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Decentralized control for linear systems with multiple input channels

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Abstract In this paper, we consider the decentralized optimal control problem for linear discrete-time systems with multiple input channels. First, under centralized control, the optimal feedback gains are given in terms of two algebraic Riccati equations. A reduced order observer is then designed using only the local input and output information. By selecting an appropriate initial value for the observer, we derive an observer-based decentralized optimal controller where the feedback gain is the same as that obtained in the centralized optimal control problem. Last but not least, we study the optimal control problem of non-homogeneous multi-agent systems as an application. A suboptimal decentralized controller is obtained and the difference between the suboptimal cost and the optimal one is given.

Keywords decentralized control, reduced order observer, multiple channels, algebraic Riccati equation

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1 Introduction

Control problems for large scale systems consisting of a number of subsystems appear in many fields, such as engineering, social, economic and biological systems [1–8]. Centralized control of these systems requires a central station and a communication network to transfer information between subsystems and the central station. This may prevent the implementation of the centralized control due to physical constraints such as high cost of cabling and limited communication bandwidth. It is thus preferred to design a decentralized control scheme using the available local input and output information. Power systems, communication networks and economic systems are examples where decentralized control is used.

A large number of scholars have tried to solve the decentralized control problem [9–15]. The main idea behind designing decentralized controllers is the use of local information to achieve some desired global performance. In [16], a reduced-order observer was proposed to implement decentralized control. A decentralized networked control system was studied in [17] where the decentralized and networked control are combined with the control loops closed through a network. Ref. [6] studied the decentralized H_2 control problem for multi-channel linear time-invariant stochastic systems governed by an Itô's differential equation in terms of a stochastic algebraic Riccati equation (ARE) and a linear matrix inequality. Large scale systems with time delay have been studied in [18]. For a multi-agent system, Ref. [14] designed an observer-based distributed control protocol and derived the necessary and sufficient condition for consensusability under this control protocol. In [19], distributed controllers were designed based on the

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topological structure of the system and suboptimal feedback gain matrices of the distributed controllers were obtained by using an 'averaged' optimization approach. For more related studies on distributed control, please refer to [6, 20, 21] and the references therein. Note that the multi-agent systems in the literature are mostly homogeneous and the study for nonhomogeneous one is much more involved.

This paper considers the decentralized optimal control problem for linear systems with multiple input channels. First, under centralized control, the optimal feedback gains are given in terms of the solutions of two AREs. It is noted that the calculation of the feedback gains is computation saving than the augmentation technique. Second, we design a reduced order observer using only the local input and output information which is available to each subsystem. It is shown that the observer-based decentralized controller is optimal by selecting an appropriate initial value for each local state observer and using the same feedback gain as the one obtained in the centralized optimal control. As an application, we study the optimal control problem for multi-agent systems which are non-homogenous. A suboptimal decentralized control is presented by using the reduced order observer.

The rest of this paper is organized as follows. The problem is formulated in Section 2. Section 3 presents the optimal feedback gains by considering the centralized optimal control problem. The decentralized optimal control problem is studied in Section 4 by a reduced order observer approach. A suboptimal control problem for multi-agent systems is considered in Section 5. One numerical example is given in Section 6. Some concluding remarks are drawn in Section 7.

2 Problem formulation

Consider the system with multiple inputs channels

$$x(k+1) = Ax(k) + \sum_{i=1}^{N} B_i u_i(k), \tag{1}$$

$$y_i(k) = C_i x(k), \quad i = 1, 2, \ldots, N, \tag{2}$$

where $x \in \mathbb{R}^n$ is the state, $u_i \in \mathbb{R}^{m_i}$ and $y_i \in \mathbb{R}^{r_i}$, $i = 1, \ldots, N$ are the i-th control input and measurement output of the system, respectively. $A \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m_i}$, $i = 1, \ldots, N$ are constant matrices. The initial condition is given by $x(0) = x_0$. In particular, it is assumed without loss of generality that $C_i = [I_{r_i}\ 0]$, where I_{r_i} is an identity matrix of dimension r_i. Note that any full row rank matrix can be transformed into this form by an orthogonal transformation matrix [16] and it means that there is no redundant measurement in the output which is the case for many real applications. The cost function is defined by

$$J = \sum_{k=0}^{\infty} \left[x'(k)Qx(k) + \sum_{i=1}^{N} u_i'(k)R_i u_i(k) \right], \tag{3}$$

where Q, R_i, $i = 1, \ldots, N$ are positive semi-definite matrices with compatible dimensions. Superscript "$'$" represents transposition.

We define the admissible set of controllers as

$$\mathcal{U}_{\rm ad}^1 \times \cdots \times \mathcal{U}_{\rm ad}^N = \{(u_1, \ldots, u_N) \in l_2 \times \cdots \times l_2, \text{ such that the associated state } x \in l_2\},$$

where l_2 denotes the set of square summable functions.

The main feature of decentralized control is that it uses only local information to produce control laws. We thus assume that the only information available to the i-th control input in system (1) is the local output y_i. Our aim is to design decentralized optimal controllers $u_i(k)$ to minimize the cost function J subject to (1) and (2) using only information available to the i-th control input.

3 Centralized optimal control problem

Before designing the decentralized optimal controllers, we first consider the centralized control problem $\min_{u_N} \cdots \min_{u_1} J$ from which the decentralized controller gains are derived. For the solvability of the centralized optimal control problem, we make two assumptions which aim to guarantee the stabilizability of the system and the solvability of a standard ARE.

Assumption 1. $(\mathcal{A}_1)$ (A, B_1) is stabilizable; $(\mathcal{A}_2)$ (A, Q) is observable.

Define the AREs

$$P = A'PA + Q - A'PB_1\Gamma_1^{-1}B_1'PA, \tag{4}$$

and

$$L = A'\Upsilon_1'\Upsilon_2' \cdots \Upsilon_N' L(I + \Phi_{N-1}L)^{-1}\Upsilon_N \cdots \Upsilon_2\Upsilon_1 A - A'\Upsilon_1'\Psi_{N-1}\Upsilon_1 A, \tag{5}$$

where

$$\begin{aligned}
&\Gamma_1 = R_1 + B_1'PB_1, \quad \Upsilon_1 = I - B_1\Gamma_1^{-1}B_1'P, \quad \Gamma_2 = R_2 + B_2'P\Upsilon_1 B_2,\\
&\Upsilon_2 = I - \Upsilon_1 B_2\Gamma_2^{-1}B_2'P, \quad \Gamma_3 = R_3 + B_3'P\Upsilon_2\Upsilon_1 B_3, \quad \Upsilon_3 = I - \Upsilon_2\Upsilon_1 B_3\Gamma_3^{-1}B_3'P,\\
&\quad\vdots\\
&\Gamma_N = R_N + B_N'P\Upsilon_{N-1}\cdots\Upsilon_1 B_N, \quad \Upsilon_N = I - \Upsilon_{N-1}\cdots\Upsilon_1 B_N\Gamma_2^{-1}B_N'P,\\
&\Phi_1 = B_1\Gamma_1^{-1}B_1' + \Upsilon_1 B_2\Gamma_2^{-1}B_2'\Upsilon_1, \quad \Psi_1 = PB_2\Gamma_2^{-1}B_2'P,\\
&\quad\vdots\\
&\Phi_{N-1} = B_1\Gamma_1^{-1}B_1' + \Upsilon_1 B_2\Gamma_2^{-1}B_2'\Upsilon_1 + \cdots + \Upsilon_{N-1}\cdots\Upsilon_1 B_N\Gamma_N^{-1}B_N'\Upsilon_1'\cdots\Upsilon_{N-1}',\\
&\Psi_{N-1} = PB_2\Gamma_2^{-1}B_2'P + \Upsilon_2'PB_3\Gamma_3^{-1}B_3'P\Upsilon_2 + \cdots + \Upsilon_2'\cdots\Upsilon_{N-1}'PB_N\Gamma_N^{-1}B_N'P\Upsilon_{N-1}\cdots\Upsilon_2.
\end{aligned}$$

Based on the above denotations, we obtain the necessary and sufficient condition for the solvability of problem $\min_{u_N} \cdots \min_{u_1} J$.

Theorem 1. Under Assumption 1, the optimization problem $\min_{u_N} \cdots \min_{u_1} J$ s.t. (1) has a unique solution if and only if the following statements hold:

(i) ARE (4) has a solution such that $\Gamma_1 > 0, \ldots, \Gamma_N > 0$;

(ii) ARE (5) has a solution such that the matrix $(I + \Phi_{N-1}L)^{-1}\Upsilon_2\Upsilon_1 A$ is stable.

In this case, the centralized optimal controllers are given by $u_i(k) = K_i x(k)$, $i = 1, \ldots, N$, where

$$K_i = -\Gamma_i^{-1}B_i'\Upsilon_1'\cdots\Upsilon_{i-1}'\Upsilon_{i+1}'\cdots\Upsilon_N'[P + L(I + \Phi_{N-1}L)^{-1}\Upsilon_N\cdots\Upsilon_{i+1}\Upsilon_{i-1}\cdots\Upsilon_1]A. \tag{6}$$

Proof. The proof is presented in Appendix A.

Remark 1. By letting $B = [B_1 \ \cdots \ B_N]$, $R = \text{diag}\{R_1, \ldots, R_N\}$ and $u(k) = [u_1'(k) \ \cdots \ u_N'(k)]'$, the optimal solution can also be given as $u(k) = -(R + B'PB)^{-1}B'PAx(k)$, where P is the solution to the standard Riccati equation $P = A'PA + Q - A'PB(R + B'PB)^{-1}B'PA$. The above is commonly referred to as augmentation approach which has more expensive computation than the presented approach, especially when the number of the input channels and/or the dimension of the inputs are large. In fact, if we apply the augmentation approach to the problem, a Riccati equation with an inversion of a matrix with dimension $\sum_{i=1}^N m_i$ is to be solved and thus the operation number (the total number of multiplication and division) can be roughly estimated as $O((\sum_{i=1}^N m_i)^3) + O(n^2\sum_{i=1}^N m_i) + O(n(\sum_{i=1}^N m_i)^2) + O(n^3)$. If we apply the presented approach in this paper, we need to solve the two Riccati equations, in which the computation cost can be roughly computed as $O(n^3) + Q(n^2\sum_{i=1}^N m_i) + Q(n(\sum_{i=1}^N m_i)^2) + Q(\sum_{i=1}^N m_i^3)$. It is easy to know that $O((\sum_{i=1}^N m_i)^3) + O(n^2\sum_{i=1}^N m_i) + O(n(\sum_{i=1}^N m_i)^2) + O(n^3) \gg O(n^3) + Q(n^2\sum_{i=1}^N m_i) + Q(n(\sum_{i=1}^N m_i)^2) + Q(\sum_{i=1}^N m_i^3)$ when $\sum_{i=1}^N m_i$ is large.

4 Decentralized optimal control problem

We now consider the decentralized control problem. Introduce the reduced order observer

$$\hat{z}_i(k+1) = E_i\hat{z}_i(k) + L_iy_i(k) + M_iB_iu_i(k), \tag{7}$$

$$\hat{x}_i(k) = V_i\hat{z}_i(k) + W_iy_i(k), \tag{8}$$

where $E_i \in \mathbb{R}^{s_i\times s_i}$, $L_i \in \mathbb{R}^{s_i\times r_i}$, $M_i \in \mathbb{R}^{s_i\times n}$, $V_i \in \mathbb{R}^{n\times s_i}$, $W_i \in \mathbb{R}^{n\times r_i}$ are constant matrices to be determined with E_i stable, and the following equalities hold:

$$M_iB_j = 0 \quad (j = 1,\dots,N, j \neq i), \tag{9}$$

$$M_iA - E_iM_i = L_iC_i, \quad V_iM_i + W_iC_i = I. \tag{10}$$

Note that the solvability of (9) and (10) is the key for the design of the observer (7) and (8). By a similar procedure in [16], Eqs. (9) and (10) can be solved if $(m - m_i)s_i + (n - r_i)s_i + (n - r_i)n \leqslant s_in$ where $m = \sum_{i=1}^N m_i$ or $m_i + r_i > m$ and $s_i \geqslant \frac{(n-r_i)n}{m_i+r_i-m}$. However, the exact solution may not always be found. An alternative procedure for solving matrices M_i and L_i has been proposed in [16]. The details are omitted here to avoid duplications. However, it is noted that a stability bound condition is needed in [16] to ensure the stability of the closed-loop system under the alternative matrices. To this end, a suboptimal algorithm will be given in Section 5.

4.1 Stability analysis

Let the decentralized controllers be

$$u_i(k) = \hat{K}_i\hat{x}_i(k), \quad i = 1,\dots,N, \tag{11}$$

where $\hat{K}_i$ are constant matrices such that $A+B_1\hat{K}_1+\cdots+B_N\hat{K}_N$ are stable. We then have the stability of the closed-loop system (1) with the controllers (11).

Lemma 1. Considering the system (1) and the control law (11), it holds that

$$\lim_{k\to\infty} x(k) = \lim_{k\to\infty} \hat{x}_i(k) = 0, \quad i = 1,\dots,N. \tag{12}$$

Proof. Let $z_i(k) = M_ix(k)$; then

$$z_i(k+1) = M_iAx(k) + M_iB_iu_i(k) = E_iz_i(k) + L_iy_i(k) + M_iB_iu_i(k). \tag{13}$$

Denote $e_i(k) = z_i(k) - \hat{z}_i(k)$; then $e_i(k+1) = E_ie_i(k)$. Using the stability of the matrix E_i, we have $\lim_{k\to\infty} e_i(k) = 0$. Combining with (10) and (8), it yields

$$\begin{aligned}\lim_{k\to\infty}[x(k) - \hat{x}_i(k)] &= \lim_{k\to\infty}[V_iM_ix(k) + W_iC_ix(k) - V_i\hat{z}_i(k) - W_iy_i(k)]\\ &= \lim_{k\to\infty}[V_iM_ix(k) - V_i\hat{z}_i(k)] = \lim_{k\to\infty}[V_ie_i(k)] = 0.\end{aligned} \tag{14}$$

In addition, applying the controller (11) to (1) yields that

$$x(k+1) = \left[A + \sum_{i=1}^N B_i\hat{K}_i\right]x(k) + \sum_{i=1}^N B_i\hat{K}_i[\hat{x}_i(k) - x(k)].$$

From the stability of $A+\sum_{i=1}^N B_i\hat{K}_i$ and (14), one has $\lim_{k\to\infty} x(k) = 0$. The proof is now completed.

Remark 2. In [11], it has been obtained that there exist decentralized linear time-invariant local control laws with dynamic compensation to stabilize a given system if and only if the system has no unstable fixed modes which correspond to the uncontrollable modes and unobservable modes in the usual centralized control case. In Lemma 1, the above conditions hold. On one hand, the feedback gains $\hat{K}_i$ are selected such that $A + B_1\hat{K}_1 + \cdots + B_N\hat{K}_N$ is stable which implies that the uncontrollable modes are stable. On the other hand, the reduced order observer (7) and (8) satisfying (9) and (10) with E_i being stable implies that the unobservable modes are stable. Hence, the system can be stabilized by a decentralized control.

4.2 Decentralized optimal controllers

Design the decentralized controllers

$$u_i = K_i\hat{x}_i(k), \quad i = 1, \ldots, N, \tag{15}$$

where the feedback gains K_i as given by (6). We now show the decentralized controllers (15) is optimal.

Theorem 2. Consider system (1) and (2) satisfying Assumption 1. The decentralized controllers (15) are optimal in minimizing the cost function (3) if the initial values of the reduced order observers (7) and (8) are given by

$$\hat{z}_i(0) = M_i x(0), \quad i = 1, \ldots, N. \tag{16}$$

Proof. From $z_i(k) = M_i x(k)$ and (10), we have $x(k) = [V_i\ W_i][\begin{smallmatrix} z_i(k) \\ y_i(k) \end{smallmatrix}]$. Using again (10) yields that

$$\begin{bmatrix} z_i(k) \\ y_i(k) \end{bmatrix} = \begin{bmatrix} M_i \\ C_i \end{bmatrix} x(k).$$

Assume that (16) holds; then $z_i(0) = \hat{z}_i(0)$. From the fact that $z_i(k+1) - \hat{z}_i(k+1) = E_i[z_i(k) - \hat{z}_i(k)]$, we have $z_i(k) - \hat{z}_i(k) = 0$, $\forall k \geqslant 0$. This implies that $\hat{x}_i(k) = x(k)$, $i = 1, \ldots, N$ for all $k \geqslant 0$. Combining with Theorem 1, the proof is now completed.

5 Application to multi-agent systems

As an application, we now consider the multi-agent system with the dynamic of the i-th agent given by

$$x_i(k+1) = A_i x_i(k) + B_i u_i(k), \tag{17}$$

$$y_i(k) = C_i x_i(k), \quad i = 1, \ldots, N, \tag{18}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^{m_i}$, $y_i \in \mathbb{R}^{r_i}$. By stacking all x_i into one variable $x(k) = [x_1(k)\ \cdots\ x_N(k)]'$, we have

$$\begin{aligned} x(k+1) &= \bar{A}x(k) + \bar{B}_1 u_1(k) + \cdots + \bar{B}_N u_N(k), \\ y_i(k) &= \bar{C}_i x(k), \end{aligned} \tag{19}$$

with $\bar{A} = \mathrm{diag}\{A_1, \ldots, A_N\}$, $\bar{B}_i = [0\ \cdots\ 0\ B_i\ 0\ \cdots\ 0]'$, $\bar{C}_i = [0\ \cdots\ C_i\ \cdots\ 0]$. The cost function of the system is defined as the general quadratic form

$$\bar{J} = \sum_{k=0}^{\infty} \left[x'(k)\bar{Q}x(k) + \sum_{i=1}^{N} u_i'(k)\bar{R}_i u_i(k) \right], \tag{20}$$

where $\bar{Q} \geqslant 0$, $\bar{R}_i > 0$, $i = 1, \ldots, N$. Similar to Assumption 1, we make Assumption 2 for the multi-agent system.

Assumption 2. System (A_i, B_i), $i = 1, \ldots, N$ is stabilizable.

Denote $\bar{B} = [\bar{B}_1\ \cdots\ \bar{B}_N]$. Then, the system $(\bar{A}, \bar{B})$ is stabilizable. This implies that the following Riccati equation admits a nonnegative definite solution:

$$\bar{P} = \bar{A}'\bar{P}\bar{A} + \bar{Q} - \bar{A}'\bar{P}\bar{B}(\mathrm{diag}\{\bar{R}_1, \ldots, \bar{R}_N\} + \bar{B}'\bar{P}\bar{B})^{-1}\bar{B}'\bar{P}\bar{A}.$$

Denote

$$[\bar{K}_1'\ \cdots\ \bar{K}_N']' = -(\mathrm{diag}\{\bar{R}_1, \ldots, \bar{R}_N\} + \bar{B}'\bar{P}\bar{B})^{-1}\bar{B}'\bar{P}\bar{A}. \tag{21}$$

Note that Eqs. (9) and (10) are unsolvable for the multi-agent system (17). We thus modify the observer as follows:

$$\hat{z}_i(k+1) = E_i\hat{z}_i(k) + L_i y_i(k) + M_i\bar{B}_i u_i(k), \tag{22}$$

$$\hat{x}_i(k) = V_i \hat{z}_i(k) + W_i y_i(k), \tag{23}$$

where

$$M_i = \begin{bmatrix} 0 \cdots 0 \ m_{i1} \ 0 \cdots 0 \\ \vdots \\ 0 \cdots 0 \ m_{is_i} \ 0 \cdots 0 \end{bmatrix} \in \mathbb{R}^{s_i N \times nN},$$

and E_i is stable. Different from (10), the matrices are assumed to satisfy $M_i \bar{A} - E_i M_i = L_i \bar{C}_i$ and that $\bar{A} + \sum_{i=1}^N \bar{B}_i \bar{K}_i (V_i M_i + W_i \bar{C}_i)$ is stable. Then, the result for multi-agent system is stated as follows.

Theorem 3. Consider system (17) and (18) satisfying Assumption 2 and the associated optimization problem of minimizing $\bar{J}$. A suboptimal decentralized controller is given by

$$u_i(k) = \bar{K}_i \hat{x}_i(k), \quad i = 1, \ldots, N, \tag{24}$$

where $\hat{x}_i(k)$ is defined by the observer (22) and (23) with the initial value $\hat{z}_i(0) = [m_{i1} \ \cdots \ m_{is_i}]' x_i(0)$ and the feedback gain $\bar{K}_i$ is given by (21). The suboptimal cost is given by

$$J^*_{\text{sub}} = x'(0) \Upsilon x(0), \tag{25}$$

where $\Upsilon = \sum_{k=0}^{\infty} [(\tilde{A}^k)'(\bar{Q} + \sum_{i=1}^N (V_i M_i + W_i \bar{C}_i)' \bar{K}_i' \bar{R}_i \bar{K}_i (V_i M_i + W_i \bar{C}_i)) \tilde{A}^k]$ and $\tilde{A} = \bar{A} + \sum_{i=1}^N \bar{B}_i \bar{K}_i (V_i M_i + W_i \bar{C}_i)$.

Proof. Reformulating (19), it yields that $x(k+1) = \bar{A}x(k) + \bar{B}\bar{u}(k)$ where $\bar{u}(k) = [u_1(k) \ \cdots \ u_N(k)]'$. Thus, the optimal controllers can be given by $u_i(k) = \bar{K}_i x(k)$ with $\bar{K}_i$ given by (21). On the other hand, applying similar procedures to the proof of Theorem 2 and letting $z_i(k) = M_i x(k)$, we have $z_i(k+1) = E_i z_i(k) + L_i y_i(k) + M_i \bar{B}_i u_i(k)$. Note that the initial value of the observer $\hat{z}_i(0) = [m_{i1} \ \cdots \ m_{is_i}]' x_i(0)$. By combining with the structure of the matrix M_i, we have $\hat{z}_i(0) = M_i x(0)$. This implies that $\hat{z}_i(0) = z_i(0)$. Together with $\hat{z}_i(k+1) - z_i(k+1) = E_i[\hat{z}_i(k) - z_i(k)]$, it yields that $\hat{z}_i(k) = z(k)$. From (23), it is obtained that

$$\hat{x}_i(k) = (V_i M_i + W_i \bar{C}_i) x(k). \tag{26}$$

Applying the decentralized control (24), the system becomes $x(k+1) = \bar{A}x(k) + \sum_{i=1}^N \bar{B}_i \bar{K}_i \hat{x}_i(k) = \tilde{A}x(k)$, where the matrix $\tilde{A}$ is stable as designed in the observer. Thus, the corresponding suboptimal cost function (25) follows. The proof is now completed.

Remark 3. It is noted that the controller (24) with the observer (22) and (23) is suboptimal rather than optimal as stated in Theorem 3. The difference between the suboptimal cost (25) and the optimal cost is given by $\Delta J = x'(0)(\Upsilon - \bar{P})x(0)$. This is due to the fact that (9) and (10) are unsolvable for the system (19), that is, $\hat{x}_i(k)$ is no longer equal to $x(k)$. Instead, we select the matrices E_i, M_i, L_i, V_i, W_i such that $\tilde{A}$ is stable and (26) holds. This gives rise to a suboptimal controller.

6 Numerical example

Consider a modular system existing in teams of vehicles flying in formation which is governed by $g_1(k+1) = g_1(k) + \bar{u}_1(k)$, $g_2(k+1) = 2g_2(k) + \bar{u}_2(k)$, $g_3(k+1) = 3g_3(k) + \bar{u}_3(k)$. The input is $\bar{u}_i(k)$ and the output is given by $\bar{y}_i(k) = g_i(k)$ for $i = 1, 2, 3$. To guarantee the subsystems maintain a special formulation, i.e., $g_1(k) \to 1$, $g_2(k) \to 2$, $g_1(k) \to 3$ for $k \to \infty$, we denote $x_1(k) = g_1(k) - 1$, $x_2(k) = g_2(k) - 2$, $x_3(k) = g_3(k) - 3$, $u_1(k) = \bar{u}_1(k)$, $u_2(k) = \bar{u}_2(k) + 2$, $u_3(k) = \bar{u}_3(k) + 6$. Then the system is reformulated as $x_1(k+1) = x_1(k) + u_1(k)$, $x_2(k+1) = 2x_2(k) + u_2(k)$, $x_3(k+1) = 3x_3(k) + u_3(k)$, where the input is $u_i(k)$ and the output is given by $y_i(k) = x_i(k)$ for $i = 1, 2, 3$. Consider the cost function (20) with the weighting matrices selected as

$$\bar{Q} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix},$$

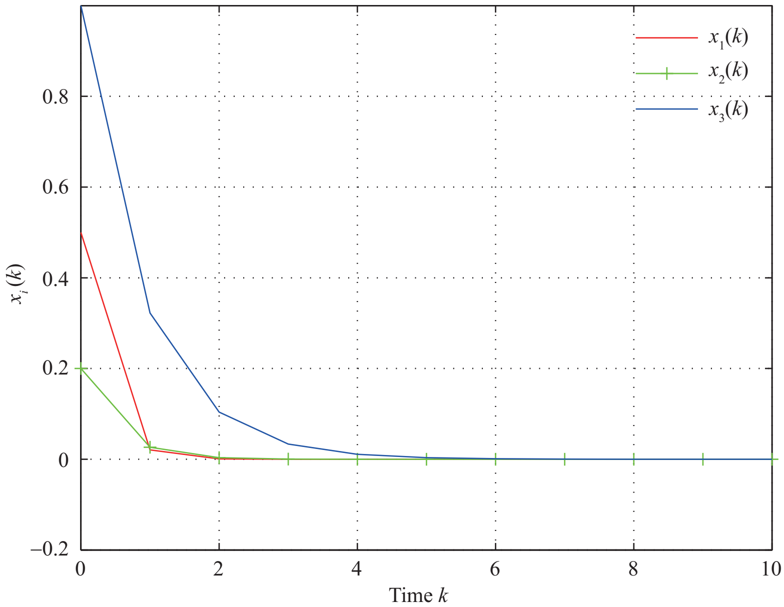

Figure 1 (Color online) The state trajectories with suboptimal distributed controller.

$\bar{R}_1 = 4$, $\bar{R}_2 = 2$, $\bar{R}_3 = 3$. Design the observer (22) and (23) with $E_i = \left[\begin{smallmatrix} 0.5 & 0.3 \\ 1 & -0.5 \end{smallmatrix}\right]$, $i = 1, 2, 3$ and

$$L_1 = [0.7 \ \ -0.5]', \quad L_2 = [-1.5 \ 24]', \quad L_3 = [2 \ 33]',$$

$$M_1 = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 10 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 10 \end{bmatrix},$$

$$V_1 = \begin{bmatrix} 2 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0 & 0 \\ 1.8 & -0.01 \\ 0 & 0 \end{bmatrix}, \quad V_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.1 & 0.1 \end{bmatrix},$$

$$W_1 = [-1 \ 0 \ 0]', \quad W_2 = [0 \ \ -0.5 \ 0]', \quad W_3 = [0 \ 0 \ \ -0.2]'.$$

According to Theorem 3, the suboptimal decentralized controller is chosen by $u_i(k) = \bar{K}_i \hat{x}_i(k)$ where $\bar{K}_1 = [-0.4795 \ 0.0730 \ 0.0502]$, $\bar{K}_2 = [0.0730 \ -1.5561 \ 0.0082]$, $\bar{K}_3 = [0.0223 \ 0.0036 \ -2.6775]$. In this case, the dynamic of the augmented system is $x(k+1) = \mathrm{diag}\{0.0409, 0.1326, 0.3225\}x(k)$. The corresponding state trajectories with the initial value $[0.5 \ 0.2 \ 1]'$ are illustrated in Figure 1 which converge to zero for all the agents, that is, $g_1(k) \to 1$, $g_2(k) \to 2$, $g_3(k) \to 3$ for $k \to \infty$.

Following Theorem 3, the suboptimal cost is 254.1311. Noting that the multi-agent systems studied in [14, 19, 20] are homogeneous, the method therein can not be applied. To make a comparison, we derive the optimal cost by using the centralized optimal control as 245.9306. It is seen that the gap between the derived suboptimal cost and the optimal cost is tiny. This indicates the effectiveness of the proposed algorithm.

7 Conclusion

In this paper, we studied the decentralized optimal control problem for linear systems with multiple input channels. The centralized optimal controller was first given in terms of two AREs. By introducing an observer involving only the available information to the subsystem, an observer-based decentralized optimal controller was derived by an appropriate initial state and using the same feedback gain as the one obtained in the centralized optimal control problem. We also obtained a suboptimal decentralized controller by applying the reduced order observer for multi-agent systems. The difference between the suboptimal cost and the optimal one was given.

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Appendix A Proof of Theorem 1

We firstly prove the case of $N = 2$, i.e., under Assumption 1, the optimization problem $\min_{u_2} \min_{u_1} J$ s.t. (1) with $N = 2$ has a unique solution if and only if ARE (4) has a solution such that $\Gamma_1 > 0$ and $\Gamma_2 > 0$, and ARE

$$L = A'\Upsilon_1'\Upsilon_2'L[I + \Phi_1 L]^{-1}\Upsilon_2\Upsilon_1 A - A'\Upsilon_1'\Psi_1\Upsilon_1 A \tag{A1}$$

has a solution such that the matrix $(I + \Phi_1 L)^{-1}\Upsilon_2\Upsilon_1 A$ is stable. In this case, the centralized optimal controllers are given by $u_1(k) = K_1 x(k)$ and $u_2(k) = K_2 x(k)$ where

$$K_1 = -\Gamma_1^{-1} B_1'\Upsilon_2'[P + L(I + \Phi_1 L)^{-1}\Upsilon_2\Upsilon_1]A, \tag{A2}$$

$$K_2 = -\Gamma_2^{-1} B_2'\Upsilon_1'[P + L(I + \Phi_1 L)^{-1}\Upsilon_2\Upsilon_1]A. \tag{A3}$$

"Necessity". The proof of the necessity mainly relies on the maximum principle, that is, the optimal controller satisfies $0 = R_1 u_1(k) + B_1'\lambda(k)$, where $\lambda(k)$ is the solution of the backward adjoint system

$$\lambda(k-1) = A'\lambda(k) + Qx(k). \tag{A4}$$

The detailed proof is divided into four parts. Firstly, we consider the LQR problem with $u_2 = 0$ which shows that ARE (4) has a solution $P \geqslant 0$ such that $\Gamma_1 > 0$. Secondly, the case of $u_2 \neq 0$ is discussed by introducing a new costate. Thirdly, the positive definiteness of Γ_2 is given. Lastly, we obtain the solvability of ARE (A1) and establish the relationship between the new costate and the original state.

(i) The unique solvability of $\min_{u_2}\min_{u_1} J$ s.t. (1) implies that $\min_{u_1} J_1$ s.t. (1) with $u_2=0$ has a unique solution. Together with Assumption 1, there exists a solution $P>0$ to the ARE (4) such that $\Gamma_1>0$ and the matrix $A-\Gamma_1^{-1}B_1'PA$ is stable. In this case, it holds that $\lambda(k)=Px(k+1)$. The detailed proof is referred to literature1).

(ii) In the case of $u_2\neq 0$, the relationship becomes nonhomogeneous, that is, there exists $\zeta(k)$ such that $\lambda(k)=Px(k+1)+\zeta(k)$. Substituting the relationship into $0=R_1u(k)+B_1'\lambda(k)$ yields that $0=\Gamma_1u_1(k)+B_1'PAx(k)+B_1'PB_2u_2(k)+B_1'\zeta(k)$. Using the fact that $\Gamma_1>0$, one has

$$u_1(k)=-\Gamma_1^{-1}[B_1'PAx(k)+B_1'PB_2u_2(k)+B_1'\zeta(k)]. \tag{A5}$$

Then the dynamic of the state becomes

$$x(k+1)=\Upsilon_1Ax(k)+\Upsilon_1B_2u_2(k)-B_1\Gamma_1^{-1}B_1'\zeta(k). \tag{A6}$$

By combining with $\lambda(k)=Px(k+1)+\zeta(k)$ and (A4), we have

$$\begin{aligned}\lambda(k-1)&=A'P\Upsilon_1Ax(k)+A'P\Upsilon_1B_2u_2(k)-A'PB_1\Gamma_1^{-1}B_1'\zeta(k)+A'\zeta(k)+Qx(k)\\&=Px(k)+A'\Upsilon_1'\zeta(k)+A'P\Upsilon_1B_2u_2(k),\end{aligned}$$

where P satisfying ARE (4) has been used in the derivation of the last equality. Accordingly, $\lambda(k)=Px(k+1)+\zeta(k)$ holds where the dynamic of ζ is given by

$$\zeta(k-1)=A'\Upsilon_1'\zeta(k)+A'P\Upsilon_1B_2u_2(k). \tag{A7}$$

We now calculate the cost function. In view of (1) and (A4), it yields that

$$\begin{aligned}x'(k)\lambda(k-1)-x'(k+1)\lambda(k)&=x'(k)Qx(k)-u_1'(k)B_1'\lambda(k)-u_2'(k)B_2'\lambda(k)\\&=x'(k)Qx(k)+u_1'(k)R_1u_1(k)-u_2'(k)B_2'\lambda(k),\end{aligned}$$

where $0=R_1u(k)+B_1'\lambda(k)$ has been inserted in the last equality. Taking summation from 0 to N and letting N tend to ∞ yields that $x'(0)\lambda(-1)=\sum_{k=0}^{\infty}[x'(k)Qx(k)+u_1'(k)R_1u_1(k)-u_2'(k)B_2'\lambda(k)]$. The cost function (3) is then reformulated as

$$J=x'(0)\lambda(-1)+\sum_{k=0}^{\infty}[u_2'(k)R_2u_2(k)+u_2'(k)B_2'\lambda(k)]. \tag{A8}$$

(iii) Consider the problem $\min_{u_2} J$ s.t. (A6) and (A7) where J is given in (A8). Using again the maximum principle, the optimal controller u_2 satisfies that

$$0=(R_2+B_2'P\Upsilon_1B_2)u_2(k)+B_2'P\Upsilon_1Ax(k)+B_2'\Upsilon_1'\zeta(k). \tag{A9}$$

It is now shown that $\Gamma_2=R_2+B_2'P\Upsilon_1B_2$ is invertible. Let $u_2(k)=0,\ k\geqslant 0$; from (A6) and (A7), one has $\zeta(k-1)=0$ and $x(k+1)=\Upsilon_1Ax(k)$ for $k\geqslant 0$. Noting that the matrix Υ_1A is stable, the zero controller $u_2(k)=0$ is stabilizing. Now consider the case of $x(0)=0$; the optimal controller must be $u_2(k)=0,\ k\geqslant 0$ with the corresponding optimal cost of 0. Selecting $u_2(s)=0,\ s>0$ and $u_2(0)\neq 0$ which is arbitrarily chosen to be stabilizing, then $\zeta(k)=0,\ k\geqslant 0$ and the optimal cost can be rewritten from (A8) as $J=u_2'(0)\Gamma_2u_2(0)$ which is strictly positive. This implies that $\Gamma_2>0$. Accordingly, from (A9), we have

$$u_2(k)=-\Gamma_2^{-1}[B_2'P\Upsilon_1Ax(k)+B_2'\Upsilon_1'\zeta(k)]. \tag{A10}$$

Substituting (A10) into (A6) and (A7) yields the Hamiltonian-Jacobi system

$$x(k+1)=\Upsilon_2\Upsilon_1Ax(k)-\Phi_1\zeta(k), \tag{A11}$$

$$\zeta(k-1)=A'\Upsilon_1'\Upsilon_2'\zeta(k)-A'\Upsilon_1'\Psi_1\Upsilon_1Ax(k). \tag{A12}$$

(iv) From the existence and uniqueness of the optimal solution to problem $\min_{u_2}\min_{u_1} J$ s.t. (1), it holds that the system (A11) and (A12) has a unique solution. In view of the stability of the matrix Υ_1A and the admissible set of u_2, we have $x(k)\in l_2$. Thus, it holds that $\lim_{k\to\infty}\zeta(k)=L\lim_{k\to\infty}x(k+1)=0$ for any matrix L. Using the induction technique, we assume that there exists a constant matrix L such that $\zeta(k)=Lx(k+1)$ holds. Substituting it into (A11) yields that $(I+\Phi_1L)x(k+1)=\Upsilon_2\Upsilon_1Ax(k)$. By combining with the uniqueness of solution to (A11) and (A12), one has that $I+\Phi_1L$ is invertible and

$$x(k+1)=(I+\Phi_1L)^{-1}\Upsilon_2\Upsilon_1Ax(k). \tag{A13}$$

Plugging (A13) into (A12), it is obtained that

$$\zeta(k-1)=[A'\Upsilon_1'\Upsilon_2'L(I+\Phi_1L)^{-1}\Upsilon_2\Upsilon_1A-A'\Upsilon_1'\Psi_1\Upsilon_1A]x(k).$$

Thus, $\zeta(k-1)=Lx(k)$ holds where L satisfies (A1). This implies that ARE (A1) has a solution. Note that $x(k)\in l_2$, and then the matrix $(I+\Phi_1L)^{-1}\Upsilon_2\Upsilon_1A$ is stable.

"Sufficiency". Assume that (4) has a positive semi-definite solution such that $\Gamma_1>0,\Gamma_2>0$, and then the matrix $A'\Upsilon_1'$ is stable under Assumption 1. The detailed proof of the sufficiency is consisting of two steps. First, we obtain the optimal controller $u_1(k)$ by completing the square. Second, based on the optimization of the controller u_1 and the corresponding state trajectory and cost function, we derive the optimal controller u_2 by the sufficient maximum principle.

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(i) First, we derive the optimal controller u_1. To this end, we introduce a new variable ζ with the following dynamic:

$$\zeta(k-1) = A'\Upsilon_1'\zeta(k) + A'\Upsilon_1'PB_2u_2(k). \tag{A14}$$

Noting that $u_2 \in l_2$, one has $\lim_{k\to\infty}\zeta(k) = 0$. Using (1), it yields that

$$\begin{aligned} x'(k+1)Px(k+1) - x'(k)Px(k) = {} & x'(k)(A'PA-P)x(k) + 2u_1'(k)B_1'PAx(k) + 2u_2'(k)B_2'PAx(k) \\ & + u_1'(k)B_1'PB_1u_1(k) + 2u_1'(k)B_1'PB_2u_2(k) + u_2'(k)B_2'PB_2u_2(k). \end{aligned}$$

From (1) and (A14), one has

$$\begin{aligned} 2x'(k+1)\zeta(k) - 2x'(k)\zeta(k-1) = {} & 2u_1'(k)B_1'\zeta(k) + 2u_2'(k)B_2'\zeta(k) \\ & + 2x'(k)A'PB_1\Gamma_1^{-1}B_1'\zeta(k) - 2x'(k)A'\Upsilon_1'PB_2u_2(k). \end{aligned}$$

Then, it yields by simple calculation that

$$\begin{aligned} & x'(k+1)Px(k+1) - x'(k)Px(k) + 2x'(k+1)\zeta(k) - 2x'(k)\zeta(k-1) \\ & = -x'(k)Qx(k) - u_1'(k)R_1u_1(k) + [u_1(k) + \Gamma_1^{-1}B_1'PAx(k) + \Gamma_1^{-1}B_1'PB_2u_2(k) + \Gamma_1^{-1}B_1'\zeta(k)]' \\ & \quad \times \Gamma_1[u_1(k) + \Gamma_1^{-1}B_1'PAx(k) + \Gamma_1^{-1}B_1'PB_2u_2(k) + \Gamma_1^{-1}B_1'\zeta(k)] + u_2'(k)B_2'P\Upsilon_1B_2u_2(k) \\ & \quad + 2u_2'(k)B_2'\Upsilon_1'\zeta(k) - \zeta'(k)B_1\Gamma_1^{-1}B_1'\zeta(k). \end{aligned}$$

Accordingly, Eq. (3) can be reformulated as

$$\begin{aligned} J = {} & x'(0)Px(0) + 2x'(0)\zeta(-1) + \sum_{k=0}^{\infty}\Big([u_1(k) + \Gamma_1^{-1}B_1'PAx(k) + \Gamma_1^{-1}B_1'PB_2u_2(k) \\ & + \Gamma_1^{-1}B_1'\zeta(k)]'\Gamma_1[u_1(k) + \Gamma_1^{-1}B_1'PAx(k) + \Gamma_1^{-1}B_1'PB_2u_2(k) + \Gamma_1^{-1}B_1'\zeta(k)] \\ & + u_2'(k)\Gamma_2u_2(k) + 2u_2'(k)B_2'\Upsilon_1'\zeta(k) - \zeta'(k)B_1\Gamma_1^{-1}B_1'\zeta(k)\Big). \end{aligned}$$

In view of the fact that $\Gamma_1 > 0$, the optimal controller of u_1 is given by

$$u_1(k) = -\Gamma_1^{-1}B_1'PAx(k) - \Gamma_1^{-1}B_1'PB_2u_2(k) - \Gamma_1^{-1}B_1'\zeta(k). \tag{A15}$$

(ii) We now aim to obtain the optimal u_2. Considering (A15), the corresponding cost becomes

$$J = x'(0)Px(0) + 2x'(0)\zeta(-1) + \sum_{k=0}^{\infty}\Big(u_2'(k)\Gamma_2u_2(k) + 2u_2'(k)B_2'\Upsilon_1'\zeta(k) - \zeta'(k)B_1\Gamma_1^{-1}B_1'\zeta(k)\Big). \tag{A16}$$

Substituting (A15) into (1), we can rewrite the states as follows:

$$x(k+1) = \Upsilon_1Ax(k) + \Upsilon_1B_2u_2(k) - B_1\Gamma_1^{-1}B_1'\zeta(k). \tag{A17}$$

Combining with (A14) yields that

$$x'(k+1)\zeta(k) - x'(k)\zeta(k-1) = u_2'(k)B_2'\Upsilon_1'\zeta(k) - \zeta'(k)B_1\Gamma_1^{-1}B_1'\zeta(k) - x'(k)A'\Upsilon_1'PB_2u_2(k).$$

Together with the fact that $\lim_{k\to\infty}\zeta(k) = 0$ and $\lim_{k\to\infty}x(k) = 0$, it is further obtained that

$$x'(0)\zeta(-1) = -\sum_{k=0}^{\infty}\Big(u_2'(k)B_2'\Upsilon_1'\zeta(k) - \zeta'(k)B_1\Gamma_1^{-1}B_1'\zeta(k) - x'(k)A'\Upsilon_1'PB_2u_2(k)\Big). \tag{A18}$$

Plugging (A18) into (A16), one has

$$J = x'(0)Px(0) + x'(0)\zeta(-1) + \sum_{k=0}^{\infty}\Big(u_2'(k)[\Gamma_2u_2(k) + B_2'\Upsilon_1'\zeta(k) + B_2'P\Upsilon_1Ax(k)]\Big). \tag{A19}$$

We then apply the sufficiency of the maximum principle and obtain that the optimal controller satisfies $0 = \Gamma_2u_2(k) + B_2'\Upsilon_1'\zeta(k) + B_2'P\Upsilon_1Ax(k)$. Combining with the assumption that $\Gamma_2 > 0$, the optimal controller of u_2 must be

$$u_2(k) = -\Gamma_2^{-1}[B_2'\Upsilon_1'\zeta(k) + B_2'P\Upsilon_1Ax(k)]. \tag{A20}$$

Provided that (A1) has a stabilizing solution, then it holds that $\zeta(k) = Lx(k+1)$. Together with (A15) and (A20), the optimal controllers can be reformulated as $u_1(k) = K_1x(k)$, $u_2(k) = K_2x(k)$ where K_1, K_2 are defined by (A2) and (A3).

For the general case of $N > 2$, the fact of $\Gamma_i > 0$ and the derivation of L in (5) can be derived similarly to that of $\Gamma_2 > 0$ in (3) and (4), respectively. The sufficiency also follows similarly to the case of $N = 2$. This completes the proof.