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Digital computation of linear canonical transform for local spectra with flexible resolution ability

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Dear editor,
The linear canonical transform (LCT) is a three-parameter linear integral transform [1]. It has many applications in optical systems, filter design, image watermarking, and other fields [2, 3]. Moreover, the discretization and fastness of the LCT is one of the most important issues in practical applications. Since the continuous LCT was introduced, there has been considerable work on the definition and fast implementation of the LCT [3–6]. These existing discrete LCT (DLCT) algorithms have advantages of high computation speed and accuracy, but they can only calculate the N-point input samples to obtain N-point output. In many applications, such as the detection and estimation of peak values, it is of more interest to study the details in a small portion of the linear canonical spectra or to calculate a single or a few output spectra with an arbitrary sampling interval. It is clear that the previous algorithms with fixed resolution cannot meet these requirements.

To solve the aforementioned problems, we present two flexible algorithms: a novel type of DLCT with zooming-in ability (ZDLCT) and a single-point LCT (Sp-LCT) based on the Goertzel algorithm. Compared to existing digital computation methods, both of our proposed algorithms are more flexible in terms of resolution and observation intervals. Also, the Sp-LCT can be suitable for used to calculate the LCT of a non-uniform sampling signal.

Before deriving the new methods for the computation of LCT, some basic preliminaries are introduced in Appendix A.

The proposed discrete LCT with zooming-in ability. In order to realize the interval selection and simplicity of the ZDLCT algorithm, we first choose an observation interval, and define a value associated with the selected interval, called shift factor as Definition 1.

Definition 1. We call λ the shift factor if it satisfies $\lambda = t_i/W$ and $-0.5 \leqslant \lambda \leqslant 0.5$, where W is a positive real number, $t_i = (t_1 + t_2)/2$ and $[t_1, t_2] \subseteqq [-W/2, W/2]$.

It is clear from its definition that the shift factor represents the relative position of the selected interval's midpoint over the entire interval. Noted that we can translate the frequency index of ZDLCT to the symmetric interval centered on 0 using the shift factor. Then, to observe the details of the spectrum in the selected interval, we define a quantization index associated with the resolution, namely zoom factor.

Definition 2. If a continuous signal $x(t)$ is band-limited in the time domain such that $x = 0$ for $|t| > W/2$, where W is a positive real number, it is sampled with sampling period T over the entire interval $[-W/2, W/2]$. If $\forall [t_1, t_2] \subseteqq [-W/2, W/2]$, the signal $x(t)$ is resampled with the sampling interval $\tilde{T}$, then the ratio $P = T/\tilde{T}$ is the zoom factor. If $P > 1$, it is called the zooming-in factor; otherwise it is called the zooming-out factor.

These two factors will be embedded in the newly

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defined ZDLCT to make the algorithm more convenient. The ZDLCT associated with λ and P is given in Theorem 1.

Theorem 1. If a signal $x(t)$ is approximately limited in the time domain such that $x(t) = 0$ for $|t| > W/2$, where W is a positive real number, the discrete time signal $x(n)$ is a sampled version of $x(t)$ with sampling period T. Then, on an arbitrary LCT frequency subinterval $[u_1, u_2] \subsetneqq [-1/(2T|\beta|), 1/(2T|\beta|) - 1/(NT|\beta|)]$, the ZDLCT admits a representation of the following form:

$$\boldsymbol{L}_A = \sqrt{\beta}\mathrm{e}^{-\mathrm{j}\pi/4}\boldsymbol{x}\boldsymbol{D}^{\gamma}_{\lambda,P}\boldsymbol{K}\boldsymbol{D}^{\alpha,\beta}_{\lambda,P}, \tag{1}$$

where $\boldsymbol{x}$ is a $1 \times N$ vector of the signal $x(n), -N/2 \leqslant n \leqslant N/2-1$, and $\boldsymbol{K}$ is an $N \times M$ matrix. $\boldsymbol{K}_{n,m} = \mathrm{e}^{\mathrm{j}\pi s\frac{(m-n)^2}{NP}}$, $\boldsymbol{D}^{\gamma}_{\lambda,P}$ and $\boldsymbol{D}^{\alpha,\beta}_{\lambda,P}$ are $N \times N$ and $M \times M$ diagonal matrices, respectively,

$$\{\boldsymbol{D}^{\gamma}_{\lambda,P}\}_{n,n} = \{\mathrm{e}^{\mathrm{j}\pi[\gamma(nT)^2 - 2\lambda sn - \frac{sn^2}{PN}]}\}_{n=-\frac{N}{2}}^{\frac{N}{2}-1}, \tag{2}$$

$$\{\boldsymbol{D}^{\alpha,\beta}_{\lambda,P}\}_{m,m} = \{\mathrm{e}^{\mathrm{j}\pi\alpha(\frac{\lambda}{T|\beta|} - \frac{m}{PNT|\beta|})^2 - \mathrm{j}\frac{\pi sm^2}{PN}}\}_{m=-\frac{M}{2}}^{\frac{M}{2}-1}, \tag{3}$$

$s = \mathrm{sign}(\beta)$, $\mathrm{sign}(\cdot)$ represents the sign function, λ is the shift factor, and P is the zoom factor.

Proof. Please refer to Appendix B for this proof.

Eq. (1) becomes the DLCT in [4] when $\lambda = 0$, $P = 1$, and $M = N$. We can adjust λ to change the observation interval, and we can change P to obtain different resolutions. Therefore, the proposed ZDLCT is more flexible in terms of observational intervals and resolution. The implementation of ZDLCT is presented in Appendix C.1. It is obvious that the ZDLCT is related to the multiplication of three matrices, with two of them, namely $\boldsymbol{D}^{\gamma}_{\lambda,P}$ and $\boldsymbol{D}^{\gamma}_{\lambda,P}$, being diagonal matrices. Thus, the dominating complexity component of the proposed ZDLCT is the multiplication of the $N \times M$ matrix with the vector $\boldsymbol{x}$. The conventional arithmetic complexity of a matrix and vector multiplication is $O(MN)$. Meanwhile, it has many interesting properties which are shown in Table 1 and $T_u^{\lambda,P} = \lambda/(T|\beta|) + m/PNT|\beta|$.

The ZDLCT is more effective when the difference between the numbers of input and output points is very small. Meanwhile, the sampling interval T_u requires uniform in the LCT domain. However, when it is necessary to calculate a single or a few output spectra or if the sampling interval is nonuniform, we derive the single-point LCT algorithm.

The proposed single-point LCT. For calculating the spectrum at an arbitrary frequency u_m in the LCT domain, we propose the Sp-LCT, which is based on the Goertzel algorithm [7]. The Goertzel algorithm is typically used to compute the summation of a triangular series through a form of iteration, which has a higher computational efficiency than direct summation. For computing the spectrum of any frequency point in the LCT domain, Eq. (4) can be implemented by performing chirp multiplication, summation of a triangular series, and a second chirp multiplication. The main arithmetic computation of this process is the summation of a triangular series. Therefore, we attempt to utilize the Goertzel algorithm to reduce the computational complexity.

We can convert the equation

$$L_A[x(n)](u) \approx \sqrt{\beta}\mathrm{e}^{-\mathrm{j}\pi/4}\mathrm{e}^{\mathrm{j}\pi\alpha u^2} \times \sum_{n=-N/2}^{N/2-1} x(n)\mathrm{e}^{\mathrm{j}\pi\gamma(nT)^2}\mathrm{e}^{-\mathrm{j}2\pi\beta unT} \tag{4}$$

into the following equation:

$$L_A[x(n)](u_m) = C\mathrm{e}^{\mathrm{j}\pi\alpha u_m^2}\mathrm{e}^{\mathrm{j}\pi\beta u_m TN}B(u_m), \tag{5}$$

where $C = \sqrt{\beta}\mathrm{e}^{-\mathrm{j}\frac{\pi}{4}}\mathrm{e}^{\mathrm{j}\pi\alpha u^2}$, $b_n = x(n)\mathrm{e}^{\mathrm{j}\pi\gamma(nT)^2}$, and $B(u_m) = \sum_{n=0}^{N-1} b_{n-N/2}\mathrm{e}^{-\mathrm{j}2\pi\beta u_m Tn}$. Note that the sequence b_n is dependent on the γ. For different values of the parameter γ, b_n must be recalculated. Due to the calculation of a complex exponential power, this will increase the computational load and make the algorithm inflexible. Thus, we use the following recursive method to overcome this disadvantage.

Take the sequence $D_n = \mathrm{e}^{\mathrm{j}\pi\gamma(2n+1)T^2}, h_n = \mathrm{e}^{\mathrm{j}\pi\gamma(nT)^2}$, $n = -N/2, \ldots, N/2$. It is easy to verify that D_n and h_n have the following recurrence relations:

$$D_{n+1} = \mathrm{e}^{\mathrm{j}2\pi T^2}D_n, \quad h_{n+1} = h_n D_n, \tag{6}$$

and h_n is an even sequence. Therefore, we only need to calculate h_n for $0 \leqslant n \leqslant \frac{N}{2}$ to derive the entire sequence h_n for $-\frac{N}{2} \leqslant n \leqslant \frac{N}{2} - 1$. The recursive method avoids recalculation of the power series for different parameters and uses less storage.

Eq. (5) can be obtained by the Gozertzel algorithm. Firstly, we calculate two the intermediate sequence $g(n)$ and $r(n)$ by iterative algorithm:

$$\begin{aligned} &g(n) = 2g(n-1)\cos(2\pi\omega_m) + r(n-1),\\ &r(n) = b_{n-N/2} - g(n-1),\\ &g(1) = b_{-N/2}, r(1) = b_{1-N/2}, n = 2, 3, \ldots, N-1. \end{aligned} \tag{7}$$

Then we apply the sequence $g(n)$, $r(n)$ to produce output sequence $y(n)$:

$$y(n) = \mathrm{e}^{-\mathrm{j}2\pi n\omega_m}[g(n)\mathrm{e}^{-\mathrm{j}2\pi\omega_m} + r(n)], \tag{8}$$

where $\omega_m = \beta u_m T, n = 1, 2, \ldots, N-1$. The output is equal to $B(u_m)$ at the time index $n =$

Table 1 The properties of ZDLCT

Linear property	$L_A[(ax_1(n)+bx_2(n))](T_u^{\lambda,P}) = aL_A[x_1(n)](T_u^{\lambda,P}) + bL_A[x_2(n)](T_u^{\lambda,P})$
Reverse property	$L_A[x(-n)](T_u^{\lambda,P}) = L_A[x(n)][-(T_u^{\lambda,P})]$
Odd-even property	$L_A\left[x(n)\right](T_u^{\lambda,P}) = L_A[x(n)][-(T_u^{\lambda,P})]$,
	or $L_A\left[x(n)\right](T_u^{\lambda,P}) = -L_A\left[x(n)\right]\left[-(T_u^{\lambda,P})\right]$
Modulation property	$L_A\left[x(n)\mathrm{e}^{\mathrm{j}2\pi\mu nT}\right](T_u^{\lambda,P}) = \mathrm{e}^{-\mathrm{j}\pi\alpha\frac{\mu^2}{\beta^2}}\mathrm{e}^{\mathrm{j}2\pi\alpha T_u^{\lambda,P}\frac{\mu}{\beta}} \times L_A[x(n)](T_u^{\lambda,P}-\frac{\mu}{\beta})$

$N-1$. Finally, the result $B(u_m)$ is multiplied by $C\mathrm{e}^{\mathrm{j}\pi\alpha u_m^2}\mathrm{e}^{\mathrm{j}\pi\beta u_m TN}$, and thus a specific LCT spectrum can be obtained.

The implementation of Sp-LCT is presented in Appendix C.2. This processing needs some phase modulation and $B(u_m)$ operation. For calculating $B(u_m)$, Eq. (7) is implemented $N-1$ times, and Eq. (8) only needs to be computed once. Thus, the corresponding calculation requires $2N+4$ real multiplies and $4N+2$ real additions. The computations of the modulation operation requires 8 real multiplies and 6 real additions. Thus, the total number of calculations (5) associated with the Goertzel algorithm are $2N+12$ real multiplications and $4N+6$ real additions. However, it requires $4N$ real multiplications and $4N-1$ real additions for the direct summation $B(u_m)$, and thus the total number of calculations (5) are $4N+8$ real multiplications and $4N+1$ real additions. This indicates that Sp-LCT has higher computational efficiency than direct summation (4). If u_m is uniform sampling, there are is $4N$ real multiplications and $2N$ real additions by ZDLCT. If the u_m is nonuniform sampling, the ZDLCT is not valid.

Based on above facts, Sp-LCT has higher calculation efficiency for small output sampling and is more flexible in the selection of sampling points than ZDLCT. Furthermore, it is easy to prove that the Sp-LCT has the same properties as ZDLCT. In order to demonstrate the effectiveness and advantages of the proposed algorithms, some simulations are discussed in Appendix D.

Conclusion. In this study, we have introduced the ZDLCT and Sp-LCT for computing local linear canonical spectra with flexible resolution. The properties of the ZDLCT and Sp-LCT have also been discussed to demonstrate their importance. As opposed to existing algorithms, the ZDLCT associated with the shift factor λ and zoom factor P has flexible resolution and selectivity of the spectrum interval. The simulation results showed that the ZDLCT can distinguish spectra when multiple frequency spectra are very closer, which means it can be applied to spectral refinement. In addition, the Sp-LCT is more effective than classical methods for the calculation of a single spectra and it overcomes the limitation of uniform sampling in the existing DLCT and ZDLCT. The simulation results showed that Sp-LCT can realize arbitrary spectrum output and also obtain perfect performance in terms of computational complexity and precision for nonuniform sampling.

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Supporting information Appendixes A–D. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

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