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Cramer-Rao lower bound-based observable degree analysis

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Dear editor,
Controllability and observability are basic concepts in modern control theory that are based on Kalman filtering (KF) [1,2]. In particular, observability is closely related to state estimation ability [2–4]. The indicator called observable degree (OD) has been used to quantitatively measure the degree of observability [5–7]. However, limited by slow progress of the observability theory, observable degree analysis (ODA) faces many difficulties in theory and applications.

In general, there are two methods through which OD can be analyzed. One is based on the observability matrix, whereas the other uses estimation error covariance, which is a performance measurement index [5–7]. Intuitively, the two methods should be closely related; however, the corresponding study was very incomplete. In one of our previous studies [7], we attempted to explore this issue. In particular, a novel ODA method based on an estimation performance measure (ODAEPM) was presented. It is based mainly on the method used for designing the traditional observability matrix. The discriminant matrix of observable degree can be constructed using the Cauchy Schwarz inequality, weighted least squares, and the Gramian matrix. Afterwards, the relation between the observable degree and the performance index of the KF is clearly disclosed. However, the following shortcomings still remain [7]:

(1) Owing to the complexity of the derivation, the process noise is not considered to simplify the computation. This directly reduces the application ability.

(2) The ODAEPM cannot effectively deal with incompletely observable systems.

Motivated by these limitations, in this study, we try to extend the result without process noise in [7] by using the Cramer-Rao lower bound (CRLB), which is a basic concept a strong tool in state estimation; it is generally used to measure the availability of the unbiased estimator [8, 9]. The main contribution of this study is to demonstrate the consistency between the ODA in [7] and the CRLB theory by obtaining the relation between the Fisher information matrix and the OD's discriminant matrix. Namely, this study shows that there is a natural relation between the two approaches. Additionally, the CRLB is an effective scheme to study and evaluate the OD for complex systems with process and measurement noises.

Review of ODAEPM [7]. The following stochastic estimation system is considered:

$$\boldsymbol{x}_k = f_k(\boldsymbol{x}_{k-1}, \boldsymbol{w}_{k-1}) = \boldsymbol{\Phi}_{k,k-1}\boldsymbol{x}_{k-1} + \boldsymbol{w}_{k-1}, \quad (1)$$

$$\boldsymbol{z}_k = h_k(\boldsymbol{x}_k, \boldsymbol{v}_k) = \boldsymbol{H}_k\boldsymbol{x}_k + \boldsymbol{v}_k, \quad (2)$$

where k ($k = 1, 2, \ldots$) is the time index, $\boldsymbol{x}_k \in \mathbb{R}^n$ is the state vector, $\boldsymbol{z}_k \in \mathbb{R}^m$ is the observation vector, $\boldsymbol{\Phi}_{k,k-1} \in \mathbb{R}^{n\times n}$ is the state transition matrix, and $\boldsymbol{H}_k \in \mathbb{R}^{m\times n}$ is the observation matrix. $\boldsymbol{w}_{k-1}$ is the process noise with dimension n, and $\boldsymbol{v}_k$ is the observation noise with dimension m; both are

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zero mean Gaussian white noises with covariances $\boldsymbol{Q}_{k-1}$ and $\boldsymbol{R}_k$, respectively.

The discriminant matrix of the OD is

$$\boldsymbol{D}_{1,k}^{*}=\boldsymbol{\Phi}_{k,1}\boldsymbol{D}_{1,k}\boldsymbol{\Phi}_{k,1}^{\mathrm{T}} \tag{3}$$

with the discriminant matrix of observability

$$\begin{aligned}\boldsymbol{D}_{1,k}&=(\boldsymbol{\mathcal{O}}_{1,k}^{\mathrm{T}}\boldsymbol{R}_{1,k}^{-1}\boldsymbol{\mathcal{O}}_{1,k})^{-1}=(\boldsymbol{G}_{1,k}^{0})^{-1}\\&=\left(\sum_{j=1}^{k}\boldsymbol{\Phi}_{j,1}^{\mathrm{T}}\boldsymbol{H}_i^{\mathrm{T}}\boldsymbol{R}_i^{-1}\boldsymbol{H}_i\boldsymbol{\Phi}_{j,1}\right)^{-1},\end{aligned} \tag{4}$$

where $\boldsymbol{\Phi}_{j,1}$ is the state transition matrix from time 1 to j, $\boldsymbol{\mathcal{O}}_{1,k}$ is the classical observability matrix, and $\boldsymbol{R}_{1,k}$ is defined in [7]. In particular, $\boldsymbol{D}_{1,k}$ is the inversion of the observability Gramian matrix. Based on this, the relation between $\boldsymbol{D}_{1,k}$ and the estimation performance of the KF was derived [7]. The local observable degree (LOD) and the global observable degree (GOD) are defined as follows:

$$\Delta\eta_i=\frac{1}{(\boldsymbol{D}_{1,k}^{*})_i},\quad \eta=\frac{1}{\mathrm{Trace}(\boldsymbol{D}_{1,k}^{*})}. \tag{5}$$

Compared with previous methods, the ODAEPM takes measurement noise into account. However, the process noise is still not considered; therefore, process noise is the focus of this study.

Motivation. Based on (1) and (2), we have

$$\boldsymbol{z}_k=\boldsymbol{H}_k\boldsymbol{\Phi}_{k,i}\boldsymbol{x}_0+\sum_{i=1}^{k}\boldsymbol{H}_i\boldsymbol{\Phi}_{k-i,1}\boldsymbol{w}_i+\boldsymbol{v}_k. \tag{6}$$

According to the classical modern control theory, random noise is not considered; the process noise was also not considered for the ODAEPM. However, Eq. (6) shows that determining the initial state value is influenced by the random variable $\boldsymbol{w}_i$. Therefore, it is necessary to consider this influence to improve the effectiveness of the ODAEPM. In addition, the ODAEPM is only applicable for completely observable environments; it cannot deal with incompletely observable cases.

As such, a novel ODM based on the CRLB is proposed to overcome the two shortcomings of the ODAEPM. The core is to find the inner relation between the Fisher information and the OD's discriminant matrices.

ODA based on the CRLB. For (1) and (2), the Fisher information matrix can be written as [8,9]:

$$\boldsymbol{J}_{k+1}=\boldsymbol{D}_k^{22}-\boldsymbol{D}_k^{21}(\boldsymbol{J}_k+\boldsymbol{D}_k^{11})^{-1}\boldsymbol{D}_k^{12}, \tag{7}$$

where

$$\begin{aligned}\boldsymbol{D}_k^{11}&=\mathrm{E}[-\Delta_{\boldsymbol{x}_k}^{\boldsymbol{x}_k}\log p(\boldsymbol{x}_{k+1}|\boldsymbol{x}_k)],\\ \boldsymbol{D}_k^{12}&=\mathrm{E}[-\Delta_{\boldsymbol{x}_k}^{\boldsymbol{x}_{k+1}}\log p(\boldsymbol{x}_{k+1}|\boldsymbol{x}_k)],\\ \boldsymbol{D}_k^{21}&=\mathrm{E}[-\Delta_{\boldsymbol{x}_{k+1}}^{\boldsymbol{x}_k}\log p(\boldsymbol{x}_{k+1}|\boldsymbol{x}_k)]=\boldsymbol{D}_k^{12\,\mathrm{T}},\\ \boldsymbol{D}_k^{22}&=\mathrm{E}[-\Delta_{\boldsymbol{x}_{k+1}}^{\boldsymbol{x}_{k+1}}\log p(\boldsymbol{x}_{k+1}|\boldsymbol{x}_k)]\\&\quad+\mathrm{E}[-\Delta_{\boldsymbol{x}_{k+1}}^{\boldsymbol{x}_k}\log p(\boldsymbol{z}_{k+1}|\boldsymbol{x}_k)].\end{aligned} \tag{8}$$

For the system in [7], we have

$$\begin{aligned}\boldsymbol{D}_k^{11}&=\boldsymbol{\Phi}_{k+1,k}^{\mathrm{T}}\boldsymbol{Q}_k^{-1}\boldsymbol{\Phi}_{k+1,k},\\ \boldsymbol{D}_k^{12}&=-\boldsymbol{\Phi}_{k+1,k}^{\mathrm{T}}\boldsymbol{Q}_k^{-1},\\ \boldsymbol{D}_k^{22}&=\boldsymbol{Q}_k^{-1}+\boldsymbol{H}_{k+1}^{\mathrm{T}}\boldsymbol{R}_{k+1}^{-1}\boldsymbol{H}_{k+1}.\end{aligned} \tag{9}$$

Based on (7) and (9) and using the matrix inversion lemma, one has

$$\begin{aligned}\boldsymbol{J}_{k+1}&=\boldsymbol{H}_{k+1}^{\mathrm{T}}\boldsymbol{R}_{k+1}^{-1}\boldsymbol{H}_{k+1}\\&\quad+(\boldsymbol{Q}_k+\boldsymbol{\Phi}_{k+1,k}^{\mathrm{T}}\boldsymbol{J}_k^{-1}\boldsymbol{\Phi}_{k+1,k})^{-1}.\end{aligned} \tag{10}$$

The CRLB is computed using $\boldsymbol{J}_{k+1}^{-1}$ [8, 9]. When $\boldsymbol{Q}_k=\boldsymbol{0}$, one can get the same discriminant matrix of observable degree with the ODAEPM:

$$\begin{aligned}\boldsymbol{D}_{1,k}^{*}&=(\boldsymbol{H}_k^{\mathrm{T}}\boldsymbol{R}_k^{-1}\boldsymbol{H}_k+\boldsymbol{\Phi}_{k,k-1}^{-\mathrm{T}}\boldsymbol{J}_k\boldsymbol{\Phi}_{k,k-1}^{-1})^{-1}\\&=\boldsymbol{\Phi}_{k,1}^{\mathrm{T}}\left(\sum_{i=1}^{k}\boldsymbol{\Phi}_{i,1}^{\mathrm{T}}\boldsymbol{H}_i^{\mathrm{T}}\boldsymbol{R}_i^{-1}\boldsymbol{H}_i\boldsymbol{\Phi}_{i,1}\right)^{-1}\boldsymbol{\Phi}_{k,1}.\end{aligned} \tag{11}$$

From (10) and (11), we can infer the following: (1) The observable degree of each state component given by the ODAEPM is the reciprocal of the correspondingly conditional, posterior CRLB. Obviously, it is also related to the Fisher information matrix of the state estimation theory.

(2) When $\boldsymbol{Q}_k=\boldsymbol{0}$, the discriminant matrix of the observable degree based on the CRLB degenerates the result obtained in [7]. In other words, the OD's discriminant matrix in [7] is a special case of the Fisher information matrix. However, the expression of the OD's discriminant matrix has more extensive coverage to indicate the observable degree than that of the ODAEPM. Namely, the ODA using the CRLB shown in (11) is more adaptive compared with the ODAEPM for more complex systems with two types of noise.

(3) There is a natural connectivity between the ODAEPM and the Fisher information matrix (or the CRLB). Both functions should be coincident. The observability (the modern control theory) and the estimation theory are also closely related.

(4) The KF is also the base of the model-based state estimation. As such, the control theory and estimation have the same basis from the KF frame.

Moreover, the ODA generally refers to the modern control theory; this naturally leads to connection between the ODA and the state estimation.

Based on (10) and (11), the CRLB can be taken as the OD's discriminant matrix of the ODA method in [7] for systems with process and measurement noises. Thereby, the CRLB can be used to assess the OD, namely, the OD's discriminant matrix of a system (Eqs. (1) and (2)) is

$$\boldsymbol{D}_{1,k}^{*c} = \boldsymbol{J}_k^{-1}, \tag{12}$$

where the CRLB $\boldsymbol{J}_k^{-1}$ is evaluated using (10). Then, the observable degree of the system and the state component are the same as in (5) for a system with the process and observation noises.

Simulation. For the CV model [7, 8], we take

$$\boldsymbol{\Phi}_{k,k-1} = \begin{bmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{H}_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

where t (s) is the sample interval and $\boldsymbol{Q}_{k-1} = 0.01 \times I_{4\times4}$, $\boldsymbol{R}_k = I_{2\times2}$.

The simulation results are shown in Figure 1. In Figure 1, the red line with circles indicates the displacement component (the first state component), whereas the blue line with stars is related to the velocity component.

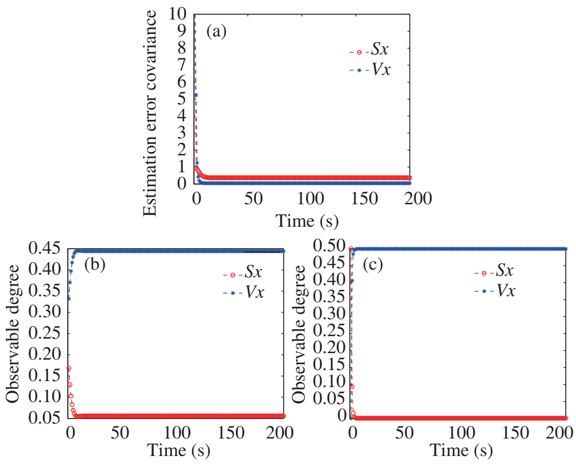

Figure 1 (Color online) (a) Estimation error covariance; (b) result of the proposed ODA; (c) result of the ODAEPM in [7].

From the simulation results, we have: (1) Figure 1(a) shows that the estimation error covariance (estimation accuracy) of the velocity is better than that of the displacement for the x direction. (2) For the proposed ODA and ODAEPM, the orders are the same for the ODs of displacement and velocity. It is also consistent with the result in Figure 1(a). It means that the two ODAs can be used to measure the estimated performance with both having the same function. (3) The OD values of the corresponding state components are different. This is because the proposed method, unlike the ODAEPM, considers the process noise using the CRLB.

Conclusion and future work. When extending the work in [7], inner connectivity is found between the observability (or ODA) and the CRLB based on the OD's discriminant matrix. This indicates that there is a link between the modern control theory and the estimation theory. This study is performed through exploring the ODA method from a different view on the CRLB. It also demonstrates the benefits of looking from multiple perspectives. The result is that the CRLB can be used to measure the observable degree. Thereby, the process noise is perfectly considered in the ODA process. The future work includes extending the study to nonlinear systems such as non-Gaussian systems, and multi-sensor systems.

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References

1 Kalman R E. A new approach to linear filtering and prediction problems. J Basic Eng Trans, 1960, 82: 35–45

2 Zhao G. Modern Control Theory. Beijing: China Machine Press, 2010

3 Ham F, Lin Y. Observability, eigenvalues, and Kalman filtering. IEEE Trans Aerosp Elec Syst, 1983, 19: 269–273

4 Ma J, Ge Q, Shao T. Impact analysis between observable degrees and estimation accuracy of Kalman filtering. In: Proceedings of International Conference on Estimation, Detection and Information Fusion, Harbin, 2015. 124–128

5 Goshen-Meskin D, Bar-Itzhack I Y. Observability analysis of piece-wise constant systems. I. Theory. IEEE Trans Aerosp Electron Syst, 1992, 28: 1056–1067

6 Ma J, Ge Q, Wang Y, et al. Comparison on system observable degree analysis methods for target tracking. In: Proceedings of 2015 IEEE International Conference on Information and Automation, Lijiang, 2015. 1037–1042

7 Ge Q, Ma J, Chen S, et al. Observable degree analysis to match estimation performance for wireless tracking networks. Asian J Control, 2017, 19: 1259–1270

8 Bar-Shalom Y, Li X R, Kirubarajan T. Estimation with Application to Tracking and Navigation. Hoboken: John Wiley and Sons, Inc., 2001

9 Schmitt L, Fichter W. Cramér-Rao lower bound for state-constrained nonlinear filtering. IEEE Signal Process Lett, 2017, 24: 1882–1885