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Stochastic stabilization using aperiodically sampled measurements

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Abstract This paper addresses the stabilization problem of sector-bounded nonlinear systems with sampled measurements via discrete-time stochastic feedback control. Unlike the previous studies, the closed-loop system is modeled as an impulsive stochastic differential equation. By developing a quasi-periodic polynomial Lyapunov function and sampling-time-dependent Lyapunov function based methods, two sufficient conditions for almost sure exponential stability are derived in terms of differential matrix inequalities (DMIs) and linear matrix inequalities (LMIs). It is shown that the DMI-based conditions can be formulated as a sum of squares (SOSs). Moreover, the obtained results are adapted to sampled-data stochastic/deterministic systems. The numerical examples illustrate the theoretical results.

Keywords stochastic stabilization, almost sure stability, sampled measurement, discrete-time feedback control

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1 Introduction

In stochastic modeling, noise is typically regarded as an undesirable disturbance to dynamical systems. However, stochastic noise can also provide positive effects if used appropriately. Stochastic stabilization, which has been successfully applied to many research fields such as finance [1], ecology [2], and automatic control [3, 4], is a typical application of the beneficial use of noise. From the perspective of control theory, the framework of stochastic stabilization is to design an artificial multiplicative noise such that the closed-loop system is almost surely asymptotically stable. Since Has'minskii [5] employed two white noise sources to stabilize an unstable system, a significant number of research results on this topic have been reported. According to Arnold et al. [6], linear time-invariant systems can be stabilized by a zero-mean stationary parameter noise if and only if the trace of its system matrix is less than zero. Mao et al. [2, 7–10] developed a general theory on stochastic stabilization and the destabilization of (functional) differential equations. Compared with [7, 10], Huang [11] proposed improved results on the stochastic stabilization and destabilization of nonlinear differential equations. Nishimura [12] utilized Gaussian white noise to control dynamical systems to meet the locally almost sure asymptotic stability. Based on the stochastic control Lyapunov function, a stabilizing controller together with the Wiener process for deterministic nonlinear systems was designed in [4]. The authors of [13] showed that unstable differential equations can be stabilized by a stochastic delay feedback controller if the time delay is sufficiently small.

All the aforementioned state-feedback controllers are designed in a continuous-time manner. Recently, for the first time, Mao [14] proposed a discrete-time stochastic feedback controller $Kx(k\Delta)\dot{w}(t)$,

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$t \in [k\Delta, (k+1)\Delta)$ to stabilize an unstable deterministic system $\dot{x}(t) = f(x(t))$ when Δ is sufficiently small, where $w(t)$ is a Wiener process. As the discrete-time stochastic feedback strategy only requires the measured information at discrete time instants, it can effectively reduce the bandwidth usage and communication cost. From a practical perspective, a larger upper bound of the sampling period would be desirable. However, the results of [14] neither provided an estimation of the upper bound of the sampling period, nor solved the controller synthesis problem. The idea of the approach [14] is based on the assumption that a continuous-time controller ($Kx(t)\dot{w}(t)$) exists such that the resulting continuous-time system ($\mathrm{d}x(t) = f(x(t))\mathrm{d}t + Kx(t)\mathrm{d}w(t)$) is stochastically stable, and subsequently analyzing the relationship between the closed-loop system ($\mathrm{d}x(t) = f(x(t))\mathrm{d}t + Kx(k\Delta)\mathrm{d}w(t)$) and the continuous-time system. Some important results [13, 15–17] are also based on this idea. In many practical systems (such as second-order systems, see Example 2), the continuous feedback controller cannot stabilize the systems but the discrete-time feedback controller can. Obviously, these approaches [13–16] are not applicable for analyzing these systems. Moreover, it is noteworthy that the input-delay approach [18] is an effective method for analyzing deterministic sampled-data systems. Note that in [19], their proof of almost sure stability of the scalar delay stochastic system $\mathrm{d}x(t) = ax(t-\delta)\mathrm{d}w(t)$ was already difficult and complicated. Hence, the input-delay approach cannot be extended to solve the stochastic stabilization problem.

Motivated by the aforementioned observations, this paper focuses on the stochastic stabilization of sector-bounded nonlinear systems under aperiodically sampled measurements. First, we represent the considered systems as an impulsive stochastic system. Subsequently, inspired by the studies of [20, 21], we propose two new approaches, namely the quasi-periodic polynomial Lyapunov function method and the sampling-time-dependent discretized Lyapunov function method, for an almost sure exponential stability analysis. The stabilization conditions are expressed in terms of sum of squares (SOSs) and linear matrix inequalities (LMIs), separately, which reveal quantitatively the effect of sampling periods on the stability performance. Moreover, our results include many existing results on the deterministic sampled-data control systems.

The paper is organized as follows. In Section 2, we model the discrete-time feedback controlled systems as an impulsive stochastic system. Two novel Lyapunov-function-based methods are proposed for the stability analysis and the synthesis of the impulsive system in Section 3. Numerical examples are provided in Section 4, and the conclusion is drawn in Section 5.

Notation. Let $\mathbb{R}$ and $\mathbb{R}_+$ denote a set of real numbers and a set of nonnegative real numbers, respectively. $\mathbb{N}$ represents a set of positive integers, and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\mathbb{R}^{n\times m}$ is the set of all $n \times m$-dimensional real matrices and the notation $A > 0$, for $A \in \mathbb{R}^{n\times n}$ means that the matrix A is positive definite. The symmetric elements of a symmetric matrix are denoted by $*$. $\|\cdot\|$ denotes the Euclidean vector norm. $\mathcal{C}_s$ denotes a set of sector-bounded nonlinear functions, i.e., $\mathcal{C}_s = \{\phi : \mathbb{R} \to \mathbb{R} \,|\, (\phi(\xi) - \kappa^-\xi)(\phi(\xi) - \kappa^+\xi) \leqslant 0,\ \phi(0) = 0,\ \forall \xi \in \mathbb{R},\ \kappa^-, \kappa^+ \in \mathbb{R},\ \kappa^- \leqslant \kappa^+\}$. For $x \in \mathbb{R}^n$, the ring of polynomials in x is denoted as $\mathbb{R}[x]$, and $\mathbb{R}^{n\times m}[x]$ denotes the ring of polynomials matrices of dimensions $n \times m$. $\sum[x] = \{p(x) \in \mathbb{R}[x] \,|\, p(x) = \sum_{i=1}^{d} g_i^2(x) \in \mathbb{R}[x]\}$ denotes the set of SOS polynomials on variable x. The set of SOS matrices of dimension n is denoted by $\sum^{n\times n}[x]$.

2 Problem formulation

To generalize our results, we consider the following Itô-type stochastic system with control input that is not only in the drift part but also in the diffusion part:

$$\begin{cases} \mathrm{d}x(t) = [A_0x(t) + A_1f(x(t)) + B_0u(t)]\,\mathrm{d}t + [Dx(t) + B_1u(t)]\,\mathrm{d}w(t), \quad t > t_0, \\ x(t_0) = x_0, \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state variable, $A_0, A_1, D \in \mathbb{R}^{n\times n}$ are constant matrices; $B_0, B_1 \in \mathbb{R}^{n\times n_b}$ are control input matrices; $w(t)$ is a one-dimensional Wiener process, which is defined on the filtered complete probability space: $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P})$ [2]. $f : \mathbb{R}^n \to \mathbb{R}^n : x_i \to f_i(x_i)$ is nonlinear vector function that satisfies Assumption 1.

Assumption 1. The scalar functions f_i, $i = 1, 2, \ldots, n$ are continuous and belong to the class $\mathcal{C}_s$, i.e., κ_i^- and κ_i^+ exist such that for any $x_i \in \mathbb{R}$,

$$\left(f_i(x_i) - \kappa_i^- x_i\right)\left(f_i(x_i) - \kappa_i^+ x_i\right) \leqslant 0.$$

We set $L_1 = \mathrm{diag}(\kappa_1^+\kappa_1^-, \kappa_2^+\kappa_2^-, \ldots, \kappa_n^+\kappa_n^-)$ and $L_2 = -\frac{1}{2}\mathrm{diag}(\kappa_1^+ + \kappa_1^-, \kappa_2^+ + \kappa_2^-, \ldots, \kappa_n^+ + \kappa_n^-)$.

Moreover, we assume the measured output of system (1) is given by

$$y_k = Cx(t_k), \quad k \in \mathbb{N}_0, \tag{2}$$

where $C \in \mathbb{R}^{n_c \times n}$, $\{t_k\}_{k\in\mathbb{N}_0}$ is the sampling time sequence belonging to $\mathcal{S}(\sigma_0, \sigma_1) \triangleq \{\{t_k\}_{k\in\mathbb{N}_0} | \sigma_0 \leqslant t_{k+1} - t_k \leqslant \sigma_1,\ t_0 = 0,\ \sigma_0, \sigma_1 \in \mathbb{R}_+,\ \sigma_0 \leqslant \sigma_1\}$. Subsequently the control input signal is constructed as

$$u(t) = Ky_k, \quad t \in [t_k, t_{k+1}),\ k \in \mathbb{N}_0, \tag{3}$$

where $K \in \mathbb{R}^{n_b \times n_c}$ is control gain matrix to be designed.

Define $z(t) = [x^{\mathrm{T}}(t)\ u^{\mathrm{T}}(t)]^{\mathrm{T}}$, $\mathcal{I}_0 = [I_{n\times n}\ 0_{n\times n_b}]$, $\mathcal{I}_1 = [0_{n_b\times n}\ I_{n_b\times n_b}]$, $\bar{A}_0 = \left[\begin{smallmatrix} A_0 & B_0 \\ 0_{n_b\times n} & 0_{n_b\times n_b}\end{smallmatrix}\right]$, $\bar{A}_1 = \mathcal{I}_0^{\mathrm{T}} A_1$, $\bar{D} = \left[\begin{smallmatrix} D & B_1 \\ 0_{n_b\times n} & 0_{n_b\times n_b}\end{smallmatrix}\right]$, and $\bar{J} = \left[\begin{smallmatrix} I_{n\times n} & 0_{n\times n_b} \\ KC & 0_{n_b\times n_b}\end{smallmatrix}\right] \triangleq J_0 + \mathcal{I}_1^{\mathrm{T}} KC\mathcal{I}_0$ with $J_0 = \left[\begin{smallmatrix} I_{n\times n} & 0_{n\times n_b} \\ 0_{n_b\times n} & 0_{n_b\times n_b}\end{smallmatrix}\right]$. Then, system (1) with measured output (2) and control input (3) can be modeled as the following impulsive stochastic system:

$$\begin{cases} \mathrm{d}z(t) = \left[\bar{A}_0 z(t) + \bar{A}_1 f(\mathcal{I}_0 z(t))\right]\mathrm{d}t + \bar{D}z(t)\mathrm{d}w(t), \quad t \neq t_k, \\ z(t) = \bar{J}z(t^-), \quad t = t_k,\ k \in \mathbb{N}, \\ z(t_0) = [x_0, KCx_0]^{\mathrm{T}} \triangleq z_0, \end{cases} \tag{4}$$

where $z(t_k) \triangleq z(t_k^+) = \lim_{s\to 0^+} z(t_k + s)$, and $z(t_k^-) = \lim_{s\to 0^-} z(t_k + s)$. Thus far, the discrete-time stochastic feedback control problem (1)–(3) is reduced to find a gain matrix K such that the impulsive stochastic system (4) is almost surely exponentially stable.

Definition 1 ([2])**.** The zero solution of the closed-loop system (1)–(3) is uniformly almost surely exponentially stable over $\mathcal{S}(\sigma_0, \sigma_1)$, if a positive scalar γ exists, such that for any sampling time sequence $\{t_k\}_{k\in\mathbb{N}_0} \in \mathcal{S}(\sigma_0, \sigma_1)$ and any $x_0 \in \mathbb{R}^n$, it holds that

$$\limsup_{t\to+\infty} \frac{1}{t} \ln \|x(t)\| \leqslant -\gamma, \text{ a.s..}$$

3 Main results

In this section, we will develop two novel Lyapunov-function-based approaches for discussing the almost sure stability and stabilization of system (4).

3.1 Stability analysis

3.1.1 *Quasi-periodic polynomial Lyapunov function approach*

Theorem 1. Given a class $\mathcal{S}(\sigma_0, \sigma_1)$ of sampling time sequences and a positive scalar γ, consider system (4). If there exist a positive definite matrix $P(\sigma) \in \mathbb{R}^{(n+n_b)\times(n+n_b)}[\sigma]$, a positive definite diagonal matrix $\Lambda(\sigma) \in \mathbb{R}^{n\times n}[\sigma]$, and a scalar c such that

$$\Xi(\sigma) = \begin{bmatrix} \Psi(\sigma) & P(\sigma)\bar{A}_1 + \mathcal{I}_0^{\mathrm{T}}\Lambda(\sigma)L_2 \\ * & -\Lambda(\sigma) \end{bmatrix} \leqslant 0 \text{ holds for } \sigma \in [0, \sigma_1], \text{ and} \tag{5}$$

$$\bar{J}^{\mathrm{T}} P(0)\bar{J} - P(\sigma) \leqslant 0 \text{ holds for } \sigma \in [\sigma_0, \sigma_1], \tag{6}$$

where $\Psi(\sigma) = \dot{P}(\sigma) + P(\sigma)(\bar{A}_0 - c\bar{D}) + (\bar{A}_0 - c\bar{D})^{\mathrm{T}} P(\sigma) + \bar{D}^{\mathrm{T}} P(\sigma)\bar{D} + (\gamma + 0.5c^2)P(\sigma) + \mathcal{I}_0^{\mathrm{T}}\Lambda(\sigma)L_1\mathcal{I}_0$, then, system (4) is uniformly almost surely exponentially stable over $\mathcal{S}(\sigma_0, \sigma_1)$.

Proof. For any $x_0 \in \mathbb{R}^n$, we define the stopping time $\tau(\omega) = \inf\{t > t_0 \,|\, z(t,\omega) = 0\}$, and denote $\Omega_1 = \{\omega \in \Omega : \tau(\omega) < +\infty\}$, $\Omega_2 = \Omega \setminus \Omega_1$. Obviously, for any $\omega \in \Omega_1$, $z(t,\omega) \equiv 0$ as $t \geqslant \tau(\omega)$, whence $z(t,\omega)$ is almost surely exponentially stable on Ω_1. Therefore, we only need to show that for all $\omega \in \Omega_2$, $z(t,\omega)$ is almost surely exponentially stable. Hence, we introduce a quasi-periodic polynomial Lyapunov function candidate for system (4): $V_1(t) \triangleq V_1(t,z(t)) = z^{\mathrm{T}}(t)\tilde{P}(t)z(t)$ with $\tilde{P}(t) = P(t-t_k)$ for $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}_0$. Applying Itô's formula to $V_1(t)$, we obtain

$$\mathrm{d}V_1(t) = \mathcal{L}V_1(t,z(t))\mathrm{d}t + \mathcal{H}V_1(t,z(t))\mathrm{d}w(t), \tag{7}$$

where $\mathcal{L}V_1(t,z) = z^{\mathrm{T}}[\dot{\tilde{P}}(t)+\tilde{P}(t)\bar{A}_0+\bar{A}_0^{\mathrm{T}}\tilde{P}(t)+\bar{D}^{\mathrm{T}}\tilde{P}(t)\bar{D}]z+2z^{\mathrm{T}}\tilde{P}(t)\bar{A}_1 f(\mathcal{I}_0 z)$ and $\mathcal{H}V_1(t,z) = z^{\mathrm{T}}[\tilde{P}(t)\bar{D}+\bar{D}^{\mathrm{T}}\tilde{P}(t)]z$. It is noteworthy that for $\omega \in \Omega_2$, $z(t,\omega) \neq 0$ for any $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}_0$. Therefore, applying Itô's formula again with (7), we obtain

$$\begin{aligned}\mathrm{d}\ln V_1(t) &= V_1^{-1}(t)\left[\mathcal{L}V_1(t,z(t)) - \frac{1}{2}V_1^{-1}(t)\left(\mathcal{H}V_1(t,z(t))\right)^2\right]\mathrm{d}t + V_1^{-1}(t)\mathcal{H}V_1(t,z(t))\mathrm{d}w(t)\\ &= V_1^{-1}(t)\big\{\mathcal{L}V_1(t,z(t)) + 0.5c^2V_1(t) - c\mathcal{H}V_1(t,z(t))\\ &\quad - 0.5V_1^{-1}(t)[\mathcal{H}V_1(t,z(t)) - cV_1(t)]^2\big\}\mathrm{d}t + V_1^{-1}(t)\mathcal{H}V_1(t,z(t))\mathrm{d}w(t).\end{aligned} \tag{8}$$

We set $\Lambda(t-t_k) = \mathrm{diag}(\alpha_1(t-t_k),\ldots,\alpha_n(t-t_k))$. Recalling Assumption 1, we obtain $\forall t \in [t_k, t_{k+1})$, $k \in \mathbb{N}_0$,

$$\begin{aligned}0 &\leqslant \sum_{i=1}^n \alpha_i(t-t_k)\left(f_i(x_i(t)) - \kappa_i^- x_i(t)\right)\left(\kappa_i^+ x_i(t) - f_i(x_i(t))\right)\\ &= x^{\mathrm{T}}(t)\Lambda(t-t_k)L_1x(t) - f^{\mathrm{T}}(x(t))\Lambda(t-t_k)f(x(t)) + 2x^{\mathrm{T}}(t)\Lambda(t-t_k)L_2f(x(t))\\ &= z^{\mathrm{T}}(t)\mathcal{I}_0^{\mathrm{T}}\Lambda(t-t_k)L_1\mathcal{I}_0z(t) - f^{\mathrm{T}}(\mathcal{I}_0z(t))\Lambda(t-t_k)f(\mathcal{I}_0z(t)) + 2z^{\mathrm{T}}(t)\mathcal{I}_0^{\mathrm{T}}\Lambda(t-t_k)L_2f(\mathcal{I}_0z(t)).\end{aligned} \tag{9}$$

Applying (9) to (8) and using condition (5), we obtain

$$\begin{aligned}\mathrm{d}\ln V_1(t) &\leqslant \left[V_1^{-1}(t)\eta^{\mathrm{T}}(t)\Xi(t-t_k)\eta(t) - \gamma\right]\mathrm{d}t + V_1^{-1}(t)\mathcal{H}V_1(t,z(t))\mathrm{d}w(t)\\ &\leqslant -\gamma\mathrm{d}t + V_1^{-1}(t)\mathcal{H}V_1(t,z(t))\mathrm{d}w(t),\end{aligned}$$

where $\eta(t) = \mathrm{col}(z(t), f(\mathcal{I}_0z(t)))$. Integrating both sides of the inequality above from t_k to t, we obtain

$$\ln V_1(t) \leqslant -\gamma(t-t_k) + \ln V_1(t_k) + \int_{t_k}^t V_1^{-1}(s)\mathcal{H}V_1(s,z(s))\mathrm{d}w(s). \tag{10}$$

Now, let us prove that $V_1(t)$ is not increasing at the sampling instant t_k, $k \in \mathbb{N}_0$. By inequality (6), we have

$$\begin{aligned}V_1(t_k) &= z^{\mathrm{T}}(t_k)\tilde{P}(t_k)z(t_k) = z^{\mathrm{T}}(t_k^-)\bar{J}^{\mathrm{T}}P(0)\bar{J}z(t_k^-)\\ &\leqslant z^{\mathrm{T}}(t_k^-)P(t_k^- - t_{k-1})z(t_k^-) = V_1(t_k^-),\end{aligned}$$

where $\ln V_1(t_k) \leqslant \ln V_1(t_k^-)$. Combining this with (10), we obtain $\ln V_1(t) \leqslant -\gamma(t-t_0) + \ln V_1(t_0) + M(t)$, where $M(t) = \int_{t_0}^t V_1^{-1}(s)\mathcal{H}V_1(s,z(s))\mathrm{d}w(s)$. As shown, $M(t)$ is a continuous local martingale with respect to $\{\mathcal{F}_t\}_{t\geqslant t_0}$ and vanishes at t_0. Moreover, its quadratic variation is given by

$$\langle M(t), M(t)\rangle = \int_{t_0}^t \left|\frac{z^{\mathrm{T}}(s)[\tilde{P}(s)\bar{D} + \bar{D}^{\mathrm{T}}\tilde{P}(s)]z(s)}{z^{\mathrm{T}}(s)\tilde{P}(s)z(s)}\right|^2 \mathrm{d}s \leqslant \beta^2(t-t_0),$$

where $\beta = \overline{\lambda}_{\tilde{P}\bar{D}}/\underline{\lambda}_{\tilde{P}}$, $\overline{\lambda}_{\tilde{P}\bar{D}} = \sup_{\sigma\in[0,\sigma_1]}\{|\lambda_{\max}(P(\sigma)\bar{D} + \bar{D}^{\mathrm{T}}P(\sigma))|, |\lambda_{\min}(P(\sigma)\bar{D} + \bar{D}^{\mathrm{T}}P(\sigma))|\}$, and $\underline{\lambda}_{\tilde{P}} = \inf_{\sigma\in[0,\sigma_1]}\{\lambda_{\min}(P(\sigma))\}$. Therefore, $\limsup_{t\to\infty}\frac{\langle M(t),M(t)\rangle}{t} = \beta^2 < \infty$. According to the strong law of large numbers, we can obtain

$$\limsup_{t\to\infty}\frac{M(t)}{t} = 0, \text{ a.s..}$$

Thus, we obtain an estimate

$$\limsup_{t\to+\infty}\frac{\|z(t)\|}{t}\leqslant-\frac{\gamma}{2},\ \text{a.s.}.$$

Therefore, system (4) is uniformly almost surely exponentially stable over $\mathcal{S}(\sigma_0,\sigma_1)$.

It is noteworthy that the sufficient conditions presented in Theorem 1 are infinite-dimensional feasibility problems that may be difficult to solve. Fortunately, the SOS approach provides an effective method for solving such problems. Therefore, those infinite-dimensional matrix inequalities of Theorem 1 must be converted to the form of SOS-based conditions. Hence, we utilize the positivstellensatz to formulate Theorem 1 into SOS programming.

Proposition 1. Theorem 1 can be verified by solving the SOS programming: for given $\varepsilon>0$ and $c\in\mathbb{R}$, find polynomial matrices $P(\sigma)\in\mathbb{R}^{(n+n_b)\times(n+n_b)}[\sigma]$, $\Lambda(\sigma)\in\mathbb{R}^{n\times n}[\sigma]$, $Q_1(\sigma)\in\sum^{(2n+n_b)\times(2n+n_b)}[\sigma]$, and $Q_2(\sigma)\in\sum^{(n+n_b)\times(n+n_b)}[\sigma]$ such that

$$\begin{aligned}
&P(\sigma)-\varepsilon I\in\textstyle\sum^{(n+n_b)\times(n+n_b)}[\sigma],\\
&\Lambda(\sigma)-\varepsilon I\in\textstyle\sum^{n\times n}[\sigma],\\
&-\Xi(\sigma)-g_0(\sigma)Q_1(\sigma)\in\textstyle\sum^{(2n+n_b)\times(2n+n_b)}[\sigma],\\
&-\bar{J}^{\mathrm{T}}P(0)\bar{J}+P(\sigma)-g_1(\sigma)Q_2(\sigma)\in\textstyle\sum^{(n+n_b)\times(n+n_b)}[\sigma],
\end{aligned}$$

where $g_0(\sigma)=\sigma(\sigma_1-\sigma)$ and $g_1(\sigma)=(\sigma-\sigma_0)(\sigma_1-\sigma)$.

Remark 1. The SOS programs can be solved by using the package SOSTOOLS [22] or YALMIP [23] together with a semidefinite programming solver, such as SeDuMi [24]. Moreover, it is well known that LMI-based conditions are not only easy to be solved but also provide a simple method to solve the controller synthesis problem. In the following, we will develop a constructive analysis method to establish an LMI-based stability criterion for the system (4).

3.1.2 *Sampling-time-dependent discretized Lyapunov function approach*

Before introducing the new Lyapunov function, we first introduce some auxiliary functions. We divide the interval $[t_k,t_{k+1})$, $k\in\mathbb{N}_0$ into N subintervals $\Delta_{ki}\triangleq[t_k+ih_k,t_k+(i+1)h_k)$, $i\in\mathrm{S}_N\triangleq\{0,1,\ldots,N-1\}$ in which $h_k=\frac{t_{k+1}-t_k}{N}$. Subsequently, we define some piecewise linear functions as follows:

$$\begin{aligned}
&\rho_{0i}(t)=\frac{t_k+(i+1)h_k-t}{h_k},\quad t\in\Delta_{ki},\quad i\in\mathrm{S}_N,\quad k\in\mathbb{N}_0,\\
&\tilde{\varrho}(t)=\frac{1}{t_{k+1}-t_k},\quad \varrho(t)=(t-t_k)\tilde{\rho}(t),\quad t\in[t_k,t_{k+1}).
\end{aligned}$$

We set $\rho_{1i}(t)=1-\rho_{0i}(t)$. It is easily shown that for any $t\in\mathbb{R}_+$, $1/h_k=N\tilde{\varrho}(t)$, $\rho_{0i}(t)\in[0,1]$, $\rho_{1i}(t)\in[0,1]$, and

$$\begin{aligned}
&\rho_{0i}(t_k+ih_k)=\rho_{1i}((t_k+(i+1)h_k)^-)=\varrho(t_k^-)=1,\\
&\rho_{0i}((t_k+(i+1)h_k)^-)=\rho_{1i}(t_k+ih_k)=\varrho(t_k)=0.
\end{aligned}$$

For any impulse sequence $\{t_k\}_{k\in\mathbb{N}_0}\in\mathcal{S}(\sigma_0,\sigma_1)$, there exists $\theta_0(t):\mathbb{R}_+\to[0,1]$ such that

$$\tilde{\varrho}(t)=\frac{\theta_0(t)}{\sigma_0}+\frac{\theta_1(t)}{\sigma_1},$$

where $\theta_1(t)=1-\theta_0(t)$ and

$$\theta_0(t)=\begin{cases}\dfrac{\tilde{\varrho}(t)-1/\sigma_1}{1/\sigma_0-1/\sigma_1}, & \text{if } \sigma_0\neq\sigma_1,\\ 1, & \text{if } \sigma_0=\sigma_1.\end{cases}$$

Now, we introduce the piecewise Lyapunov function candidate for system (4) described by

$$V_2(t) \triangleq V_2(t, z(t)) = \psi(t) z^{\mathrm{T}}(t) P(t) z(t), \tag{11}$$

where $\psi(t) = \mu^{\varrho(t)}$ with $\mu > 0$, $P(t) = \sum_{i=0}^{N-1} P_i(t)\chi_i(t)$, $P_i(t) = \sum_{j=0}^{1} \rho_{ji}(t) P_{i+j}$, $t \in \Delta_{ki}$, $i \in \mathrm{S}_N$ with $P_{\bar{i}} > 0$, $\bar{i} = 0, 1, 2, \ldots, N$, and $\chi_i(t)$ is the indicator function of the interval Δ_{ki}, for $i \in \mathrm{S}_N$. It can be easily verified that $V_2(t)$ is continuous in every sampling time interval (t_k, t_{k+1}), $k \in \mathbb{N}_0$.

Theorem 2. Given a class $\mathcal{S}(\sigma_0, \sigma_1)$ of sampling time sequences and the number of discretized subintervals N, consider system (4). If for a prescribed positive scalar μ, there exist positive definite matrices $P_{\bar{i}} \in \mathbb{R}^{(n+n_b)\times(n+n_b)}$, positive definite diagonal matrices $\Lambda_{\ell,\bar{i}} \in \mathbb{R}^{n\times n}$, $\bar{i} = 0, 1, \ldots, N$, $\ell = 0, 1$, and a scalar c such that the following LMIs hold for any $i \in \mathrm{S}_N$ and $j, \ell = 0, 1$,

$$\tilde{\Xi}_{ij\ell} = \begin{bmatrix} \tilde{\Psi}_{ij\ell} & P_{i+j}\bar{A}_1 + \mathcal{I}_0^{\mathrm{T}} \Lambda_{\ell,i+j} L_2 \\ * & -\Lambda_{\ell,i+j} \end{bmatrix} < 0, \tag{12}$$

$$\begin{bmatrix} -\mu P_N & \bar{J}^{\mathrm{T}} P_0 \\ * & -P_0 \end{bmatrix} \leqslant 0, \tag{13}$$

where $\tilde{\Psi}_{ij\ell} = (0.5c^2 + \frac{\ln(\mu)}{\sigma_\ell}) P_{i+j} + \frac{N}{\sigma_\ell}(P_{i+1} - P_i) + P_{i+j}(\bar{A}_0 - c\bar{D}) + (\bar{A}_0 - c\bar{D})^{\mathrm{T}} P_{i+j} + \bar{D}^{\mathrm{T}} P_{i+j} \bar{D} + \mathcal{I}_0^{\mathrm{T}} \Lambda_{\ell,i+j} L_1 \mathcal{I}_0$, then system (4) is uniformly almost surely exponentially stable over $\mathcal{S}(\sigma_0, \sigma_1)$.

Proof. Condition (12) implies that there exists a sufficiently small positive scalar γ such that $\tilde{\Xi}_{ij\ell}(\gamma) < 0$, where $\tilde{\Xi}_{ij\ell}(\gamma)$ is derived from $\tilde{\Xi}_{ij\ell}$, in which $0.5c^2 + \frac{\ln(\mu)}{\sigma_\ell}$ is replaced by $\gamma + 0.5c^2 + \frac{\ln(\mu)}{\sigma_\ell}$. It follows that

$$\tilde{\Xi}_i(t) \triangleq \sum_{j,\ell=0}^{1} \rho_{ji}(t)\theta_\ell(t)\tilde{\Xi}_{ij\ell}(\gamma) < 0, \quad i \in \mathrm{S}_N. \tag{14}$$

For notational brevity, we define $\Lambda_i(t) \triangleq \mathrm{diag}(\alpha_{i1}(t), \alpha_{i2}(t), \ldots, \alpha_{in}(t)) = \sum_{j,\ell=0}^{1} \rho_{ji}(t)\theta_\ell(t)\Lambda_{\ell,i+j}$.

For $t \in [t_k + ih_k, t_k + (i+1)h_k)$ with any given $k \in \mathbb{N}_0$, and $i \in \mathrm{S}_N$, by applying the Itô's formula to $V_2(t)$, we obtain

$$\mathrm{d}V_2(t) = \mathcal{L}V_2(t, z(t))\mathrm{d}t + \mathcal{H}V_2(t, z(t))\mathrm{d}w(t), \tag{15}$$

where $\mathcal{L}V_2(t,z) = \psi(t) z^{\mathrm{T}}[\ln(\mu)\tilde{\varrho}(t)P_i(t) + N\tilde{\varrho}(t)(P_{i+1} - P_i) + P_i(t)\bar{A}_0 + \bar{A}_0^{\mathrm{T}} P_i(t) + \bar{D}^{\mathrm{T}} P_i(t)\bar{D}]z + 2\psi(t) z^{\mathrm{T}} P_i(t) \bar{A}_1 f(\mathcal{I}_0 z)$ and $\mathcal{H}V_2(t,z) = \psi(t) z^{\mathrm{T}}[P_i(t)\bar{D} + \bar{D}^{\mathrm{T}} P_i(t)]z$.

Recalling Assumption 1, we obtain $\forall t \geqslant t_0$,

$$\begin{aligned} 0 &\leqslant \sum_{q=1}^{n} \alpha_{iq}(t) \left(f_q(x_q(t)) - \kappa_q^- x_q(t)\right)\left(\kappa_q^+ x_q(t) - f_q(x_q(t))\right) \\ &= z^{\mathrm{T}}(t)\mathcal{I}_0^{\mathrm{T}} \Lambda_i(t) L_1 \mathcal{I}_0 z(t) - f^{\mathrm{T}}(\mathcal{I}_0 z(t))\Lambda_i(t) f(\mathcal{I}_0 z(t)) + 2z^{\mathrm{T}}(t)\mathcal{I}_0^{\mathrm{T}} \Lambda_i(t) L_2 f(\mathcal{I}_0 z(t)). \end{aligned} \tag{16}$$

For any $\omega \in \Omega_2$, and any $t \in [t_k + ih_k, t_k + (i+1)h_k)$, applying Itô's formula to $\ln V_2(t)$ along with (15) and using (16), we obtain

$$\begin{aligned} \ln V_2(t) &\leqslant \ln V_2(t_k + ih_k) - \gamma(t - t_k - ih_k) + \int_{t_k+ih_k}^{t} \frac{\psi(s)}{V_2(s)} z^{\mathrm{T}}(s)\tilde{\Xi}_i(s) z(s)\mathrm{d}s + M_2(t, t_k + ih_k) \\ &\leqslant \ln V_2(t_k + ih_k) - \gamma(t - t_k - ih_k) + M_2(t, t_k + ih_k), \end{aligned} \tag{17}$$

where $M_2(t, v) = \int_v^t \frac{2\psi(s)}{V_2(s)} z^{\mathrm{T}}(s) P(s)\bar{D} z(s)\mathrm{d}w(s)$.

Because $V_2(t)$ is continuous on the impulse time interval $[t_k, t_{k+1})$, for any $t \in [t_k, t_{k+1})$, inequality (17) can be deduced into

$$\ln V_2(t) \leqslant \ln V_2(t_k) - \gamma(t - t_k) + M_2(t, t_k). \tag{18}$$

Next, we will estimate $\ln V_2(t)$ at the impulsive instant t_k, $k \in \mathbb{N}_0$. From the definition of $P_i(t)$ and $\psi(t)$, we obtain

$$P_0(t_k) = P_0, \quad P_{N-1}(t_k^-) = P_N, \quad \psi(t_k) = 1, \quad \psi(t_k^-) = \mu.$$

Subsequently, using (13), for any $k \in \mathbb{N}_0$, we obtain

$$V_2(t_k) = z^{\mathrm{T}}(t_k)P_0z(t_k) = z^{\mathrm{T}}(t_k^-)\bar{J}^{\mathrm{T}}P_0\bar{J}z(t_k^-)$$
$$\leqslant \mu z^{\mathrm{T}}(t_k^-)P_Nz(t_k^-) = V_2(t_k^-),$$

whence $\ln V_2(t_k) \leqslant \ln V_2(t_k^-)$. By jointly applying this inequality and (18), we obtain $\ln V_2(t) \leqslant \ln V_2(t_0) - \gamma(t - t_0) + M_2(t, t_0)$. Subsequently, using the same technique used in the proof of Theorem 1, we obtain

$$\limsup_{t\to\infty} \frac{\ln \|z(t)\|}{t} \leqslant -\frac{\gamma}{2}, \text{ a.s..}$$

Therefore, by Definition 1, we conclude that the zero solution of system (1) is uniformly almost surely exponentially stable over $\mathcal{S}(\sigma_0, \sigma_1)$.

Remark 2. In [25], Hu and Mao showed that a linear stochastic system can be almost surely stabilized by a continuous state-feedback controller ($u(t) = Kx(t)$), if there exist a matrix $P > 0$ and a scalar $\tilde{c} \geqslant 0$ such that

$$-\tilde{c}P + P\tilde{A}_0 + \tilde{A}_0^{\mathrm{T}}P + \tilde{D}^{\mathrm{T}}P\tilde{D} < 0, \tag{19}$$

and either

$$\tilde{D}^{\mathrm{T}}P + P\tilde{D} - \sqrt{2\tilde{c}}P > 0, \tag{20}$$

or

$$\tilde{D}^{\mathrm{T}}P + P\tilde{D} + \sqrt{2\tilde{c}}P < 0, \tag{21}$$

where $\tilde{A}_0 = A_0 + B_0K$ and $\tilde{D} = D + B_1K$. However, based on Theorem 2, we have a novel almost sure stabilization criterion

$$0.5c^2P + P(\tilde{A}_0 - c\tilde{D}) + (\tilde{A}_0 - c\tilde{D})^{\mathrm{T}}P + \tilde{D}^{\mathrm{T}}P\tilde{D} < 0. \tag{22}$$

In fact, inequalities (19)–(21) imply (22). By setting $c^2 = 2\tilde{c}$, and using (20) and (21), we obtain

$$-c(\tilde{D}^{\mathrm{T}}P + P\tilde{D}) + c^2P < 0.$$

Applying the inequality above to (19), we obtain (22). It is noteworthy that the stability conditions (19)–(21) are conservative and difficult to be used in designing stabilizing controllers by stochastic noise. This is because condition (20) or (21) requires that $\tilde{D}$ or $-\tilde{D}$ should be a Hurwitz matrix. This also indicates that the diffusion term of the considered system is controllable. However, our result does not imposing this restriction.

Remark 3. It is noteworthy that both the continuous- and discrete-time dynamics of the impulsive system (4) are unstable. The results achieved by the time-invariant quadratic Lyapunov function approach cannot be applied to this class of systems. Therefore, the time-dependent Lyapunov functions used herein are critical in deriving Theorems 1 and 2. It is noteworthy that the Lyapunov function has two important features. First, the infinitesimal generator $\mathcal{L}V_1(t, z)$ ($\mathcal{L}V_2(t, z)$) of the time-dependent Lyapunov functions along the trajectories of the unstable continuous-time dynamic of system (4) leads to the term $\dot{\tilde{P}}(t)$ ($N\tilde{\rho}(t)(P_{i+1} - P_i)$), which compensates the infinitesimal generator of the Lyapunov function to be negative. Next, because the time-varying matrix $\tilde{P}(t)$ ($P(t)$ in (11)) can select different values at the left and right limits of t_k, i.e., $P(0) \neq P(t_k^- - t_{k-1})$ ($P_0 \neq P_N$), the time-dependent Lyapunov functions are discontinuous at the sampling times t_k, $k \in \mathbb{N}_0$, which are consistent with the dynamic behaviors exhibited by the impulsive system (4). Furthermore, because impulsive systems involve impulses at variable times, they are a class of quasi-periodic systems. Thus, the time-dependent Lyapunov functions are suitable to characterize the dynamic behavior of the impulsive systems. It should be emphasized that the time-dependent Lyapunov function proposed in [21, 26] is a special case of (11) and the looped Lyapunov functional introduced in [27] cannot be extended to study the almost sure stability of system (4) because this Lyapunov functional cannot guarantee the positive definiteness in the impulse interval.

Remark 4. It is known from Lemma 1 of [28] that a $2d$-degree SOS polynomial $Q(\sigma) \in \sum^{n\times n}[\sigma]$ has $\mathcal{N}(n,d) \triangleq 0.5n^2(1+d)^2 + 0.5n(1+d) - 0.5n(n+1)(1+2d)$ scalar variables. Let $P(\sigma) = \sum_{i=0}^{2d} P_i\sigma^i$ with symmetrical matrix $P_i \in \mathbb{R}^{(n+n_b)\times(n+n_b)}$, and $\Lambda(\sigma) = \sum_{i=0}^{2d} \Lambda_i\sigma^i$ with diagonal matrix $\Lambda_i \in \mathbb{R}^{n\times n}$. Accordingly, the number of decision variables used to test the stability of Theorem 1 is $(0.5(n+n_b)(n+n_b+1)+2n)(2d+1)+\mathcal{N}(n+n_b,d)+\mathcal{N}(2n+n_b,d)+1$. In addition, the number of decision variables of Theorem 2 is $(0.5(n+n_b)(n+n_b+1)+2n)(N+1)+2$. Therefore, a higher degree polynomial of $P(\sigma)$ or a larger partition number N would increase the computation cost.

Remark 5. The free weight parameter μ introduced in Theorem 2 may reduce the number of interpolated nodes. For the periodic sampling case, i.e., $\sigma_0 = \sigma_1$, Theorem 2 with $\mu = 1$ can be regarded as a linear approximation of Theorem 1. Moreover, mean-square exponential stability of system (1) can be calculated by selecting $c = 0$ in Theorem 1 or 2.

3.2 Controller design

Based on the previous stability results, we can now solve the problem of the sampled-data controller synthesis.

Theorem 3. Given a class $\mathcal{S}(\sigma_0,\sigma_1)$ of sampling time sequences and scalars $\gamma > 0$, $\alpha \in \mathbb{R}$, $d \in \mathbb{N}$. If there exist matrices $P_i \in \mathbb{R}^{(n+n_b)\times(n+n_b)}$, $i = 1,\ldots,2d$, $0 < Y_1 \in \mathbb{R}^{n\times n}$, $0 < Y_2 \in \mathbb{R}^{n_b\times n_b}$, $\bar{K} \in \mathbb{R}^{n_b\times n_c}$, a diagonal matrix $0 < \Lambda(\sigma) \in \mathbb{R}^{n\times n}[\sigma]$, and a scalar c such that (5) and the following inequality

$$\begin{bmatrix} -P(\sigma) & J_0^{\mathrm{T}}P_0 + \mathcal{I}_0^{\mathrm{T}}C^{\mathrm{T}}[\alpha\bar{K}^{\mathrm{T}} \ 0_{n_c\times(n-n_b)} \ \bar{K}^{\mathrm{T}}] \\ * & -P_0 \end{bmatrix} \leqslant 0 \text{ holds for } \sigma \in [\sigma_0,\sigma_1], \tag{23}$$

where $P(\sigma) = \sum_{i=0}^{2d} P_i\sigma^i$, $P_0 = \left[\begin{smallmatrix} Y_1 & \alpha\bar{Y}_2 \\ * & Y_2 \end{smallmatrix}\right]$, and $\bar{Y}_2 = \left[\begin{smallmatrix} Y_2 \\ 0_{(n-n_b)\times n_b} \end{smallmatrix}\right]$, then the sampled-data controller (3) with $K = Y_2^{-1}\bar{K}$ almost surely exponentially stabilizes system (1).

Proof. The proof is straightforward by that the matrix inequality (23) implies (6).

Theorem 4. Given a class $\mathcal{S}(\sigma_0,\sigma_1)$ of sampling time sequences and the number of discretized subintervals N, if for the prescribed scalars $\mu > 0$, $\alpha \in \mathbb{R}$, there exist positive definite matrices $P_{\bar{i}} \in \mathbb{R}^{(n+n_b)\times(n+n_b)}$, $\bar{i} = 1,\ldots,N$, $Y_1 \in \mathbb{R}^{n\times n}$, $Y_2 \in \mathbb{R}^{n_b\times n_b}$, positive definite diagonal matrices $\Lambda_{\ell,\tilde{i}} \in \mathbb{R}^{n\times n}$, $\tilde{i} = 0,1,\ldots,N$, $\ell = 0,1$, a matrix $\bar{K} \in \mathbb{R}^{n_b\times n_c}$, and a scalar c such that (12) and the following LMI holds:

$$\begin{bmatrix} -\mu P_N & J_0^{\mathrm{T}}P_0 + \mathcal{I}_0^{\mathrm{T}}C^{\mathrm{T}}[\alpha\bar{K}^{\mathrm{T}} \ 0_{n_c\times(n-n_b)} \ \bar{K}^{\mathrm{T}}] \\ * & -P_0 \end{bmatrix} \leqslant 0,$$

where $P_0 = \left[\begin{smallmatrix} Y_1 & \alpha\bar{Y}_2 \\ * & Y_2 \end{smallmatrix}\right]$, and $\bar{Y}_2 = \left[\begin{smallmatrix} Y_2 \\ 0_{(n-n_b)\times n_b} \end{smallmatrix}\right]$, then the sampled-data controller (3) with $K = Y_2^{-1}\bar{K}$ almost surely exponentially stabilizes system (1).

4 Illustrative examples

Example 1. Consider the unstable scalar system: $\dot{x}(t) = x(t)$. It has been shown that this system can be stabilized via $Kx(t)\dot{w}(t)$ if $K > \sqrt{2}$ or $K < -\sqrt{2}$. Now, our purpose is to design a sampled-data stochastic controller $Kx(t_k)\dot{w}(t)$, $t \in [t_k, t_{k+1})$ such that the closed-loop system $\mathrm{d}x(t) = x(t)\mathrm{d}t + Kx(t_k)\mathrm{d}w(t)$ is almost surely exponentially stable. We set $K = 2$ and $c = 3.5$. First, by applying a two-dimensional search approach to Theorem 2.1 of [14], we obtain the maximum values of the sampling period as 0.0106. Subsequently, for different degrees of $P(\sigma)$ and N given in Table 1, by applying Theorems 1 and 2, we obtain the maximum values of single sampling period σ_0 that preserve the stability. As shown, our results can significantly improve the result of [14], and the SOS approach can yield better results than the LMI-based method.

Next, we consider the aperiodic sampling problem. Choosing $\deg(P(\sigma)) = 8$ and $c = 3.5$, by applying Theorem 1, the stability region of the aperiodic sampling time sequence can be calculated, which is shown

Table 1 The maximum values of sampling period σ_0 for different N

Theorem	N or $\deg(P(\sigma))$				
	2	4	6	10	100
Theorem 1	0.061	0.086	0.086	0.086	0.086
Theorem 2	0.048	0.062	0.068	0.075	0.085

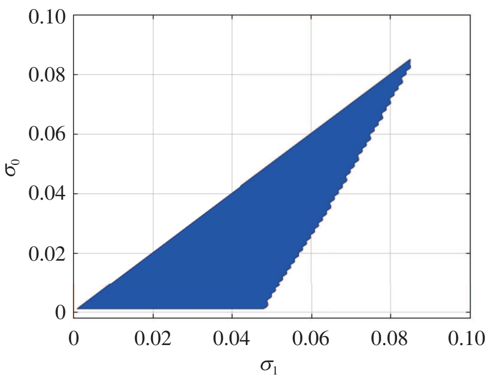

Figure 1 (Color online) Stability region for admissible sampling time sequences.

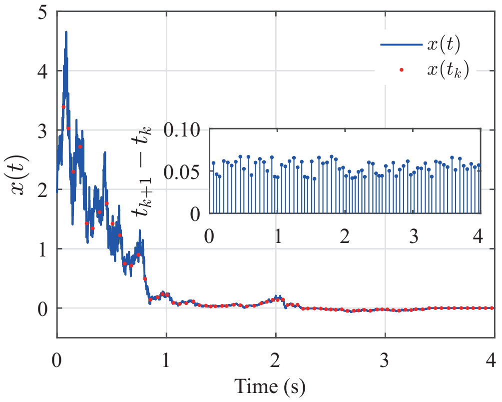

Figure 2 (Color online) Sample-path trajectory of the stochastic system described in Example 1.

Table 2 The maximum values of σ_1 for different approaches

σ_0	Result		
	Theorem 2 of [29]	Theorem 2	Theorem 1
$\sigma_0 = 0.21$	0.43	0.60	0.72
$\sigma_0 = 0.40$	1.25	1.64	1.82
$\sigma_0 = 1.25$	1.57	1.96	2.02

in Figure 1. Let the sampling period be randomly selected from $[0.04, 0.067]$ and $x_0 = 2$. The sample-path state response is depicted in Figure 2, which shows that the trajectory converges to zero under the designed sampled-data stochastic controller.

Example 2. Consider the following system [29]:

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \\ y_k = [1\ 0]x(t_k). \end{cases} \tag{24}$$

This system with continuous-time static output-feedback $u(t) = Ky(t)$ ($y(t) = x_1(t)$, $K = 1$) is unstable. Thus, the results of [13, 15, 16] are not applicable. In fact, Seuret [29] had proven that this system can be stabilized under discrete-time output feedback $u(t) = Ky_k$, $t \in [t_k, t_{k+1})$ when the constant sampling period in $[0.21, 1.62]$. For given $\deg(P(\sigma)) = 8$ and $N = 30$, by applying Theorems 1 and 2, we obtain the value of the constant sampling period that can preserve the stability as $\sigma_0 \in [0.21, 2.02]$. Meanwhile, the aperiodic sampling periods are listed in Table 2. The table indicates that our results are less conservative than the result of [29].

Next, we assume that the control input is interfered by white noise in the implementation of the controller, i.e., the control input has changed as

$$u(t) = Ky_k + \delta y_k \xi(t), \quad t \in [t_k, t_{k+1}), \tag{25}$$

where $\delta = 0.2$ is the noise intensity and $\xi(t)$ is Gaussian while noise, that satisfies $\int_{t_0}^{t} \xi(s)\mathrm{d}s = w(t)$, $t \geqslant t_0$. Subsequently, the closed-loop system (24) and (25) can be rewritten as a closed-loop system

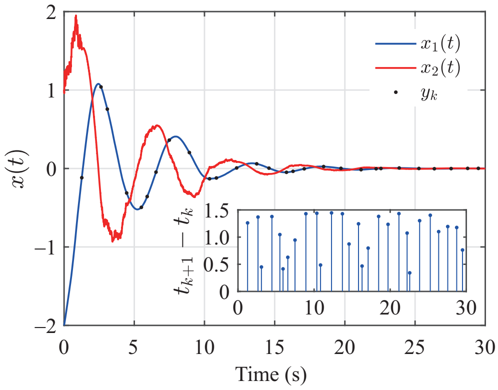

Figure 3 (Color online) Sample-path trajectories of stochastic system described in Example 2 under sampled-data control law with $K=1$ and $\{t_k\}_{k\in\mathbb{N}_0}\in\mathcal{S}(0.3,1.48)$.

(1)–(3) with parameters

$$A_0=\begin{bmatrix}0 & 1\\ -2 & 0.1\end{bmatrix},\quad B_0=\begin{bmatrix}0\\1\end{bmatrix},\quad B_1=\begin{bmatrix}0\\0.2\end{bmatrix},$$

$A_1=D=0$, $C=[1\ 0]$, and $K=1$. Applying Theorem 2 with the choices of $(\sigma_0,c,\mu,N)=(0.3,0,0.91,30)$, we found that the maximum values of σ_1 as 1.34. While applying Theorem 1 with the choices of $(c,\deg(P(\sigma)))=(0,8)$, we obtain the maximum values of σ_1 is 1.48. For the simulation studies, we let the sampling period to be randomly selected from $[0.3,1.48]$, and the initial value chosen as $x(0)=[-2\ 1]^{\mathrm{T}}$. The sample-path trajectories of the sampled-data control system are shown in Figure 3.

5 Conclusion

By employing the impulsive system modeling together with time-dependent Lyapunov function methods, we addressed the almost sure exponential stabilization problem of continuous-time differential equations by artificial multiplicative noise based on variable sampled measurements. The obtained results are formulated as SOS-based conditions and LMI-based conditions, thus providing a solution for designing the controller gain matrix. Compared with the existing results [14–16], our results not only significantly enlarged the upper bound of the sampling period but could also be applied to aperiodic sampled-data stochastic/deterministic systems.

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