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# Bifurcation and chaos in digital filters: identification of periodic solutions

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Dear editor,

It is well known that second-order digital filters with two's complement algorithm may exhibit complex dynamics, including periodic solutions and chaos. In various eminent papers [1–3], the periodic and chaotic trajectories of second-order digital filters have been studied.

Though for certain filter parameters digital filters may exhibit periodic behavior, the relationship between the filter parameters, initial point, trajectory period, and motion pattern has not been fully studied [4–6]. This study has also received widespread attention from researchers in various disciplines [7, 8].

We classify the periodic behavior of second-order digital filters and discuss the relationship between filter parameters, periods of periodic orbits, and trajectory travel patterns. A complete classification of the periodic behavior is given. Interesting dynamic behavior has been demonstrated through meticulous simulation studies.

*Topological transformation of the system.* We study the two-dimensional nonlinear map $\mathcal{F}$ that maps a point $\boldsymbol{x} = (x_1, x_2)^{\mathrm{T}} \in I^2 = [-1, 1)\times[-1, 1)$ into $I^2 = [-1, 1) \times [-1, 1)$, defined as

$$\boldsymbol{x}(k+1) = \mathcal{F}(\boldsymbol{x}(k)) = A\boldsymbol{x}(k) + Bs_k. \tag{1}$$

Here

$$A = \begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 2 \end{bmatrix},$$

and

$$s_k = \begin{cases} -1, & \text{if } bx_1(k) + ax_2(k) \geqslant 1, \\ 1, & \text{if } bx_1(k) + ax_2(k) < -1, \\ 0, & \text{otherwise.} \end{cases}$$

Here $s_k$ is an integer-valued function whose role is to translate the image of the point $\boldsymbol{x}(k) \in I^2$ back into $I^2$. We choose parameters $a$, $b$ as $-2 < a < 2$, $b = -1$ because they are in the stability margin.

For system (1), there exists a linear transformation

$$T = \begin{bmatrix} 1 & 0 \\ \cos\theta & \sin\theta \end{bmatrix},$$

such that

$$A = T \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} T^{-1},$$

where $\cos\theta = a/2,\ 0 < \theta < \pi$ corresponds with the range $-2 < a < 2$.

Defining transformation

$$\begin{pmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \end{pmatrix} = T^{-1} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix},$$

we have

$$\hat{\boldsymbol{x}}(k+1) = \Phi\hat{\boldsymbol{x}}(k) + \hat{B}s_k, \tag{2}$$

where

$$\boldsymbol{x}(k) = \begin{pmatrix} \hat{x}_1(k+1) \\ \hat{x}_2(k+1) \end{pmatrix}, \quad \Phi = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

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$$\hat{B}=\begin{pmatrix}0\\ \dfrac{2}{\sin\theta}\end{pmatrix},$$

and

$$s_k=\begin{cases}-1, & \text{if } \cos 2\theta\hat{x}_1(k)+\sin 2\theta\hat{x}_2(k)\geqslant 1,\\ 1, & \text{if } \cos 2\theta\hat{x}_1(k)+\sin 2\theta\hat{x}_2(k)<-1,\\ 0, & \text{otherwise.}\end{cases}$$

As the linear transformation $T^{-1}$ is one-to-one, reversible, and continuous, its inverse transformation is also continuous. Thus, the linear transformation $T^{-1}$ is a homeomorphic mapping. Hence, the phase portraits of systems (1) and (2) are in fact topologically equivalent. We can understand the behaviors of system (1) by studying the behaviors of system (2).

The phase plane of system (1) is divided into three regions that are defined as follows:

$$\begin{aligned}&D_0=\{(x_1,x_2)\,|\,-1<bx_1+ax_2<1,\ s_k=0\},\\ &D_+=\{(x_1,x_2)\,|\,bx_1+ax_2<-1,\ s_k=1\},\\ &D_-=\{(x_1,x_2)\,|\,bx_1+ax_2>1,\ s_k=-1\}.\end{aligned}\qquad(3)$$

In addition the corresponding phase plane of system (2) is divided into three regions that are defined as follows:

$$\begin{aligned}&\hat{D}_0=\\ &\{(\hat{x}_1,\hat{x}_2)|-1<\cos 2\theta\hat{x}_1+\sin 2\theta\hat{x}_2<1,\ s_k=0\},\\ &\hat{D}_+=\{(\hat{x}_1,\hat{x}_2)|\cos 2\theta\hat{x}_1+\sin 2\theta\hat{x}_2<-1,\ s_k=1\},\\ &\hat{D}_-=\{(\hat{x}_1,\hat{x}_2)|\cos 2\theta\hat{x}_1+\sin 2\theta\hat{x}_2>1,\ s_k=-1\}.\end{aligned}\qquad(4)$$

*Analysis of periodic trajectories that lie on one ellipse.* When

$$(\hat{x}_1(0),\hat{x}_2(0))\in\hat{\Pi}_0=\{(\hat{x}_1,\hat{x}_2)|\hat{x}_1^2+\hat{x}_2^2<1\},\qquad(5)$$

there is no overflow in the system (2); $(\hat{x}_1(i),\hat{x}_2(i))$ $(1\leqslant i)$ lies inside of $\hat{\Pi}_0$. The trajectory of system (2) is on a circle with a radius equal to $d=\sqrt{\hat{x}_1(0)^2+\hat{x}_2(0)^2}$ and the corresponding trajectory of system (1) is on an ellipse: $\Pi_0=\{(x_1,x_2)|x_1^2+(-\frac{\cos\theta}{\sin\theta}x_1+\frac{1}{\sin\theta}x_2)^2=d^2\}$.

*Analysis of trajectories travelling periodically among two ellipses.* Our next step is to determine other periodic solutions for the system.

**Proposition 1.** For system (1), if $\cos\theta>0$, periodic behavior with a period of 2 occurs that satisfies $s_ks_{k+1}<0$ and the periodic points are $(-\frac{1}{1+\cos\theta},\frac{1}{1+\cos\theta})\in D_-$ and $(\frac{1}{1+\cos\theta},-\frac{1}{1+\cos\theta})\in D_+$.

It is obvious that if the initial point chosen is close to the point with periodic 2, it should also satisfy $s_ks_{k+1}<0$, and may also have the same periodic behavior. We have the following theorem.

**Theorem 1.** For system (1), if $\cos\theta>0$, and the initial point $(x_1(0),x_2(0))\in\Pi_2^+\cup\Pi_2^-$, where

$$\begin{aligned}\Pi_2^+=\Bigg\{(x_1,x_2)\,\Bigg|\,&\left(x_1-\frac{1}{1+\cos\theta}\right)^2\\ &+\left(-\frac{\cos\theta}{\sin\theta}x_1+\frac{1}{\sin\theta}x_2+\frac{1}{1+\cos\theta}\right)^2\\ &<\left(\frac{\cos\theta}{1+\cos\theta}\right)^2\Bigg\},\end{aligned}$$

$$\begin{aligned}\Pi_2^-=\Bigg\{(x_1,x_2)\,\Bigg|\,&\left(x_1+\frac{1}{1+\cos\theta}\right)^2\\ &+\left(-\frac{\cos\theta}{\sin\theta}x_1+\frac{1}{\sin\theta}x_2-\frac{1}{1+\cos\theta}\right)^2\\ &<\left(\frac{\cos\theta}{1+\cos\theta}\right)^2\Bigg\},\end{aligned}$$

then $s_ks_{k+1}<0$. In addition for $\theta=2\pi q/p$ (where $p$ and $q$ are positive integers satisfying $2q<p$ and $\gcd(p,q)=1$, where gcd stands for greatest common divisor), (i) if $p$ is even, the system has a periodic behavior with a period of $p$; (ii) if $p$ is odd, the system has a periodic behavior with a period of $2p$.

**Remark 1.** In most studies, it is only assumed that $s_ks_{k+1}<0$, and then the corresponding conclusion is obtained. In this study, we can give the area where $(x_1(0),x_2(0))$ exists, which guarantees $s_ks_{k+1}<0$.

*Analysis of trajectories travelling periodically among three ellipses.*

**Proposition 2.** For system (1), if $\cos\theta>\frac{1}{2}$, there exist two periodic behaviors with period 3.

(i) If $\bar{s}=(-\ +\ 0)$, the periodic points are $(0,\frac{2}{1+2\cos\theta})\in D_-$, $(\frac{2}{1+2\cos\theta},\frac{-2}{1+2\cos\theta})\in D_+$, and $(\frac{-2}{1+2\cos\theta},0)\in D_0$.

(ii) If $\bar{s}=(+\ -\ 0)$, the periodic points are $(0,-\frac{2}{1+2\cos\theta})\in D_+$, $(-\frac{2}{1+2\cos\theta},\frac{2}{1+2\cos\theta})\in D_-$, and $(\frac{2}{1+2\cos\theta},0)\in D_0$.

**Theorem 2.** For system (1), if $\cos\theta>1/2$, and the initial point $(x_1(0),x_2(0))\in\Pi_{31}^+\cup\Pi_{31}^-\cup\Pi_{31}^0$, where

$$\begin{aligned}\Pi_{31}^-=\Bigg\{(x_1,x_2)\,\Bigg|\,&\left(x_1+\frac{2}{1+2\cos\theta}\right)^2\\ &+\left(-\frac{\cos\theta}{\sin\theta}x_1+\frac{1}{\sin\theta}x_2-\frac{2}{1+2\cos\theta}\right)^2\\ &<\left(\frac{2\cos\theta-1}{1+2\cos\theta}\right)^2\Bigg\},\end{aligned}$$

$$\Pi_{31}^0=\Bigg\{(x_1,x_2)\,\Bigg|\,\left(x_1-\frac{2}{1+2\cos\theta}\right)^2$$

$$+\left(-\frac{\cos\theta}{\sin\theta}x_1+\frac{1}{\sin\theta}x_2\right)^2<\left(\frac{2\cos\theta-1}{1+2\cos\theta}\right)^2\Bigg\},$$

$$\Pi_{31}^{+}=\Bigg\{(x_1,x_2)|x_1^2+\left(-\frac{\cos\theta}{\sin\theta}x_1+\frac{1}{\sin\theta}x_2+\frac{2}{1+2\cos\theta}\right)^2<\left(\frac{2\cos\theta-1}{1+2\cos\theta}\right)^2\Bigg\},$$

then $\bar{s}=(-+0)$ for $r=q/p$ (where $p$ and $q$ are positive integers satisfying $3q<p$ and $\gcd(p,q)=1$), and (i) system (1) exhibits periodic behaviors with period $p$ if $p=3n$, $n=1,2,\ldots$; (ii) system (1) exhibits periodic behaviors with period $3p$ if $p\neq 3n$, $n=1,2,\ldots$.

**Theorem 3.** For system (1), if $\cos\theta>1/2$, and the initial point $(x_1(0),x_2(0))\in\Pi_{32}^{+}\cup\Pi_{32}^{-}\cup\Pi_{32}^{0}$, where

$$\Pi_{32}^{+}=\Bigg\{(x_1,x_2)\,\Bigg|\left(x_1-\frac{2}{1+2\cos\theta}\right)^2+\left(-\frac{\cos\theta}{\sin\theta}x_1+\frac{1}{\sin\theta}x_2+\frac{2}{1+2\cos\theta}\right)^2<\left(\frac{2\cos\theta-1}{1+2\cos\theta}\right)^2\Bigg\},$$

$$\Pi_{32}^{0}=\Bigg\{(x_1,x_2)\,\Bigg|\left(x_1+\frac{2}{1+2\cos\theta}\right)^2+\left(-\frac{\cos\theta}{\sin\theta}x_1+\frac{1}{\sin\theta}x_2\right)^2<\left(\frac{2\cos\theta-1}{1+2\cos\theta}\right)^2\Bigg\},$$

$$\Pi_{32}^{-}=\Bigg\{(x_1,x_2)|x_1^2+\left(-\frac{\cos\theta}{\sin\theta}x_1+\frac{1}{\sin\theta}x_2-\frac{2}{1+2\cos\theta}\right)^2<\left(\frac{2\cos\theta-1}{1+2\cos\theta}\right)^2\Bigg\},$$

then $\bar{s}=(+-0)$ for $r=q/p$ (where $p$ and $q$ are positive integers satisfying $3q<p$ and $\gcd(p,q)=1$), and (i) system (1) exhibits periodic behaviors with period $p$ if $p=3n$, $n=1,2,\ldots$; (ii) system (1) exhibits periodic behaviors with period $3p$ if $p\neq 3n$, $n=1,2,\ldots$.

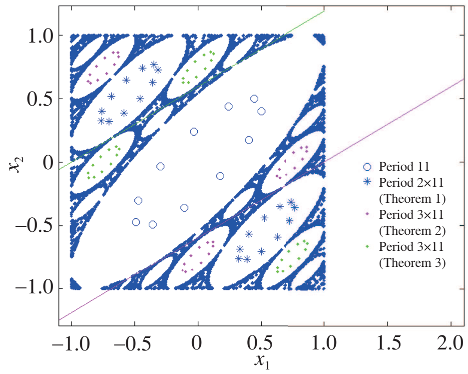


**Figure 1** (Color online) When $\theta=2\pi\frac{1}{p}$ (we choose $p=11$ here), system (1) exhibits periodic behaviors with periods $p$, $2p$, and $3p$ for different initial points.

*Discussion and conclusion.* A framework for studying the dynamic behavior of second-order digital filters has been developed. Conditions to avoid overflow have been derived and various cyclical behaviors have been explored (Figure 1).

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