T(t)$ is non-decreasing with respect to T and that $\hat{P}^T(t)$ is non-increasing with respect to t .

Next, we show the uniform boundedness of $\hat{P}^T(t)$. As there exists an optimal and stabilizing controller, there also exists a positive constant c such that

$$J_T^t(x_0; u) \leq \int_0^\infty (x'(t)Qx(t) + u'(t)Ru(t)) dt \leq c\|x_0\|^2.$$

Combining this with (E5), it follows that $\hat{P}^T(t) \leq cI$. As all the system matrices are time-invariant, $\hat{P}^T(t)$ is also time-invariant, i.e., $\hat{P}^T(t) = \hat{P}^{T-t}(0)$. Recalling (E6) and (E7), this shows that the limit $\lim_{T \rightarrow \infty} \hat{P}^T(t) = \hat{P}$ exists. Moreover, by letting $t \rightarrow \infty$ in $\hat{P}^T(t) = P^T(t) + P_1^T(t)$, we see that \hat{P} satisfies

$$0 = A'\hat{P} + \hat{P}A + Q - \hat{P}BR^\dagger B'\hat{P}.$$

This is exactly the same equation for P ; hence, Eq. (27) is solvable. This further implies that Eq. (32) admits a solution P_1 and that $\hat{P} = P + P_1$. Likewise, letting $t \rightarrow \infty$ in (19) yields $C_0 + B_0'P_1 = 0$, which is exactly (33).

Finally, by applying the maximum principle, the optimal solution satisfies

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{E8}$$

$$\dot{p}(t) = -A'p(t) - Qx(t), \tag{E9}$$

$$0 = Ru(t) + B'p(t), \tag{E10}$$

with $\lim_{t \rightarrow \infty} p(t) = 0$ and $x(0) = x_0$. Recalling that the optimal solution is also stabilizing, we obtain

$$\lim_{t \rightarrow \infty} x(t) = 0,$$

and hence that

$$\lim_{t \rightarrow \infty} Px(t) = 0, \tag{E11}$$

$$\lim_{t \rightarrow \infty} P_1x(t) = 0. \tag{E12}$$

Let

$$p(t) = Px(t) + \Theta(t), \tag{E13}$$

where $P(t)$ obeys (27) and $\Theta(t)$ is to be determined. From (E11) and $\lim_{t \rightarrow \infty} p(t) = 0$, we then have $\lim_{t \rightarrow \infty} \Theta(t) = 0$.

By substituting (E13) into (E10), we obtain

$$0 = Ru(t) + B'Px(t) + B'\Theta(t).$$

This implies that

$$u(t) = -R^\dagger (B'Px(t) + B'\Theta(t)) + (I - R^\dagger R)z(t) \tag{E14}$$

and

$$C_0x(t) + B'_0\Theta(t) = 0, \tag{E15}$$

where $z(t)$ is an arbitrary vector of compatible dimension.

Substituting (E14) into (1) reduces the state dynamics to

$$\begin{aligned} \dot{x}(t) &= Ax(t) - BR^\dagger (B'Px(t) + B'\Theta(t)) + B(I - R^\dagger R)z(t) \\ &= A_0x(t) + D_0\Theta(t) + B_0u_1(t), \end{aligned} \tag{E16}$$

where we have used $T_0(I - R^\dagger R)z(t) = \begin{bmatrix} 0 \\ \tau_{T_0} \end{bmatrix} z(t) \triangleq \begin{bmatrix} 0 \\ u_1(t) \end{bmatrix}$ in the derivation of the last equality. Taking the derivative of (E13) yields

$$\dot{p}(t) = P\dot{x}(t) + \dot{\Theta}(t) = P(A_0x(t) + D_0\Theta(t) + B_0u_1(t)) + \dot{\Theta}(t).$$

Comparing this with (E9) and using (32), we have

$$\dot{\Theta}(t) = -A'_0\Theta(t) - C'_0u_1(t). \tag{E17}$$

We now prove that the solution to the FBDEs (E15)–(E17) is $\Theta(t) = P_1x(t)$, where $x(t)$ satisfies (34). By taking the derivative of $P_1x(t)$, we obtain

$$\begin{aligned} \frac{d}{dt}(P_1x(t)) &= P_1(A_0 + D_0P_1)x(t) + P_1B_0u_1(t) \\ &= -A'_0P_1x(t) + P_1B_0u_1(t) \\ &= -A'_0P_1x(t) - C'_0u_1(t), \end{aligned} \tag{E18}$$

where we have used (33) to derive the last equality. By comparing (34), (E18), and (33) with (E15)–(E17), we immediately obtain

$$\Theta(t) = P_1x(t).$$

Accordingly, the state dynamics is given by (34). To ensure the state's stability, the state dynamics must be stabilizable, giving us the third condition. This completes the proof.