

Variation of a signal in Schwarzschild spacetime

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Abstract In this paper, the variation of a signal in Schwarzschild spacetime is studied and a general equation for frequency shift parameter (FSP) is presented. The FSP is found to depend on the gravitationally modified Doppler effects and the gravitational effects of observers. In addition, the time rates of a transmitter and receiver may differ. When the FSP is a function of the receiver time, the FSP contributed through the gravitational effect (GFSP) or the gravitationally modified Doppler effect (GMDSP) may convert a bandlimited signal into a non-bandlimited signal. Using the general equation, the FSP as a function of receiver time is calculated in three scenarios: (a) a spaceship leaving a star at constant velocity communicating with a transmitter at a fixed position; (b) a spaceship moving around a star with different conic trajectories communicating with a transmitter at a fixed position; and (c) a signal transmitted from a fixed position in a star system to a receiver following an elliptic trajectory in another star system. The studied stars are a Sun-like star, a white dwarf, and a neutron star. The theory is illustrated with numerical examples.

Keywords deep space communications, general relativity, Schwarzschild spacetime, bandlimited signal, gravitationally modified Doppler effect

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1 Introduction

With the development of space technologies, humans have sent exploration spacecraft into the far reaches of space. After exploring the planets in the solar system, Voyager 1 of NASA's Voyager interstellar mission escaped the solar system and entered the interstellar medium on August 25 of 2012. Voyager 1 is currently journeying into outer space [1]. Voyager 2 was at a distance of 119 AU from the Sun on November 2 of 2018. Once it reaches the escape velocity, Voyager 2 will also leave the solar system and explore interstellar space [2]. Inspired by the success of its Chang'E missions, China launched its first Mars program on January 11 of 2016 [3]. Many aspects of deep space communication (but ignoring the gravitational effect) have been discussed recently, including communication network design [3–5], channel models [6], channel coding [7], and laser communications [8]. One can easily imagine the exploration of much deeper space by high-speed spaceships in the near future. Therefore, the study of deep space communications has gained importance.

After Einstein proposed his field equations of general relativity [9], Schwarzschild developed the first solution, called the Schwarzschild metric [10, 11], which describes the curved spacetime around a static space object with spherical symmetry. The Schwarzschild metric predicts that when an electromagnetic wave with a certain frequency at a fixed position propagates to another position with a different gravity, the frequency of the wave changes. If the gravity is larger (smaller) at the latter position than at the

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former position, the rate of time is smaller (larger) at the latter position and the frequency is blue (red)-shifted. These phenomena have already been proven in a series of experiments [12–15], and accounted for in satellite tracking [16], global positioning systems [17] and X-ray pulsar-based navigation by weak-field approximation [18–20].

The Doppler effect plays an important role in near-Earth wireless communications and the frequency of the communication signal depends on the relative velocity of the receiver. However, the gravitational effect has been rarely considered, as the Earth's gravitation is too weak to noticeably affect a signal's spectrum. As we shall demonstrate, the Doppler effect is also important in deep-space wireless communications (see also [21]). However, the Doppler formula should be modified and the gravitational effect becomes much more important as it is contributed by masses much greater than the Earth: the Sun, white dwarfs, neutron stars, and other massive space objects. The frequency variation by the gravitational effect depends not on the relative velocity but on the positions of the transmitter and receiver.

This paper studies the gravitational effect on communication signals. The analysis is based on the Schwarzschild metric of general relativity. The studied stars are a Sun-like star, a white dwarf, and a neutron star. Three communication scenarios are considered: (a) a spaceship leaving a star at constant velocity communicating with a transmitter at a fixed position; (b) a spaceship moving around a star with different conic trajectories communicating with a transmitter at a fixed position; and (c) a signal transmitted from a star system to another star system where the receiver follows an elliptic trajectory. This study may improve our understanding of signal models for deep space communications.

The remainder of this paper is organized as follows. Section 2 formulates a general equation of the frequency shift parameter (FSP) in Schwarzschild spacetime. Based on the general FSP equation developed in Section 2, Section 3 investigates the behaviors of a signal traveling in Schwarzschild spacetime in scenarios (a)–(c). Section 4 presents numerical examples of the three scenarios. Section 5 concludes the paper.

Some notations used in this paper are listed below [22, 23]:

T : time coordinate, Newtonian time, time of a static observer at infinity in Schwarzschild spacetime, or time of a static observer in Minkowski spacetime.

t : time of a static observer at distance r in Schwarzschild spacetime.

τ : time of a moving observer at distance r in Schwarzschild spacetime.

$\frac{\partial}{\partial T}$: a coordinate basis vector of the partial derivative operator along direction T .

A boldface English letter is a vector, which can be represented by four coordinate basis vectors. For example, $\mathbf{V} = V_0 \frac{\partial}{\partial T} + V_1 \frac{\partial}{\partial r} + V_2 \frac{\partial}{\partial \theta} + V_3 \frac{\partial}{\partial \varphi}$, where V_m is real-valued and the m th component of the vector \mathbf{V} ($m = 0, \text{time-coordinate component}; m = 1, 2, 3, \text{space-coordinate components}$).

dT : the dual vector of $\frac{\partial}{\partial T}$ along direction T , which is a linear operator on a vector space, i.e., $dT(\frac{\partial}{\partial T}) = \frac{\partial T}{\partial T} = 1$, and $dT(\mathbf{V}) = V_0 \frac{\partial T}{\partial T} + V_1 \frac{\partial T}{\partial r} + V_2 \frac{\partial T}{\partial \theta} + V_3 \frac{\partial T}{\partial \varphi} = V_0 + V_1 \cdot 0 + V_2 \cdot 0 + V_3 \cdot 0 = V_0$.

$dT \otimes dT$: a coordinate basis tensor defining the tensor product of dT and dT and a bilinear operator on a vector space, i.e., $dT \otimes dT(\frac{\partial}{\partial T}, \frac{\partial}{\partial T}) = dT(\frac{\partial}{\partial T}) \cdot dT(\frac{\partial}{\partial T}) = 1 \cdot 1 = 1$, and $dT \otimes dT(\mathbf{V}, \mathbf{W}) = dT(\mathbf{V}) \cdot dT(\mathbf{W}) = V_0 W_0$.

g : the Schwarzschild metric in Schwarzschild spacetime, a bilinear operator on a vector space represented by a set of coordinate basis tensors (see (1) in Section 2).

g_{mn} : the (m, n) th component of the metric g .

$g(\mathbf{V}, \mathbf{W}) = \sum_{m,n} g_{mn} V_m W_n$.

$g(\mathbf{V}, \mathbf{V}) = \sum_{m,n} g_{mn} V_m V_n$. If this sum is positive, negative, or zero, then \mathbf{V} is a space-like vector, a time-like vector, or a light-like vector, respectively (and vice versa). Note that a time-like vector may have space-coordinate components and a space-like vector may have a time-coordinate component.

$g(\mathbf{V}, \mathbf{V}) = 1$: \mathbf{V} is a unit space-like vector.

$g(\mathbf{V}, \mathbf{V}) = -1$: \mathbf{V} is a unit time-like vector.

h : the Minkowski metric in Minkowski spacetime, a bilinear operator on a vector space, which can be represented by a set of coordinate basis tensors (see (15) in Section 2).

2 Frequency change in Schwarzschild spacetime

In Newtonian mechanics, the world is described by three-dimensional space and one-dimensional time. The rate of time is fixed for all observers. Space and time are treated separately. A signal is a function of time and its spectrum changes by the Doppler effect. In contrast, special relativity describes the world by four-dimensional Minkowski spacetime, in which a moving observer's own time is related to the observer's velocity so that space and time are inextricably connected. The rates of time are equal for all static observers but differ from those of moving observers. A signal is a function of its observer's own time, which is determined by the observer's velocity, and its spectrum follows the special relativity version of the Doppler equation. General relativity also describes the world by four-dimensional spacetime with combined space and time, but takes into account of gravitational effects. In this theory, the rates of time of static observers may vary at different positions. A signal is a function of its observer's own time, and is determined by both the observer's velocity and position. The Doppler equation should be further modified (as shown later) to accommodate the gravitational effect in the spectrum change. This more complex Doppler shift will be derived next.

Schwarzschild spacetime is one solution of Einstein's field equations in general relativity. It describes space and time around a static object with spherical symmetry. Schwarzschild spacetime is characterized by the Schwarzschild metric as follows [10, 11, 22, 23]:

$$\begin{aligned} g &\triangleq g_{00}dT \otimes dT + g_{11}dr \otimes dr + g_{22}d\theta \otimes d\theta + g_{33}d\varphi \otimes d\varphi \\ &= -c^2 \left(1 - \frac{2GM}{c^2 r}\right) dT \otimes dT + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr \otimes dr + r^2(d\theta \otimes d\theta + \sin^2 \theta d\varphi \otimes d\varphi), \end{aligned} \quad (1)$$

where G is the gravitational constant, c represents the speed of light, and M denotes the mass of the spherical static object (the Sun, white dwarfs, and neutron stars are regarded as static and spherically symmetric objects, although neither property strictly holds in real star systems). $\{r, \theta, \phi\}$ are the spherical coordinates of the reference frame whose origin is the center of the object. For simplicity, we also define $g = (g_{mn})_{0 \leq m, n \leq 3}$ and $\alpha \triangleq 2GM/c^2$. In general, α is smaller than r ; the exceptions are black holes, which are not discussed in this paper. Therefore, $g_{00} < 0$, $g_{nn} > 0$ for $n = 1, 2, 3$ and $g_{mn} = 0$ for $m \neq n$. For this reason, the "quared length" (weighted by g_{mn}) $g(\mathbf{V}, \mathbf{V})$ of a vector \mathbf{V} can be zero or even negative.

From (1), we obtain the unit vectors of the four coordinates in Schwarzschild spacetime. Note that a unit time-like vector \mathbf{V} and a unit space-like vector \mathbf{W} satisfy $g(\mathbf{V}, \mathbf{V}) = -1$ and $g(\mathbf{W}, \mathbf{W}) = 1$, respectively. The unit time-coordinate vector and the three unit space-coordinate vectors are, respectively, given by

$$\begin{aligned} \mathbf{Z} &= \frac{1}{c} \frac{\partial}{\partial t} = \frac{1}{c} (1 - \alpha/r)^{-\frac{1}{2}} \frac{\partial}{\partial T}, \\ &(1 - \alpha/r)^{\frac{1}{2}} \frac{\partial}{\partial r}, \\ &r^{-1} \frac{\partial}{\partial \theta}, \\ &r^{-1} \sin^{-1} \theta \frac{\partial}{\partial \varphi}. \end{aligned} \quad (2)$$

Note that, T is not only the time coordinate but also represents the time at infinity where there is no gravitation (or Newtonian time that ignores gravitation effects). The vector \mathbf{Z} is simply the unit vector of a static observer's time direction at r , by which the observer measures time, and t is the time of the observer. The dual vector of \mathbf{Z} is

$$cdt = c(1 - \alpha/r)^{\frac{1}{2}} dT. \quad (3)$$

From the dual vector, we obtain [22]

$$dt = (1 - \alpha/r)^{\frac{1}{2}} dT, \quad (4)$$

where the relation between dt and dT is r -dependent. This shows that if the rate of time at infinity (or the rate of Newtonian time) is 1, the relative rate of time at r is $(1 - \alpha/r)^{\frac{1}{2}}$. Therefore, the rates of time depend on r . More specifically, the larger is the r , the larger is the rate of time.

As is well known, an electromagnetic wave is described by two parts: an angular frequency ω and a spatial wave vector \mathbf{k} . In Schwarzschild spacetime, these two parts are combined into a four-dimensional wave vector [23]:

$$\mathbf{K} = \omega \mathbf{Z} + c \mathbf{k}. \quad (5)$$

In the geometrical optics approximation, \mathbf{K} is a light-like vector [23], i.e., $g(\mathbf{K}, \mathbf{K}) = 0$. Meanwhile, \mathbf{Z} is a unit time-like vector without a space-coordinate component, i.e., $g(\mathbf{Z}, \mathbf{Z}) = -1$ and \mathbf{k} is a space-like vector without a time-coordinate component, i.e., $g(\mathbf{Z}, \mathbf{k}) = 0$. Thus, it is not difficult to obtain

$$g(\mathbf{k}, \mathbf{k}) = \omega^2 / c^2. \quad (6)$$

When two static observers occupy different positions, say $\{r_1, r_2\}$, and the first observer at r_1 transmits an electromagnetic wave with angular frequency ω_1 to the second observer, the frequency ω_2 of the received wave at r_2 is given by [23]

$$\omega_2 = \omega_1 \left(\frac{1 - \alpha/r_1}{1 - \alpha/r_2} \right)^{\frac{1}{2}}. \quad (7)$$

Eq. (5) shows that the angular frequency ω is actually the result of projecting \mathbf{K} along \mathbf{Z} , i.e., the unit vector of a static observer's time direction. Therefore, if the electromagnetic wave vector transmitted by a static observer at r_1 is

$$\mathbf{K}_{r_1} = \omega_1 \mathbf{Z}_{r_1} + c \mathbf{k}_{r_1}, \quad (8)$$

then, the electromagnetic wave vector received by a static observer at r_2 is

$$\mathbf{K}_{r_2} = \omega_2 \mathbf{Z}_{r_2} + c \mathbf{k}_{r_2}, \quad (9)$$

where \mathbf{Z}_{r_i} and \mathbf{k}_{r_i} are the unit time-coordinate vector and spatial wave vector for the static observer at r_i ($i = 1, 2$), respectively. In addition, from (4) and (7), we obtain

$$\omega_2 / \omega_1 = dt_1 / dt_2, \quad (10)$$

where t_i is the time of the static observer at r_i ($i = 1, 2$). Therefore, if both observers are static, the angular frequency ω is inversely proportional to the rate of time. Obviously, if $r_2 > r_1$, the frequency red-shifts; conversely, if $r_1 < r_2$, the frequency blue-shifts.

Now consider that a new observer at r_2 is not static but has a four-dimensional velocity (also called a 4-velocity [23]):

$$\begin{aligned} U &= \frac{1}{c} \frac{\partial}{\partial \tau} \\ &= \frac{dt_2}{d\tau} \mathbf{Z}_{r_2} + \frac{1}{c} \left(\frac{dr_2}{d\tau} \frac{\partial}{\partial r_2} + \frac{d\theta}{d\tau} \frac{\partial}{\partial \theta} + \frac{d\varphi}{d\tau} \frac{\partial}{\partial \varphi} \right) \\ &= \frac{dt_2}{d\tau} \mathbf{Z}_{r_2} + \frac{1}{c} \left(\frac{dr_2/dt_2}{d\tau/dt_2} \frac{\partial}{\partial r_2} + \frac{d\theta/dt_2}{d\tau/dt_2} \frac{\partial}{\partial \theta} + \frac{d\varphi/dt_2}{d\tau/dt_2} \frac{\partial}{\partial \varphi} \right) \\ &= \frac{dt_2}{d\tau} \left[\mathbf{Z}_{r_2} + \frac{1}{c} \mathbf{u}_g \right], \end{aligned} \quad (11)$$

where

$$\begin{aligned} \mathbf{u}_g &\triangleq \frac{dr_2}{dt_2} \frac{\partial}{\partial r_2} + \frac{d\theta}{dt_2} \frac{\partial}{\partial \theta} + \frac{d\varphi}{dt_2} \frac{\partial}{\partial \varphi} \\ &= (1 - \alpha/r_2)^{-\frac{1}{2}} \left(\frac{dr_2}{dT} \frac{\partial}{\partial r_2} + \frac{d\theta}{dT} \frac{\partial}{\partial \theta} + \frac{d\varphi}{dT} \frac{\partial}{\partial \varphi} \right) \end{aligned} \quad (12)$$

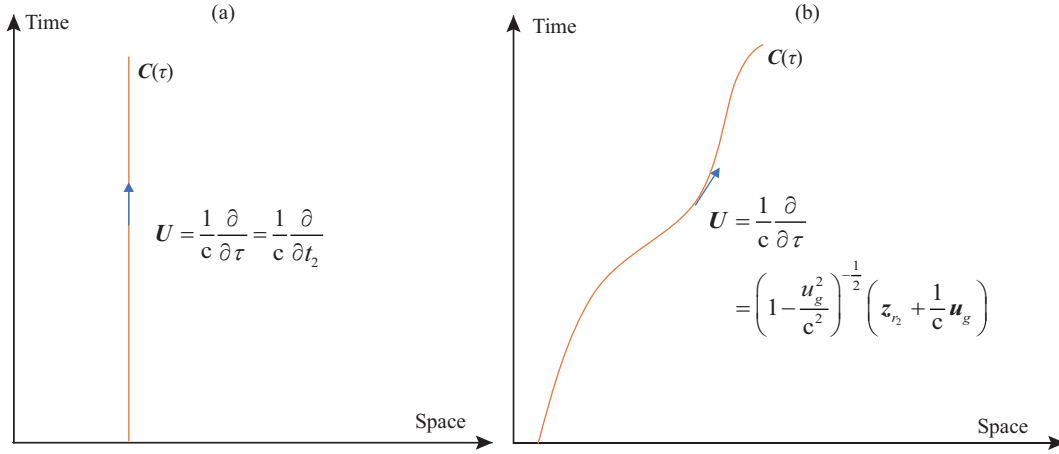


Figure 1 (Color online) World lines of (a) a static observer and (b) a moving observer in Schwarzschild spacetime.

is the spatial velocity of the new observer in Schwarzschild spacetime. This velocity is a space-like vector without a time-coordinate component. The second equality in (12) is Eq. (4) with $r = r_2$ and $t = t_2$. Let $u_g = [g(\mathbf{u}_g, \mathbf{u}_g)]^{\frac{1}{2}}$ define the magnitude of the spatial velocity \mathbf{u}_g at r_2 in Schwarzschild spacetime [23]. Then, we have

$$\begin{aligned} u_g &= [(1 - \alpha/r_2)^{-1}(dr_2/dt_2)^2 + r_2^2(d\theta/dt_2)^2 + r_2^2 \sin^2 \theta (d\varphi/dt_2)^2]^{\frac{1}{2}} \\ &= (1 - \alpha/r_2)^{-\frac{1}{2}} [(1 - \alpha/r_2)^{-1}(dr_2/dT)^2 + r_2^2(d\theta/dT)^2 + r_2^2 \sin^2 \theta (d\varphi/dT)^2]^{\frac{1}{2}}, \end{aligned} \quad (13)$$

where the term $(1 - \alpha/r_2)^{-1}$ in the first equality is the element g_{11} in \mathbf{g} .

Note that \mathbf{U} is the unit vector of the new observer's time direction. The new observer measures time τ based on \mathbf{U} . To demonstrate this, we illustrate two examples in Figure 1. The coordinates in Figure 1 are represented in four-dimensional spacetime, but only two dimensions are depicted for simplicity. The coordinates describe a curve $C(\tau)$ in the time versus space plot. This curve, called the observer's world line, is parametrized by τ or characterized by tangent vector $\frac{1}{c} \frac{\partial}{\partial \tau}$. A time-like curve means that every tangent vector of the curve is a unit time-like vector [22]. As the speed of every observer cannot exceed the light speed, the world lines of all observers are time-like curves. The time τ is precisely the time of the observer along the curve, and $\frac{1}{c} \frac{\partial}{\partial \tau}$ is a metaphorical ruler by which the observer measures time. In Figure 1(a), the observer at r_2 is static and the tangent vector of every position on the curve is $\frac{1}{c} \frac{\partial}{\partial t_2}$, so the observer's time is t_2 . In Figure 1(b), the observer moves at different velocities through different positions and its tangent vector is constantly changing. However, at every position of its world line, the observer can measure the time along the curve using the tangent vector as a ruler. As the 4-velocity \mathbf{U} is a unit time-like vector, \mathbf{Z}_{r_2} is a unit time-like vector without space-coordinate components, and \mathbf{u}_g is a space-like vector without a time-coordinate component, we have $g(\mathbf{U}, \mathbf{U}) = -1$, $g(\mathbf{Z}_{r_2}, \mathbf{Z}_{r_2}) = -1$, and $g(\mathbf{Z}_{r_2}, \mathbf{u}_g) = 0$. Then, it is not difficult to obtain

$$dt_2/d\tau = (1 - u_g^2/c^2)^{-\frac{1}{2}} \triangleq \gamma_g. \quad (14)$$

By comparison, the four-dimensional spacetime in special relativity is Minkowski spacetime, which is characterized by the so-called Minkowski metric (also called the flat metric) [23]:

$$\mathbf{h} = -c^2 dT \otimes dT + dr \otimes dr + r^2 [d\theta \otimes d\theta + \sin^2 \theta d\varphi \otimes d\varphi], \quad (15)$$

and let \mathbf{u}_h be the spatial velocity in Minkowski spacetime. Then, we have

$$\mathbf{u}_h = \frac{dr_2}{dT} \frac{\partial}{\partial r_2} + \frac{d\theta}{dT} \frac{\partial}{\partial \theta} + \frac{d\varphi}{dT} \frac{\partial}{\partial \varphi}, \quad (16)$$

and the magnitude u_h of the spatial velocity \mathbf{u}_h in Minkowski spacetime can be calculated as

$$u_h = [\mathbf{h}(\mathbf{u}_h, \mathbf{u}_h)]^{\frac{1}{2}} = [(dr_2/dT)^2 + r_2^2(d\theta/dT)^2 + r_2^2 \sin^2 \theta (d\varphi/dT)^2]^{\frac{1}{2}}. \quad (17)$$

From (12) and (16), one observes that the spatial velocities in the two spacetimes differ in two respects. First, the Newtonian time T (a static observer's time at r_2 in Minkowski spacetime) in (16) is replaced by the time t_2 (a static observer's time at r_2 in Schwarzschild spacetime) in (12). Both velocities have the same direction. Second, the magnitudes of the spatial velocities differ in the two spacetimes. Comparing (13) and (17), there is one more coefficient $(1 - \alpha/r_2)^{-1}$ in the first equality of (13) which derives from g_{11} in \mathbf{g} , besides the different times T and t_2 .

From (14), and (4) setting $r = r_2$ and $t = t_2$, we have

$$d\tau = (1 - u_g^2/c^2)^{\frac{1}{2}}(1 - \alpha/r_2)^{\frac{1}{2}}dT. \quad (18)$$

As the frequency ω'_2 at r_2 for the new observer is the component of its own time direction, one must project \mathbf{K}_{r_2} along \mathbf{U} (the unit vector of the new observer's time direction). The projected value is given by

$$\begin{aligned} \omega'_2 &= -\mathbf{g}(\mathbf{U}, \mathbf{K}_{r_2}) \\ &= -\gamma_g \mathbf{g}(\mathbf{Z}_{r_2} + \mathbf{u}_g/c, \omega_2 \mathbf{Z}_{r_2} + c \mathbf{k}_{r_2}) \\ &= \gamma_g \omega_2 - \gamma_g \mathbf{g}(\mathbf{u}_g, \mathbf{k}_{r_2}) \\ &= \gamma_g \omega_2 - \gamma_g u_g \frac{\omega_2}{c} \cos \psi \\ &= \omega_1 \gamma_g \left(1 - \frac{u_g}{c} \cos \psi\right) \left(\frac{1 - \alpha/r_1}{1 - \alpha/r_2}\right)^{\frac{1}{2}}, \end{aligned} \quad (19)$$

where ω_2 is the frequency observed by the static observer at r_2 , and

$$\cos \psi = \frac{\mathbf{g}(\mathbf{u}_g, \mathbf{k}_{r_2})}{\sqrt{\mathbf{g}(\mathbf{k}_{r_2}, \mathbf{k}_{r_2}) \mathbf{g}(\mathbf{u}_g, \mathbf{u}_g)}}. \quad (20)$$

The static observer shares the same \mathbf{K}_{r_2} as the moving observer at r_2 but with a different time direction. The second equality in (19) follows from (9), (11), and (14), and the second-to-last and last equalities follow from (6) with $\mathbf{k} = \mathbf{k}_{r_2}$ and (7), respectively. We also define the FSP β :

$$\beta \triangleq \gamma_g \left(1 - \frac{u_g}{c} \cos \psi\right) \left(\frac{1 - \alpha/r_1}{1 - \alpha/r_2}\right)^{\frac{1}{2}}, \quad (21)$$

which is independent of angular frequency ω_1 . In terms of the signal frequency ω_1 of the transmitter, the frequency ω'_2 of the moving observer at r_2 is then given by

$$\omega'_2 = \beta \omega_1. \quad (22)$$

Clearly, the FSP β in (21) can be separated into two parts. The first part,

$$\beta_1 \triangleq \gamma_g \left(1 - \frac{u_g}{c} \cos \psi\right) \quad (23)$$

is similar to the special relativity version of the Doppler equation [23], namely

$$\beta'_1 = \gamma_h \left(1 - \frac{u_h}{c} \cos \psi'\right). \quad (24)$$

Here, γ_h is similar to γ_g in (14) but with u_g replaced by u_h , and ψ' is the intersection angle between \mathbf{u}_h and \mathbf{k}_{r_2} . When $r_1, r_2 \rightarrow \infty$, Eqs. (23) and (24) are equivalent. Furthermore, if $u_h \ll c$, then $\gamma_h \approx 1$ and we retrieve the conventional equation

$$\beta'_1 = \left(1 - \frac{u_h}{c} \cos \psi'\right), \quad (25)$$

Therefore, Eq. (23) is merely a gravitationally modified Doppler equation. The parameter β_1 is called the gravitationally modified Doppler frequency shift parameter (GMDFPS).

The second part is purely derived from the gravitational effect (or Einstein effect in [24]), and is called the gravitational frequency shift parameter (GFSP):

$$\beta_2 \triangleq \left(\frac{1 - \alpha/r_1}{1 - \alpha/r_2} \right)^{\frac{1}{2}}. \quad (26)$$

Thus, the FSP can be rewritten as

$$\beta = \beta_1 \beta_2. \quad (27)$$

Clearly, when β is very far from 1, ω'_2 and ω_1 in (22) are widely different, meaning that the signal frequencies of a moving observer at r_2 and a transmitter at r_1 are widely different (implying a large frequency shift). The contributions of the two parts of β_1 and β_2 can be considered separately. If the difference between β_1 and 1 (or β_2 and 1) is large or small, the gravitationally modified Doppler effect (gravitational effect) is significant or negligible, respectively.

For a static observer at r_2 , we have $u_g = 0$ and $\gamma_g = 1$, and the FSP β in (21) is constant and equal to β_2 . In this case, Eq. (19) reduces to (7). Moreover, when u_g or ψ in (23) changes over time τ , β_1 is also a function of τ . Similarly, when r_1 or r_2 in (26) changes over time τ , β_2 is a function of τ . Consequently, both β in (21) and the frequency ω'_2 in (22) are functions of τ . In general, we also have $0 < \beta_1(\tau) < \infty$ (as $u_g < c$), and $0 < \beta_2(\tau) < \infty$ (as $\alpha < r_1$ and r_2). Therefore, $0 < \beta(\tau) < \infty$.

Eq. (21) is the basic equation for calculating the FSP β . This equation depends on the parameters r_1 , u_g , r_2 , and $\cos \psi$, which are related to the transmitter position and receiver trajectory. Fixing the transmitter position, the FSP β for different receiver trajectories will be calculated in Section 3.

3 A signal in Schwarzschild spacetime

Section 2 established the basic principles of the FSP when an electromagnetic wave is propagating through Schwarzschild spacetime. From these principles, one can easily derive the variation of a signal transmitted through Schwarzschild spacetime in particular scenarios. For consistency with Section 2, we here denote the times of the transmitter at a fixed position r_1 and a moving receiver at r_2 as t_1 and τ , respectively, and the time at infinity as T . When a single-frequency complex exponential signal $e^{j\omega_1 t_1}$ is transmitted at r_1 to a spaceship at r_2 in the absence of noise, a different signal will be received by the spaceship; that is,

$$e^{j\omega_1 t_1} \rightarrow e^{j\omega_1 B(\tau)}, \quad (28)$$

where $B(\tau) = \int_0^\tau \beta(\tau') d\tau'$, and $\beta(\tau')$ is the FSP in (21). If β is constant, $B(\tau) = \beta\tau$. A periodical signal $f_1(t_1)$ with fundamental angular frequency ω can be expanded as a Fourier series:

$$f_1(t_1) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t_1}, \quad (29)$$

with

$$a_k = \frac{\omega}{2\pi} \int_{-\pi/\omega}^{\pi/\omega} f(t_1) e^{-jk\omega t_1} dt_1. \quad (30)$$

The received signal is then given by

$$f_2(\tau) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega B(\tau)}. \quad (31)$$

Regarding the fundamental frequency of a non-periodic signal as infinitesimal ($\Delta\omega$), the transmitted signal can be expanded as [25]

$$f_1(t_1) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Delta\omega t_1}, \quad (32)$$

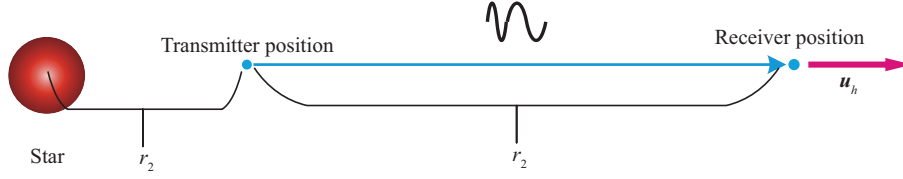


Figure 2 (Color online) A spaceship leaving a star and communicating with a transmitter at a fixed position.

with

$$a_k = \frac{\Delta\omega}{2\pi} \int_{-\pi/\Delta\omega}^{\pi/\Delta\omega} f_1(t_1) e^{-jk\Delta\omega t_1} dt_1. \quad (33)$$

Then, the received signal is

$$f_2(\tau) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Delta\omega\tau} = \sum_{k=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} \int_{-\pi/\Delta\omega}^{\pi/\Delta\omega} f_1(t_1) e^{-jk\Delta\omega t_1} dt_1 e^{jk\Delta\omega B(\tau)}. \quad (34)$$

As $\Delta\omega \rightarrow 0$, the received signal can be rewritten as

$$f_2(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(t_1) e^{-j\omega t_1} dt_1 e^{j\omega B(\tau)} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}_1(\omega) e^{j\omega B(\tau)} d\omega = f_1(B(\tau)), \quad (35)$$

where $\hat{f}_1(\omega)$ is the Fourier transform of $f_1(t_1)$.

In general, a bandlimited communication signal with bandwidth W is given by

$$f_1(t_1) = \frac{1}{2\pi} \int_{-W}^W \hat{f}_1(\omega) e^{j\omega t_1} d\omega. \quad (36)$$

When β is a non-zero constant, from (35), the received signal is

$$f_2(\tau) = \frac{1}{2\pi} \int_{-\beta W}^{\beta W} \frac{1}{\beta} \hat{f}_1\left(\frac{\omega}{\beta}\right) e^{j\omega\tau} d\omega. \quad (37)$$

Then, the spectrum of the received signal is broadened when $\beta < 1$, narrowed when $\beta > 1$, and unchanged when $\beta = 1$. When β is a function of time τ , the received signal is $f_1(B(\tau))$. As inferred from [26], the signal $f_1(B(\tau))$ is bandlimited only when $\beta(\tau)$ is constant. Thus, a bandlimited signal transmitted through Schwarzschild spacetime becomes a non-bandlimited signal. Clearly, the non-bandlimited-ness increases with the varying rate of $\beta(\tau)$. The non-bandlimited-ness is contributed by the gravitationally modified Doppler effect and the gravitational effect. A large (small) varying rate of $\beta_1(\tau)$ (or $\beta_2(\tau)$) implies a large (possibly negligible) non-bandlimited-ness contributed by $\beta_1(\tau)$ (or $\beta_2(\tau)$).

3.1 Variation of a signal from an object moving along a straight trajectory

This subsection considers the simple case of a receiver moving away from a star at constant $dr_2/dT \triangleq u_h$. The transmitter position r_1 , receiver position r_2 , and center of the star are along the same line, as shown in Figure 2. Suppose that at time $\tau = 0$, T is zero and the receiver begins detecting a transmitted signal at position r_0 . At the Newtonian time $T(\tau)$ when the receiver's time is τ , we have

$$r_2 = r_0 + u_h T(\tau). \quad (38)$$

From (13) and (38),

$$u_g = (1 - \alpha/(r_0 + u_h T(\tau)))^{-1} dr_2/dT = (1 - \alpha/(r_0 + u_h T(\tau)))^{-1} u_h. \quad (39)$$

As a function of τ , the Newtonian time $T(\tau)$ can be obtained from (18), (38), and (39):

$$d\tau = (1 - u_g^2/c^2)^{\frac{1}{2}} (1 - \alpha/r_2)^{\frac{1}{2}} dT = \left[1 - \alpha/(r_0 + u_h T(\tau)) - (1 - \alpha/(r_0 + u_h T(\tau)))^{-1} \frac{u_h^2}{c^2} \right]^{\frac{1}{2}} dT, \quad (40)$$

where the relation between τ and T depends on u_h and r_0 . When u_h and r_0 are provided, T is a fixed function of τ . Because $\cos\psi = 1$ in (19), the GMDFSP and GFSP are, respectively, derived from (23), (26), (38), (39) and (40) as follows:

$$\begin{aligned}\beta_1(\tau) &= (1 - u_g^2/c^2)^{-\frac{1}{2}}(1 - u_g/c) \\ &= [1 - (1 - \alpha/(r_0 + u_h T(\tau)))^{-2} u_h^2/c^2]^{-\frac{1}{2}} \times [1 - (1 - \alpha/(r_0 + u_h T(\tau)))^{-1} u_h/c] \\ &= \left(\frac{1 - \alpha/(r_0 + u_h T(\tau)) - u_h/c}{1 - \alpha/(r_0 + u_h T(\tau)) + u_h/c} \right)^{\frac{1}{2}}, \\ \beta_2(\tau) &= \left(\frac{1 - \alpha/r_1}{1 - \alpha/(r_0 + u_h T(\tau))} \right)^{\frac{1}{2}}.\end{aligned}\quad (41)$$

As $T(\tau)$ depends only on the coefficients u_h and r_0 , $\beta_1(\tau)$, $\beta_2(\tau)$, and $\beta(\tau) = \beta_1(\tau)\beta_2(\tau)$ also depend only on u_h and r_0 .

The speed u_h cannot exceed the speed of light, i.e., $u_h < c$. If $u_h \ll c$, we have

$$\left(\frac{1 - \alpha/(r_0 + u_h T(\tau)) - u_h/c}{1 - \alpha/(r_0 + u_h T(\tau)) + u_h/c} \right)^{\frac{1}{2}} \approx 1 - (1 - \alpha/(r_0 + u_h T(\tau)))^{-1} u_h/c, \quad (42)$$

where $(1 - \alpha/(r_0 + u_h T(\tau)))^{-1}$ is generally close to 1, and $\beta(\tau)$ can be written as

$$\beta(\tau) \approx \left(1 - \frac{u_h/c}{1 - \alpha/(r_0 + u_h T(\tau))} \right) \left(\frac{1 - \alpha/r_1}{1 - \alpha/(r_0 + u_h T(\tau))} \right)^{\frac{1}{2}}. \quad (43)$$

We also have

$$d\tau \approx (1 - \alpha/(r_0 + u_h T(\tau)))^{\frac{1}{2}} dT. \quad (44)$$

As the studied stars are not black holes, $\alpha < r_0$ (see Section 2). If $\alpha \ll r_0$, the star is far from being a black hole. Eq. (43) then gives

$$\begin{aligned}\beta(\tau) &\approx (1 - \alpha/r_1)^{\frac{1}{2}} \left(1 - \frac{u_h}{c} - \frac{\alpha u_h/c}{r_0 + u_h T(\tau)} \right) \left(1 + \frac{\alpha/2}{r_0 + u_h T(\tau)} \right) \\ &\approx (1 - \alpha/r_1)^{\frac{1}{2}} \left[1 - \frac{u_h}{c} + \left(1 - \frac{3u_h}{c} \right) \frac{\alpha/2}{r_0 + u_h T(\tau)} \right].\end{aligned}\quad (45)$$

From (44), we also have

$$d\tau \approx (1 - \alpha/[2(r_0 + u_h T(\tau))]) dT. \quad (46)$$

Since $T(0) = 0$,

$$\tau \approx T(\tau) - \frac{\alpha}{2u_h} \ln(1 + u_h T(\tau)/r_0). \quad (47)$$

Furthermore, if $u_h T(\tau) \ll r_0$, i.e., the spaceship flies a relatively short distance throughout the communication time, (45) gives

$$\begin{aligned}\beta(\tau) &\approx (1 - \alpha/r_1)^{\frac{1}{2}} \left[1 - \frac{u_h}{c} + \frac{1}{2} \left(1 - \frac{3u_h}{c} \right) \frac{\alpha/r_0}{1 + u_h T(\tau)/r_0} \right] \\ &\approx (1 - \alpha/r_1)^{\frac{1}{2}} \left[1 - \frac{u_h}{c} + \frac{\alpha}{2r_0} \left(1 - \frac{3u_h}{c} \right) \left(1 - \frac{u_h T(\tau)}{r_0} \right) \right] \\ &= C_0 + C_1 T(\tau),\end{aligned}\quad (48)$$

where

$$\begin{aligned}C_0 &\triangleq (1 - \alpha/r_1)^{\frac{1}{2}} \left[1 - \frac{u_h}{c} + \frac{\alpha}{2r_0} \left(1 - \frac{3u_h}{c} \right) \right], \\ C_1 &\triangleq -\frac{u_h \alpha}{2r_0^2} (1 - \alpha/r_1)^{\frac{1}{2}} \left(1 - \frac{3u_h}{c} \right).\end{aligned}\quad (49)$$

From (47), we also have

$$\tau \approx (1 - \alpha/(2r_0))T(\tau) = C_2T(\tau), \quad (50)$$

where $C_2 \triangleq 1 - \alpha/(2r_0)$. Consequently, a signal $f_1(t_1)$ transmitted at r_1 is received by the spaceship at $r_2(\tau)$ as

$$f_2(\tau) \approx f_1(C_0\tau + C_3\tau^2), \quad (51)$$

where $C_3 \triangleq C_1/(2C_2)$. From [26], one observes that a bandlimited signal $f_1(t_1)$ becomes a non-bandlimited signal $f_2(\tau)$ at the receiver. This raises the following question: Is $f_1(t_1)$ bandlimited in the fractional Fourier domain [27,28]? To answer this question, we first consider a single-frequency signal

$$f_1(t_1) = e^{j\omega t_1},$$

which is received by the spaceship as

$$f_2(\tau) \approx e^{j\omega(C_0\tau + C_3\tau^2)}.$$

The received signal is a chirp signal with initial frequency $C_0\omega$ and frequency rate $2C_3\omega$. Therefore, $f_2(\tau)$ is bandlimited in the fractional Fourier domain with the frequency rate $2C_3\omega$. However, if $f_1(t_1)$ contains multiple frequency, then $f_2(\tau)$ is not bandlimited in any fractional Fourier domain. To see this, let us consider the following two-frequency signal:

$$f_1(t_1) = e^{j\omega_1 t_1} + e^{j\omega_2 t_2} = p_1(t_1) + q_1(t_1),$$

with $\omega_1 \neq \omega_2$. The received signal by the spaceship is then

$$f_2(\tau) \approx e^{j\omega_1(C_0\tau + C_3\tau^2)} + e^{j\omega_2(C_0\tau + C_3\tau^2)} = p_2(\tau) + q_2(\tau).$$

Notably, $p_2(\tau)$ and $q_2(\tau)$ are both chirp signals but with different frequency rates; therefore, a common fractional domain in which both fractional Fourier transforms of $p_2(\tau)$ and $q_2(\tau)$ are bandlimited cannot be found [29]. Consequently, $f_2(\tau)$ is not bandlimited in the fractional Fourier domain. The same rule applies to any multiple-frequency signal.

3.2 Variation of a signal transmitted to a conic trajectory

In general, when a spacecraft moves around a star with no extra force, it follows a conic trajectory. The change in a signal received by a spaceship following such a trajectory is worthy of attention. A conic trajectory may be elliptic, parabolic or hyperbolic (see Figure 3), depending on the mechanical energy of the spaceship (i.e., the summed gravitational potential energy (< 0) and kinetic energy (> 0)). If the mechanical energy is smaller than, larger than, or equal to 0, the trajectory is elliptic, parabolic, or hyperbolic, respectively.

Consider that the transmitter is fixed at a location close to a star and that a spaceship with an onboard receiver moves around the star with a conic trajectory. The trajectory radius relative to the center of the star is [30]

$$r_2 = p/(1 + e \cos \varphi), \quad (52)$$

where $p = h^2/(GM)$ with

$$h = r_2^2 d\varphi/dT, \quad (53)$$

$e = [1 + 2h^2E/(G^2M^2m)]^{\frac{1}{2}}$ is the eccentricity of the trajectory ($0 < e < 1$ for an ellipse, $e = 1$ for a parabola, or $e > 1$ for a hyperbola), and m is the mass of the spaceship. The mechanical energy E and angular momentum h are constants satisfying mechanical energy conservation and angular momentum conservation, respectively. T is the Newtonian time and M is the mass of the star. If the semi-major axis a of the spaceship orbit is known, then we have

$$p = \begin{cases} a(1 - e^2), & \text{for } 0 < e < 1, \\ 2a, & \text{for } e = 1, \\ a(e^2 - 1), & \text{for } e > 1. \end{cases} \quad (54)$$

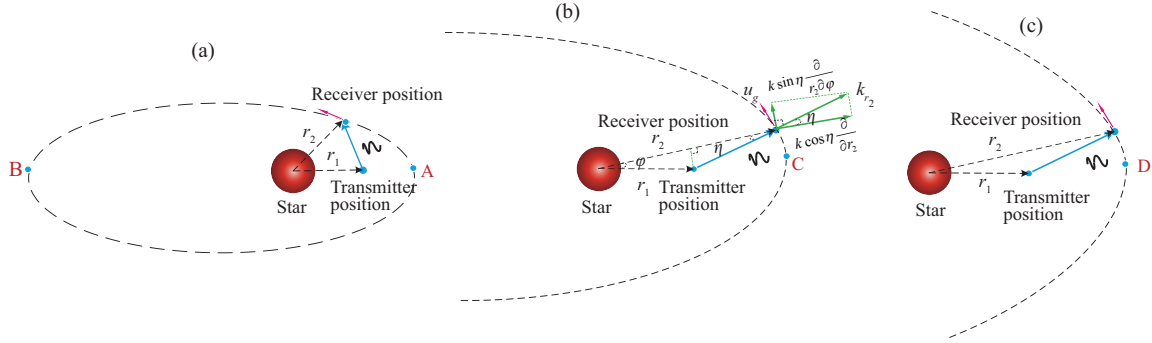


Figure 3 (Color online) A spaceship moving around a star with (a) an elliptic, (b) a parabolic, and (c) a hyperbolic trajectory while communicating with a transmitter at a fixed position.

Combining (52) and (53), the angular velocity in Newtonian time is given by

$$\frac{d\varphi}{dT} = hr_2^{-2} = \frac{h}{p^2} (1 + e \cos \varphi)^2. \quad (55)$$

The radial velocity of the spaceship relative to the center of the star in Newtonian time is

$$\frac{dr_2}{dT} = \frac{pe \sin \varphi}{(1 + e \cos \varphi)^2} \frac{d\varphi}{dT} = \frac{h}{p} e \sin \varphi. \quad (56)$$

As $\theta = \pi/2$, from (12), (13), (52), (55), and (56), we get

$$\mathbf{u}_g = \frac{h}{p} (1 - \alpha/r_2)^{-\frac{1}{2}} [e \sin \varphi \partial/\partial r_2 + pr_2^{-2} \partial/\partial \varphi] \quad (57)$$

and

$$u_g = \frac{h}{p} (1 - \alpha/r_2)^{-\frac{1}{2}} [(1 - \alpha/r_2)^{-1} e^2 \sin^2 \varphi + (1 + e \cos \varphi)^2]^{\frac{1}{2}}. \quad (58)$$

From (18), (52), (55), and (58), we then obtain

$$d\tau = \left\{ 1 - \alpha/r_2 - \frac{h^2}{c^2 p^2} [(1 - \alpha/r_2)^{-1} e^2 \sin^2 \varphi + (1 + e \cos \varphi)^2] \right\}^{\frac{1}{2}} \frac{p^2}{h} (1 + e \cos \varphi)^{-2} d\varphi. \quad (59)$$

Supposing that the transmitted signal approximately propagates along straight lines and given that $\theta = \pi/2$ in (9), we let

$$\mathbf{k}_{r_2} = k \cos \eta \partial/\partial r_2 + k \sin \eta r_2^{-1} \partial/\partial \varphi, \quad (60)$$

where k is a parameter and η is the intersection angle between \mathbf{k}_{r_2} and the radial direction at r_2 . As clarified in Figure 3(b), we have

$$\cos \eta = \frac{r_2 - r_1 \cos \varphi}{(r_1^2 + r_2^2 - 2r_1 r_2 \cos \varphi)^{\frac{1}{2}}}, \quad \sin \eta = \frac{r_1 \sin \varphi}{(r_1^2 + r_2^2 - 2r_1 r_2 \cos \varphi)^{\frac{1}{2}}}. \quad (61)$$

Setting $\mathbf{k} = \mathbf{k}_{r_2}$ and $\omega = \omega_2$ in (6) and using (60) and (61), we get

$$k = \frac{\omega_2}{c} \frac{(r_1^2 + r_2^2 - 2r_1 r_2 \cos \varphi)^{\frac{1}{2}}}{[(1 - \alpha/r_2)^{-1} (r_2 - r_1 \cos \varphi)^2 + r_1^2 \sin^2 \varphi]^{\frac{1}{2}}}. \quad (62)$$

Applying (57), (60)–(62), $\cos \psi$ in (19) is given by

$$\cos \psi = \frac{g(\mathbf{k}_{r_2}, \mathbf{u}_g)}{\sqrt{g(\mathbf{k}_{r_2}, \mathbf{k}_{r_2})g(\mathbf{u}_g, \mathbf{u}_g)}} = \frac{g(\mathbf{k}_{r_2}, \mathbf{u}_g)}{u_g \omega_2 / c}, \quad (63)$$

where

$$g(\mathbf{k}_{r_2}, \mathbf{u}_g)$$

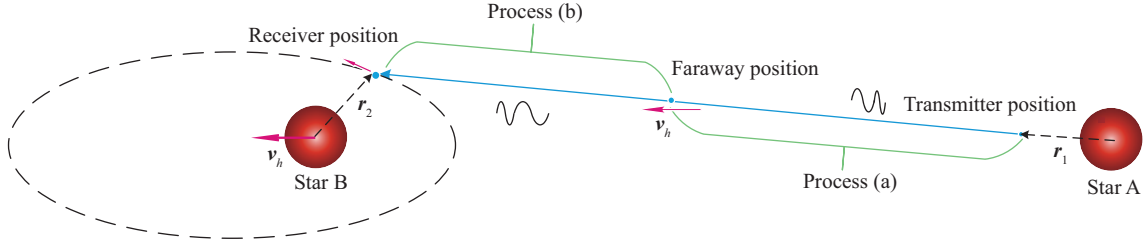


Figure 4 (Color online) A signal propagating from star system A to another faraway star system B.

$$= \frac{h\omega_2}{pc} (1 - \alpha/r_2)^{-\frac{3}{2}} \sin \varphi [r_2 e - r_1 (1 + e \cos \varphi) \alpha / r_2 + r_1] [(1 - \alpha/r_2)^{-1} (r_2 - r_1 \cos \varphi)^2 + r_1^2 \sin^2 \varphi]^{-\frac{1}{2}}. \quad (64)$$

When the initial value of φ and the values of h and p (or a and e) are given, φ as a function of τ can be obtained from (52) and (59). As shown in (52), (58) and (63), r_2 , u_g , and $\cos \psi$ are functions of φ ; therefore, they can be also solved as functions of τ . Furthermore, as shown in (21), (23), and (26), the FSP, G MDFSP, and GFSP depend only on r_1 , r_2 , u_g , and $\cos \psi$; therefore, they can be also solved as functions of τ .

3.3 Variation of a signal during interstellar communications

Conceivably, aliens in a faraway star system might someday send electromagnetic waves containing information to our solar system. Moreover, mankind can conceivably reach other star systems in the future. These events might elicit frequent communications between different star systems. Therefore, calculating the signal changes between different star systems is a worthwhile task.

Consider a signal transmitted from a star system A to another faraway star system B. Suppose that the relative velocity \mathbf{v}_h between the two stars has a constant magnitude v_h along a fixed direction in Newtonian time T (see Figure 4). As the two stars are distantly separated, the gravitation of one star can be neglected if the signal is much closer to the other star. When the signal approaches the transmitter, the gravitational effect is almost totally caused by star A. When the signal approaches the receiver, the gravitational effect is almost totally caused by star B. When the signal is far from both stars, the gravitation can be neglected. Therefore, the whole process can be separated into two parts: (a) propagation of the signal from the transmitter position near star A to a moving position that is at the same velocity as \mathbf{v}_h and far away from either star (hereafter, faraway position), and (b) propagation of the signal from the faraway position to the receiver position near star B. Process (a) is identical to that considered in Subsection 3.1 and shown in Figure 2. Now set the parameter of the frequency shift as β_a . The faraway position $r_\infty \rightarrow \infty$ and moves away from the transmitter position with velocity \mathbf{v}_h . If the transmitter position is fixed, then we have (21)

$$\beta_a = (1 - v_h^2/c^2)^{-\frac{1}{2}} (1 - v_h/c) (1 - \alpha_1/r_1)^{\frac{1}{2}}, \quad (65)$$

where $\alpha_1 = 2GM_1/c^2$, M_1 is the mass of star A, and r_1 is the radial distance of the transmitter.

Process (b) is similar to that in Subsection 3.2 and shown in Figure 3(a). Now set the parameter of the frequency shift as β_b , and suppose an elliptic trajectory of the receiver position r_2 . By (52), we then obtain

$$r_2 = p/(1 + e \cos \varphi), \quad (66)$$

where $0 < e < 1$. Letting \mathbf{u}_g be the spatial velocity of the receiver in Schwarzschild spacetime, (58) gives

$$u_g = \frac{h}{p} (1 - \alpha_2/r_2)^{-\frac{1}{2}} [(1 - \alpha_2/r_2)^{-1} e^2 \sin^2 \varphi + (1 + e \cos \varphi)^2]^{\frac{1}{2}}. \quad (67)$$

Substituting r_1 with $r_\infty = \infty$ in (63) gives

$$\cos \psi = \sin \varphi [1 - (1 - \alpha/r_2)^{-1} \alpha e \cos \varphi / r_2] [(1 - \alpha/r_2)^{-1} \cos^2 \varphi$$

$$+ \sin^2 \varphi]^{-\frac{1}{2}} [(1 - \alpha/r_2)^{-1} e^2 \sin^2 \varphi + (1 + e \cos \varphi)^2]^{-\frac{1}{2}}. \quad (68)$$

By (59), we have

$$d\tau = \left\{ 1 - \alpha_2/r_2 - \frac{h^2}{c^2 p^2} [(1 - \alpha_2/r_2)^{-1} e^2 \sin^2 \varphi + (1 + e \cos \varphi)^2] \right\}^{\frac{1}{2}} \frac{p^2}{h} (1 + e \cos \varphi)^{-2} d\varphi, \quad (69)$$

where τ is the time of the receiver and φ can be solved as a function of τ . As shown in (66)–(68) that r_2 , u_g and $\cos \psi$ are functions of φ , and hence, of τ . Therefore, by (21), β_b can be also solved as a function of τ :

$$\beta_b(\tau) = (1 - u_g^2/c^2)^{-\frac{1}{2}} \left(1 - \frac{u_g}{c} \cos \psi \right) (1 - \alpha_2/r_2)^{-\frac{1}{2}}. \quad (70)$$

Combining processes (a) and (b), the FSP of the whole process is given as

$$\beta(\tau) = \beta_a \beta_b(\tau) = (1 - v_h^2/c^2)^{-\frac{1}{2}} (1 - u_g^2/c^2)^{-\frac{1}{2}} \left(1 - \frac{v_h}{c} \right) \left(1 - \frac{u_g}{c} \cos \psi \right) \times \left(\frac{1 - \alpha_1/r_1}{1 - \alpha_2/r_2} \right)^{\frac{1}{2}}. \quad (71)$$

The GMDFSP and GFSP are then given as

$$\beta_1(\tau) = (1 - v_h^2/c^2)^{-\frac{1}{2}} (1 - u_g^2/c^2)^{-\frac{1}{2}} (1 - v_h/c) \left(1 - \frac{u_g}{c} \cos \psi \right) \quad (72)$$

and

$$\beta_2(\tau) = \left(\frac{1 - \alpha_1/r_1}{1 - \alpha_2/r_2} \right)^{\frac{1}{2}}, \quad (73)$$

respectively.

3.4 Variation of a signal transmitted from a moving position to a moving spaceship

In the above studies, the transmitter was spatially fixed. The results for a stationary transmitter are easily extended to those of a moving transmitter as follows. The frequency of a moving transmitter is simply the projection along its own 4-velocity. Suppose that when the moving transmitter is at r_1 , it transmits a signal with frequency ω'_1 . The relation between the frequency of the static observer at r_1 and the moving transmitter at r_1 is then obtained by (19), with r_2 replaced by r_1 :

$$\omega_1 = \omega'_1 (1 - u_{1g}^2/c^2)^{\frac{1}{2}} \left(1 - \frac{u_{1g}}{c} \cos \psi_1 \right)^{-1}, \quad (74)$$

where $\cos \psi_1 = \mathbf{g}(\mathbf{k}_{r_1}, \mathbf{u}_{1g}) / \sqrt{\mathbf{g}(\mathbf{k}_{r_1}, \mathbf{k}_{r_1}) \mathbf{g}(\mathbf{u}_{1g}, \mathbf{u}_{1g})}$, \mathbf{k}_{r_1} is the spatial wave vector at r_1 , and \mathbf{u}_{1g} is the spatial velocity (with magnitude u_{1g}) of the moving transmitter in Schwarzschild spacetime. Applying unmodified Eq. (19), which relates the frequencies of a moving observer at r_2 and a static observer at r_1 , the frequency of the moving receiver at r_2 is given by

$$\omega'_2 = \omega_1 (1 - u_{2g}^2/c^2)^{-\frac{1}{2}} \left(1 - \frac{u_{2g}}{c} \cos \psi_2 \right) \left(\frac{1 - \alpha/r_1}{1 - \alpha/r_2} \right)^{\frac{1}{2}}, \quad (75)$$

where $\cos \psi_2 = \mathbf{g}(\mathbf{k}_{r_2}, \mathbf{u}_{2g}) / \sqrt{\mathbf{g}(\mathbf{k}_{r_2}, \mathbf{k}_{r_2}) \mathbf{g}(\mathbf{u}_{2g}, \mathbf{u}_{2g})}$, \mathbf{k}_{r_2} is the spatial wave vector at r_2 , and \mathbf{u}_{2g} is the spatial velocity (with magnitude u_{2g}) of the moving receiver in Schwarzschild spacetime. Then, we have

$$\omega'_2 = \omega'_1 \left(\frac{c^2 - u_{1g}^2}{c^2 - u_{2g}^2} \right)^{\frac{1}{2}} \left(\frac{c - u_{2g} \cos \psi_2}{c - u_{1g} \cos \psi_1} \right) \left(\frac{1 - \alpha/r_1}{1 - \alpha/r_2} \right)^{\frac{1}{2}}. \quad (76)$$

Eq. (76) is extended from (19). As determined in the three scenarios above, the specific parameters in (76) are related to the trajectories of the transmitter and receiver.

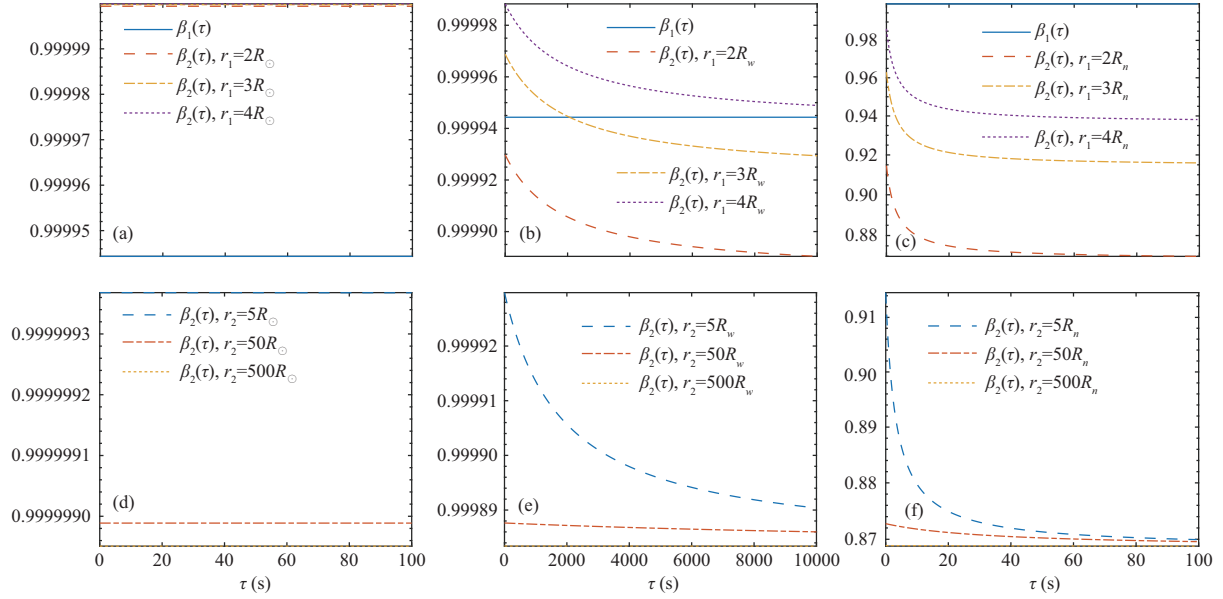


Figure 5 (Color online) FSPs from gravitational effect (GFSP; top panels) and gravitationally modified Doppler effect (GMDFSP; bottom panels) vs. time for a spaceship moving away from (a), (d) the Sun, (b), (e) a white dwarf, and (c), (f) a neutron star. Results are plotted for different initial positions of the receiver and different transmitter positions.

4 Numerical examples

This section illustrates the above analyses with some numerical examples. The studied stars are a Sun-like star, a white dwarf, and a neutron star. The approximate mass M_{\odot} and radius R_{\odot} of the Sun are 1.9891×10^{30} kg and 6.955×10^8 m, respectively [31]. The mass and radius of the Sun-like star are set to the Sun's mass and radius, respectively. The mass M_w and radius R_w of the white dwarf are set to M_{\odot} and $0.9\%R_{\odot}$ (nearly the radius of the Earth), respectively. The mass M_n and radius R_n of the neutron star M_n are set to $2.01M_{\odot}$ and $1.71 \times 10^{-5}R_{\odot}$ or 12 km, respectively. All of these settings are consistent with [32–39].

4.1 Spaceship moving away from a star

In this subsection, we consider a signal transmitted from a fixed position to a spaceship moving away from the Sun, a white dwarf, or a neutron star as shown in Figure 2. In all examples, $u_h = dr_2/dT$ is set to the third cosmic velocity (escape velocity of the solar system) 16.7 km/s.

In Figure 5(a)–(c), the initial distances of the receiver to the center of the Sun, white dwarf, and neutron star, at which the receiver obtains its first signals, are $5R_{\odot}$, $5R_w$, and $5R_n$, respectively. In the solar system, the gravitational effect is negligibly smaller than gravitationally modified Doppler effect, because GFSP $\beta_2(\tau)$ is very close to 1 in all three cases. As $\beta_1(\tau)$ and $\beta_2(\tau)$ are both nearly constant, the non-bandlimited-ness due to the gravitationally modified Doppler effect and gravitational effect is negligible in this system. In contrast, when the transmitter and receiver are both near the white dwarf, the gravitational effect and gravitationally modified Doppler effect are comparable and the non-bandlimited-ness contributed by the gravitational effect is noticeable. When the transmitter is close to the neutron star, the gravitationally modified Doppler effect is negligibly smaller than the gravitational effect, and the frequency rate contributed by $\beta_2(\tau)$ is huge, causing a high non-bandlimited-ness. When the receiver is placed far from the star, $\beta_2(\tau)$ is nearly constant in the communications of all star systems (solar, white dwarf, and neutron star systems; see the curves of $r_2 = 500R_{\odot}$, $500R_w$, and $500R_n$ in Figure 5(d)–(f), respectively). When the receiver is far away from the star, it is negligibly affected by the star's gravitation. Consequently, the frequency rate is very small and the non-bandlimited-ness is barely noticeable.

Figure 6 shows the difference between the gravitationally modified Doppler effect $\beta_1(\tau)$ and the Doppler effect (special relativity version) $\beta'_1(\tau)$, and the difference between the time τ of a moving receiver and the

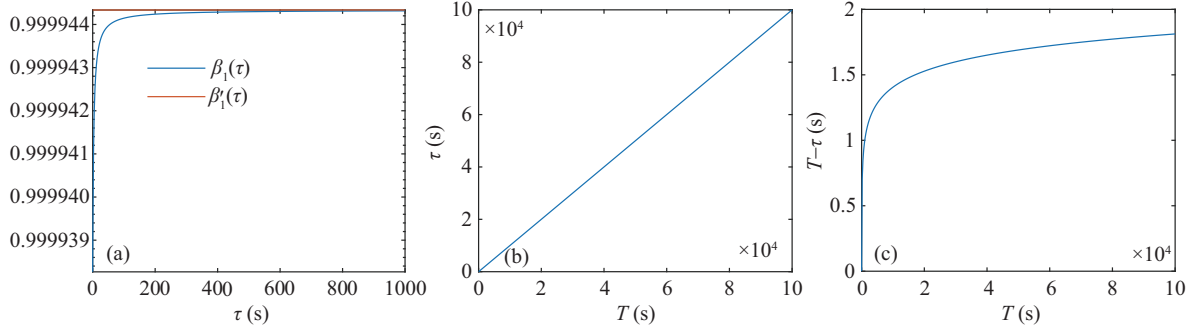


Figure 6 (Color online) (a) Difference between the Doppler effect $\beta_1'(\tau)$ (special relativity version) and the gravitationally modified Doppler effect $\beta_1(\tau)$ as a spaceship moves away from a neutron star at a fixed Newtonian velocity; (b) time τ of the moving spaceship vs. Newtonian time T ; (c) difference between τ and T vs. T . The receiver is initially located at $5R_n$.

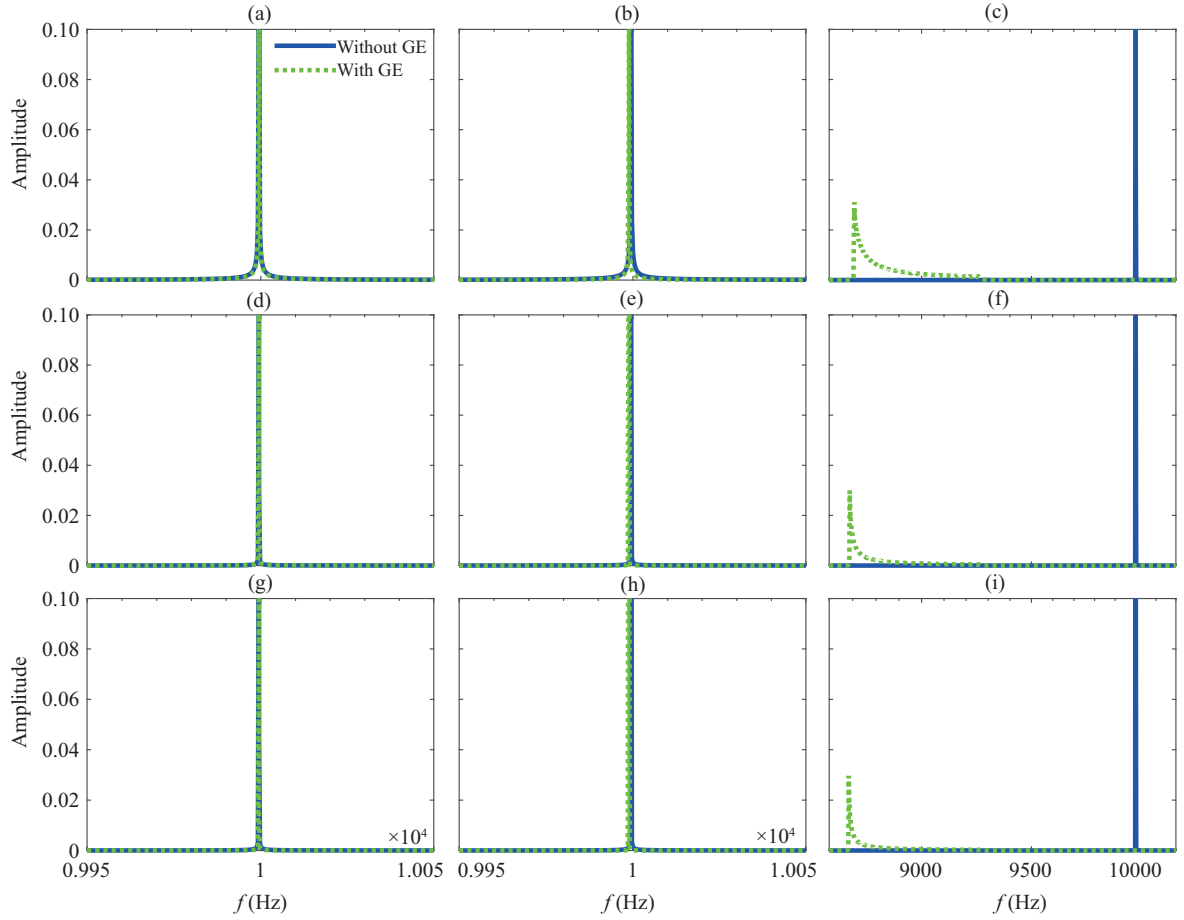


Figure 7 (Color online) FFTs of a 10^4 Hz signal transmitted in (a), (d), (g) the solar system, (b), (e), (h) a white dwarf system, and (c), (f), (i) a neutron star system with or without gravitational effects at $\tau = 50$ s (top panels), 150 s (center panels), and 250 s (bottom panels). The transmitters in the solar, white dwarf, and neutron star systems are located at $2R_\odot$, $2R_w$, and $2R_n$, respectively. The initial positions of the receivers are $4R_\odot$, $4R_w$, and $4R_n$, respectively. The sampling rate is 2.5 times the signal frequency.

Newtonian time T . As the gravitation is much higher in a neutron star system than in other star systems, these differences are much more evident in the neutron star system. Therefore, the neutron star system is selected for highlighting these differences. As shown in Figure 6(a), the gravitationally modified Doppler effect $\beta_1(\tau)$ exceeds the Doppler effect $\beta_1'(\tau)$ because $\beta_1'(\tau)$ is close to 1, but the difference gradually tends to zero as the receiver moves away from the neutron star. Although τ increases approximately linearly with T (Figure 6(b)), the difference between τ and T increases dramatically at the beginning and later

increases linearly with T (Figure 6(c)).

Figure 7 demonstrates the non-bandlimited-ness of a signal in these star systems. The FFTs of a special bandlimited signal — a single-frequency complex exponential signal are calculated at different times during signal transmission through the solar system, the white dwarf system, and the neutron star system. In the solar system, the FFTs with and without gravitational effects are nearly identical in the same time periods. The bands with and without the gravitational effect are both very narrowly distributed around the centric frequency 10^4 Hz. In the white dwarf system, the FFTs are also closely located in frequency, but the FFTs slightly translate to the left under the gravitational effect. The bands with and without gravitational effect are both very narrowly centered about their slightly different frequencies. At small time periods in the neutron star system, the FFT peaks at much lower frequency and spans over a much broader frequency band under the gravitational effect than that without the gravitational effect. At later times, the band under the gravitational effect becomes narrower.

4.2 Spaceship moving around a star with conic trajectories

This subsection considers a signal transmitted from a fixed position to a spaceship moving around the Sun, a white dwarf, or a neutron star with an elliptic, a parabolic, or a hyperbolic trajectory as shown in Figure 3.

As shown in Figure 8(a), (d), and (g), the gravitational effect is negligibly smaller than the gravitationally modified Doppler effect in all three cases in the solar system. Accordingly, the non-bandlimited-ness contributed by the gravitational effect is negligible in this system. In the white dwarf system (Figure 8(b), (e), and (g)), the differences between $\beta(\tau)$ and $\beta_1(\tau)$ are small but noticeable for all three trajectories. Therefore, the gravitational effect should be considered in the white dwarf system. However, the non-bandlimited-ness contributed by the gravitational effect is negligible in all trajectories because the variation in $\beta_2(\tau)$ is small. When the spaceship moves around the neutron star (Figure 8(c), (f), and (i)), the gravitational effect becomes significant. The variations in $\beta_2(\tau)$ are very noticeable, especially in the elliptic trajectory, and the non-bandlimited-ness contributed by the gravitational effect cannot be ignored. However, the variation in $\beta_1(\tau)$ is more significant, so the non-bandlimited-ness contributed by the gravitationally modified Doppler effect is more significant than that contributed by the gravitational effect.

In all systems (the Sun, white dwarf, and neutron star systems), the non-bandlimited-ness is mainly contributed by the gravitationally modified Doppler effect. Furthermore, in the elliptic trajectories, when the spaceship is located its closest or farthest distance to the transmitter (see positions A and B in Figures 3 and 8), the frequency rate derived from $\beta_1(\tau)$ is high and leads to a high non-bandlimited-ness. In the parabolic and hyperbolic trajectories, the frequency rates are also high when the spaceship is located near its closest position to the transmitter (see positions C and D in Figures 3 and 8), but are nearly zero when the spaceship moves far from the transmitter. The levels of non-bandlimited-ness differ among the systems, being highest in the neutron star system, slight in the white dwarf system, and negligible in the solar system.

As the gravitational effects in the solar system and white dwarf system are insignificant and the variation in $\beta_2(\tau)$ is more significant in the elliptic trajectory than that in the other trajectories, we analyze only the spectrum of a signal transmitted to a spaceship following an elliptic trajectory in the neutron star system. The spectral results are presented in Figure 9. When accounting for the gravitational effect $\beta_2(\tau)$ only, the single-frequency complex exponential signal expands over a noticeable frequency band. This confirms that the gravitational effect contributes non-negligibly to the non-bandlimited-ness. When considering the gravitationally modified Doppler effect $\beta_1(\tau)$ only, the signal expands over even broader frequency band. The spectrum of the whole effect $\beta(\tau)$ appears similar to that of the gravitationally modified Doppler effect, confirming that non-bandlimited-ness is mainly determined by the gravitationally modified Doppler effect.

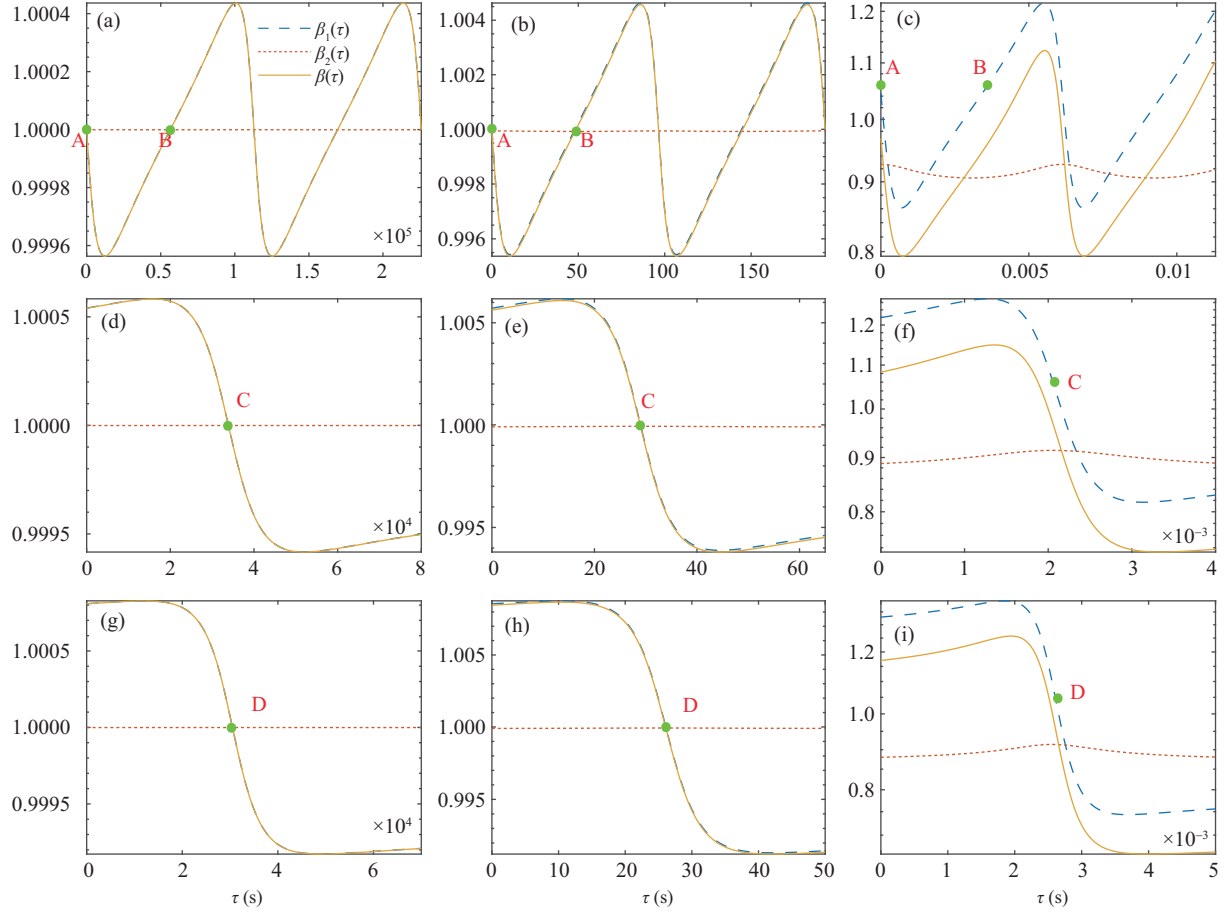


Figure 8 (Color online) FSPs of the gravitationally modified Doppler effect ($\beta_1(\tau)$), gravitational effect ($\beta_2(\tau)$), and the whole frequency shift parameter ($\beta(\tau)$), as functions of time. The spaceship moves around (a), (d), (g) the Sun, (b), (e), (h) a white dwarf, and (c), (f), (i) a neutron star with different conic trajectories: an ellipse ($e = 0.2$, top panels), a parabola ($e = 1$, center panels) and a hyperbola ($e = 2$, bottom panels) with $a = 5R_\odot$, $5R_w$ and $5R_n$, respectively.

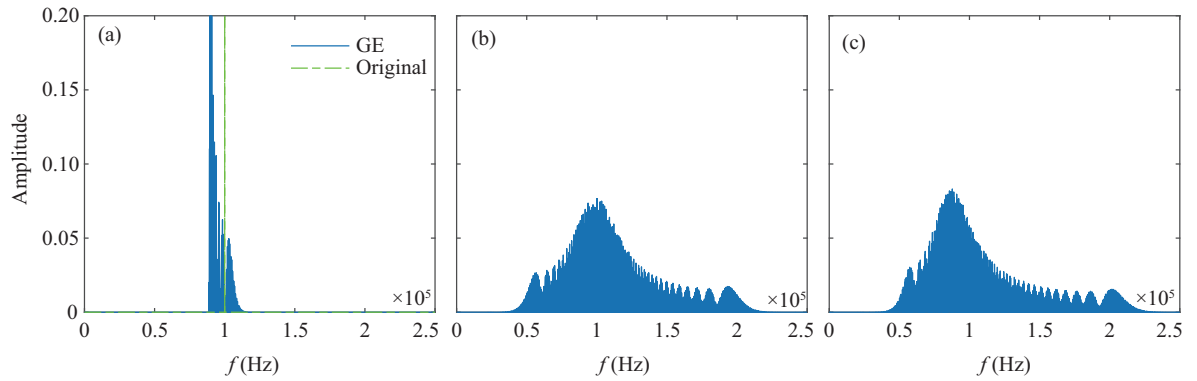


Figure 9 (Color online) FFTs of a 10^5 Hz signal under (a) gravitational effects only ($\beta_2(\tau)$), (b) gravitationally modified Doppler effect only ($\beta_1(\tau)$), and (c) the whole effect ($\beta(\tau)$). The signals are transmitted in the neutron star system from $2R_n$ to the spaceship following an elliptic trajectory with $e = 0.7$ and semi-major axis $a = 5R_n$. The sampling rate is 2.5 times the signal frequency.

4.3 Communications between different star systems

This subsection computes the gravitational effect on a signal transmitted from a Sun-like star, a white dwarf, and a neutron star system to another Sun-like star, white dwarf, and neutron star system, respectively, as shown in Figure 4. The receiver moves with an elliptic trajectory. Assuming that the

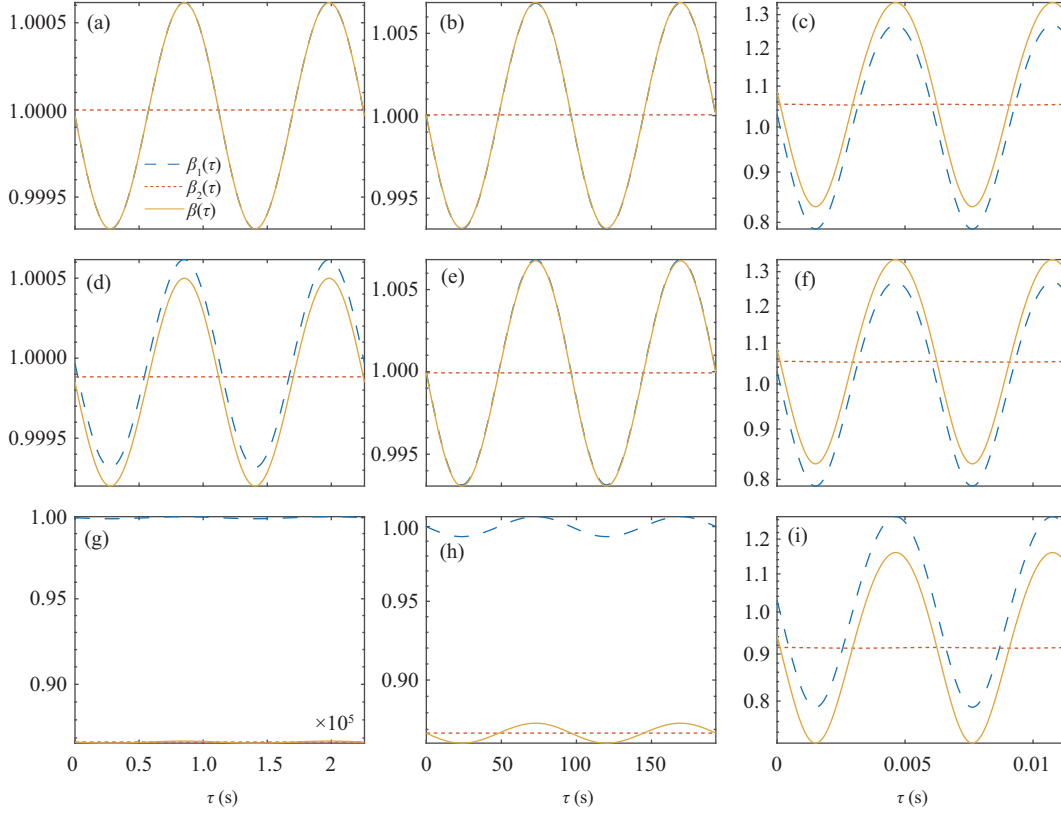


Figure 10 (Color online) FSPs of the gravitationally modified Doppler effect ($\beta_1(\tau)$), gravitational effect ($\beta_2(\tau)$), and the whole frequency shift parameter ($\beta(\tau)$), as functions of time. The signal is received at a position with an elliptic trajectory ($e = 0.0167$) in (a), (d), (g) a Sun-like star system, (b), (e), (h) a white dwarf system, and (c), (f), (i) a neutron star system, and is transmitted from a fixed position in another Sun-like star, white dwarf, and neutron star systems.

communications last from a few seconds to a few minutes, v_h in (71) is set to 10^4 km/s.

In Figure 10(a), (d), and (g), a signal sent by an intelligent species in another system enters the solar system for a relatively short period of time. If the signal comes from a Sun-like system, the gravitational effect can be neglected. If the signal comes from a white dwarf system, the gravitational effect is comparable to the gravitationally modified Doppler effect. Meanwhile, if the signal comes from a neutron star system, the gravitational effect is significant and the gravitationally modified Doppler shift is relatively small. However, regardless of the signal source system, the FSP $\beta(\tau)$ is nearly constant over short time periods. Consequently, the frequency rate is nearly zero and the non-bandlimited-ness is negligible.

In Figure 10(b), (e), and (h), a signal from a star system is transmitted to a white dwarf system. On one hand, if the source star system is a Sun-like star system or a white dwarf system, the gravitational effect can be neglected. On the other hand, if the source star is a neutron star, the gravitational effect is more significant than the gravitationally modified Doppler effect. Because $\beta_2(\tau)$ is nearly constant, the non-bandlimited-ness contributed by the gravitationally modified Doppler effect dominates in this case. Therefore, neither the gravitational nor the gravitationally modified Doppler effect can be ignored. The gravitational effect provides a large frequency shift, whereas the gravitationally modified Doppler effect generates high non-bandlimited-ness. In any case, the frequency rate is noticeable and the non-bandlimited-ness is significant.

In Figure 10(c), (f), and (i), a neutron star system receives a signal from a star system. Both the gravitationally modified Doppler effect and the non-bandlimited-ness contributed by this effect are significant. However, while $\beta_2(\tau)$ differs largely from 1 and the gravitational effect is significant in all three cases, $\beta_2(\tau)$ is much less variable than $\beta_1(\tau)$. Therefore, the non-bandlimited-ness is mainly contributed by the gravitationally modified Doppler effect.

5 Conclusion

We studied here the variation of signals in Schwarzschild spacetime and derived a general equation for the FSP. The FSP was divided into a G MDFSP component and a GFSP component. In addition, the time rates of a transmitter and receiver may differ. The FSP was calculated as a function of the receiver's time in three scenarios (a)–(c) as described in Subsections 3.1–3.3, respectively. Relative to the gravitationally modified Doppler effect, the gravitational effect in scenario (a) was negligible in the solar system, comparable in the white dwarf system, and more significant in the neutron star system. These results suggests that for deep space missions in the solar system, such as Mars exploration, the gravitational effect can be neglected. In this scenario, non-bandlimited-ness was only contributed by the gravitational effect. The non-bandlimited-ness contributed by the gravitational effect was more significant in the neutron star system than in the other systems. When a receiver was close to, for example, a neutron star, the difference between the gravitationally modified Doppler effect and the special relativity version of the Doppler effect, and that between the time τ of a moving receiver and the Newtonian time T , were both significant. Similarly to scenario (a), the gravitational effect in scenario (b) was insignificant in the solar system, lightly significant in the white dwarf system, and decidedly significant in the neutron star system. The non-bandlimited-ness was mainly contributed by the gravitationally modified Doppler effect in all star systems, but the contribution from the gravitational effect cannot be ignored in the neutron star system. In scenario (c), when the communications were passed between a Sun-like star system and another Sun-like star system (or a white dwarf system and another white dwarf system), the gravitational effect is negligible; however, when a white dwarf communicated with a Sun-like star, the gravitational effect was insignificant only if the signals were received (not transmitted) in the white dwarf star system. If the receiver was located in a Sun-like star system, the gravitational effect became noticeable. In a neutron star, the gravitational effect was noticeable and comparable to gravitationally modified Doppler effect. In all cases of scenario (c), the non-bandlimited-ness was mainly contributed by the gravitationally modified Doppler effect.

Although transmitter position is usually fixed, all results are easily extendible to the case of moving transmitters (see Subsection 3.4). Finally, the investigations will assist our understanding of signal models in deep space communications.

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