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Input-to-state stability of coupled hyperbolic PDE-ODE systems via boundary feedback control

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Abstract We herein investigate the boundary input-to-state stability (ISS) of a class of coupled hyperbolic partial differential equation-ordinary differential equation (PDE-ODE) systems with respect to the presence of uncertainties and external disturbances. The boundary feedback control of the proportional type acts on the ODE part and indirectly affects the hyperbolic PDE dynamics via the boundary input. Using the strict Lyapunov function, some sufficient conditions in terms of matrix inequalities are obtained for the boundary ISS of the closed-loop hyperbolic PDE-ODE systems. The feedback control laws are designed by combining the line search algorithm and polytopic embedding techniques. The effectiveness of the designed boundary control is assessed by applying it to the system of interconnected continuous stirred tank reactor and a plug flow reactor through a numerical simulation.

Keywords hyperbolic PDE-ODE systems, input-to-state stability, boundary control, Lyapunov function, PFR-CSTR models

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1 Introduction

Many complex engineering processes are represented by hyperbolic partial differential equations (PDEs) coupled with ordinary differential equations (ODEs). Typical examples include the hydraulic model in oil well drilling [1], gas transport model in diesel engines [2,3], freeway traffic model with on-ramp vehicle dynamics [4–6], the screw extrusion process in three-dimensional printing [7], and chemical reaction of interconnected continuous stirred tank reactor (CSTR) and plug flow reactor (PFR) [8].

The backstepping method is primarily used to control the coupled hyperbolic PDE-ODE systems. Recently, a coupled system including a first-order PDE and second-order ODE system is stabilized using this approach in [9]. The predictor-like feedback control laws are designed in [10] for a diffusion PDE (the heat equation) coupled with an LTI ODE model in cascade. In [1], an observer was designed for a class of hyperbolic PDE-ODE cascade systems with the boundary measurement. An infinite-dimensional backstepping method is constructed in [11] to guarantee that the closed-loop PDE-ODE system is exponentially stable. Meanwhile, a strict Lyapunov function based method has been used in [2] to prove that the hyperbolic PDE-ODE system is stable under different time scales. A general framework of methods for proving the stability is given in [12–14], thus allowing the study of a wide class of nonlinear systems.

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It is noteworthy that the input-to-state stability (ISS) has been used in newest research pertaining to the infinite-dimensional systems [15–17]. These studies are parallel to the work of [18] where a linearization principle is applied for a class of infinite-dimensional systems in a Banach space. In [19], the equivalence between the hyperbolic systems and integral delay systems has been established, and sufficient conditions for the ISS are given. For the time-varying hyperbolic PDEs, ISS Lyapunov functions are constructed in [20]. When focusing on the quantized control of linear hyperbolic PDEs, the work to compute the ISS Lyapunov functions also has been performed [21]. The ISS properties of communication networks when operating at some optimal equilibria are studied in [22]. Ref. [23] further derived the ISS bounds for the one-dimensional parabolic systems in the presence of boundary disturbances.

We herein consider a class of hyperbolic distributed parameter systems interacting with a lumped parameter system through a dynamic boundary condition, e.g., the CSTR-PFR models in [8]. The first contribution of this paper is the sufficient matrix inequality conditions for the boundary ISS by means of designing the strictly ISS-Lyapunov function with the weighting parameters restricted to be positive. The numerical computing of the inequality conditions is performed by combining the line search algorithm and the polytopic embedding techniques. Moreover, the proportional boundary control is applied to stabilize the hyperbolic PFR-CSTR model with the spatial un-uniform equilibria. Theoretical analyses guarantee the ISS of the coupled hyperbolic PDE-ODE model with respect to the presence of the uncertainty disturbances.

This paper is organized as follows. In Section 2, we introduce the ISS and ISS-Lyapunov function for a class of coupled hyperbolic PDE-ODE systems. Our primary result on the sufficient condition for the ISS is derived in Section 3. Subsequently, in Section 4, the numerical computational conditions are obtained using the line search algorithm and the polyhedron approximating techniques. Finally, in Section 5, we present an application to the PFR-CSTR configuration as a coupled hyperbolic PDE-ODE system.

2 Coupled hyperbolic PDE-ODE systems

We herein consider a coupled hyperbolic PDE-ODE system of the form:

$$\partial_t \xi(x,t) + \Lambda \partial_x \xi(x,t) = M(x)\xi(x,t) + \delta(x,t), \tag{1}$$

$$\dot{\eta}(t) = A\eta(t) + Bu(t) + \epsilon(t), \tag{2}$$

with the following boundary and initial conditions:

$$\xi(0,t) = \eta(t), \tag{3}$$

$$\xi(x,0) = \xi_0(x), \tag{4}$$

$$\eta(0) = \eta_0, \tag{5}$$

where $x \in (0,L)$, $t \in [0,\infty)$; $\xi : [0,L] \times [0,+\infty) \to \mathbb{R}^n$, and $\eta : [0,+\infty) \to \mathbb{R}^n$ denote the state variables for the hyperbolic PDE and the ODE systems, respectively; $\delta \in L^2(0,L)$ and $\epsilon \in \mathbb{R}^n$ are the bounded disturbances; $u(t) \in \mathbb{R}^n$ is the input variable acting on the ODE part and indirectly affecting the boundary of hyperbolic PDEs at $x = 0$; $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\} > 0$, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$; $M(x)$ is a continuous matrix whose entries are functions in $L_\infty(0,L)$; $\xi_0(x) \in L^2(0,L)$ is a given function and η_0 is a constant vector.

We consider the feedback control $u(t)$ for the coupled system (1)–(5) using only the boundary measurement of hyperbolic PDEs at $x = L$, i.e.,

$$u(t) = K\xi(L,t), \quad t \in [0,\infty), \tag{6}$$

where $K \in \mathbb{R}^{m\times n}$. Our control objective is to design the suitable feedback gain K to achieve the stabilization of the closed-loop system with respect to the presence of the external disturbance $(\delta,\epsilon) \in L^2(0,L) \times \mathbb{R}^n$.

Remark 1. The existence and uniqueness of the solution of such a coupled system (1)–(5) with (6) have been studied in many researches. For instance, according to [24, Theorem A.6], for every $(\xi_0, \eta_0) \in L^2(0,L)\times\mathbb{R}^n$, the Cauchy problem (1)–(6) has one and only one solution $(\xi,\eta) \in C^0([0,\infty), L^2(0,L)\times\mathbb{R}^n)$.

Hereafter, we define the state space X for the system (1)–(5) as the Hilbert space $X = L^2(0,L) \times \mathbb{R}^n$ equipped with the norm

$$\|(\xi,\eta)\|_X^2 = \|\xi\|_{L^2(0,L)}^2 + |\eta|^2, \tag{7}$$

for every $(\xi,\eta) \in X$. Subsequently, we introduce the notion of ISS and the ISS-Lyapunov function for the coupled hyperbolic PDE-ODE system (1)–(5) considered herein (see for instance [20, Definition 1] given in the infinite dimensional context).

Definition 1. The coupled hyperbolic PDE-ODE system (1)–(6) is thought to be input-to-state stable with respect to the disturbance (δ,ϵ), if there exists a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that, for any initial state $(\xi_0,\eta_0) \in X$, the solution (ξ,η) of (1)–(6) satisfies

$$\|(\xi,\eta)\|_X \leqslant \beta(\|(\xi_0,\eta_0)\|_X) + \gamma\left(\sup_{0\leqslant\tau\leqslant t} \|(\delta(\cdot,\tau),\epsilon(\tau))\|_X\right). \tag{8}$$

Definition 2. Let $V : X \times [0,\infty) \to \mathbb{R}$ be a continuously differentiable function such that

$$\alpha_1(\|(\xi,\eta)\|_X) \leqslant V(\xi,\eta,t) \leqslant \alpha_2(\|(\xi,\eta)\|_X), \tag{9}$$

$$\dot{V}(\xi,\eta,t) \leqslant -\lambda V(\xi,\eta,t) + \alpha_3(\|(\delta,\epsilon)\|_X), \tag{10}$$

for all $(\xi,\eta,t) \in X \times [0,\infty)$, and $(\delta,\epsilon) \in X$, where α_1, α_2 are class $\mathcal{K}_\infty$ functions, α_3 is a class $\mathcal{K}$ function, and $\lambda > 0$ is a positive real number. Subsequently, the function V is thought to be an ISS-Lyapunov function for (1)–(5).

When the coupled hyperbolic PDE-ODE system (1)–(5) with (6) admits an ISS-Lyapunov function V satisfying (9)–(10) of definition 2, the following inequality

$$\|(\xi,\eta)\|_X \leqslant \alpha_1^{-1}(2\mathrm{e}^{-\lambda t}\alpha_2(\|(\xi_0,\eta_0)\|_X)) + \alpha_1^{-1}\left(\frac{2}{\lambda}\sup_{0\leqslant\tau\leqslant t}\alpha_3(\|(\delta(\cdot,\tau),\epsilon(\tau))\|_X)\right) \tag{11}$$

holds, for all solutions $(\xi,\eta) \in X$. Let $\beta(\cdot) = \alpha_1^{-1}(2\mathrm{e}^{-\lambda t}\alpha_2(\cdot))$, and $\gamma(\cdot) = \alpha_1^{-1}(2\lambda^{-1}\alpha_3(\cdot))$; using Definition 1 we find that the coupled hyperbolic PDE-ODE system (1)–(6) is input-to-state stable in the norm X. This inequality (11) provides an estimation of the influence of disturbance (δ,ϵ) on the solutions of the coupled systems (1)–(5) with the boundary control (6).

3 ISS stability for hyperbolic PDE-ODE systems

In this section, we first present a sufficient condition for the ISS of the coupled hyperbolic system (1)–(5) using Lyapunov-based techniques. It is solved with the following theorem.

Theorem 1. Consider the coupled hyperbolic PDE-ODE system (1)–(5). If there exist a feedback gain $K \in \mathbb{R}^{m\times n}$, constants $\mu > 0$, $\kappa_i > 0$, and diagonal positive definite matrices $P_i > 0$, $i = 1,2$, such that

$$\begin{bmatrix} A^{\mathrm{T}}P_2 + P_2A + \Lambda P_1 + \kappa_2\lambda_{P_2}I_n + \mu P_2 & P_2BK \\ K^{\mathrm{T}}B^{\mathrm{T}}P_2 & -\mathrm{e}^{-\mu L}\Lambda P_1 \end{bmatrix} \leqslant 0, \tag{12}$$

$$-\mu\Lambda P_1 + M^{\mathrm{T}}(x)P_1 + P_1M(x) + \kappa_1\lambda_{P_1}I_n < 0, \tag{13}$$

where λ_{P_i} is the largest eigenvalue of P_i, the coupled system (1)–(5) with the boundary feedback control (6) is input-to-state stable.

Proof. We begin the proof by choosing an appropriate ISS-Lyapunov function candidate $V: X \to \mathbb{R}$ defined as

$$V(\xi,\eta)=\int_0^L \xi^{\mathrm{T}}P_1\xi \mathrm{e}^{-\mu x}\mathrm{d}x+\eta^{\mathrm{T}}P_2\eta. \tag{14}$$

Computing the time derivative of V along with the solution to (1)–(6) and using the integration by parts, we obtain

$$\begin{aligned}\dot{V} &= -\left[\xi^{\mathrm{T}}\Lambda P_1\xi \mathrm{e}^{-\mu x}\right]_0^L+\int_0^L \xi^{\mathrm{T}}[-\mu\Lambda P_1+M^{\mathrm{T}}(x)P_1+P_1M(x)]\xi\mathrm{e}^{-\mu x}\mathrm{d}x\\ &\quad +\eta^{\mathrm{T}}(A^{\mathrm{T}}P_2+P_2A)\eta+2\xi^{\mathrm{T}}(L,t)K^{\mathrm{T}}B^{\mathrm{T}}P_2\eta+\int_0^L 2\xi^{\mathrm{T}}P_1\delta\mathrm{e}^{-\mu x}\mathrm{d}x+2\eta^{\mathrm{T}}P_2\epsilon\\ &= V_1+V_2+V_3,\end{aligned} \tag{15}$$

with

$$\begin{aligned}V_1 &\triangleq -\left[\xi^{\mathrm{T}}\Lambda P_1\xi \mathrm{e}^{-\mu x}\right]_0^L+\eta^{\mathrm{T}}(A^{\mathrm{T}}P_2+P_2A)\eta+2\xi^{\mathrm{T}}(L,t)K^{\mathrm{T}}B^{\mathrm{T}}P_2\eta+\mu\eta^{\mathrm{T}}P_2\eta,\\ V_2 &\triangleq \int_0^L \xi^{\mathrm{T}}[-\mu\Lambda P_1+M^{\mathrm{T}}(x)P_1+P_1M(x)]\xi\mathrm{e}^{-\mu x}\mathrm{d}x-\mu\eta^{\mathrm{T}}P_2\eta,\\ V_3 &\triangleq \int_0^L 2\xi^{\mathrm{T}}P_1\delta\mathrm{e}^{-\mu x}\mathrm{d}x+2\eta^{\mathrm{T}}P_2\epsilon.\end{aligned}$$

Owing to the Cauchy-Schwarz inequality, it follows that, for all $\kappa_i>0$, $i=1,2$,

$$\begin{aligned}V_3 &\leqslant \lambda_{P_1}\int_0^L\left(\kappa_1\xi^{\mathrm{T}}\xi+\frac{1}{\kappa_1}\delta^{\mathrm{T}}\delta\right)\mathrm{e}^{-\mu x}\mathrm{d}x+\lambda_{P_2}\left(\kappa_2\eta^{\mathrm{T}}\eta+\frac{1}{\kappa_2}\epsilon^{\mathrm{T}}\epsilon\right)\\ &\leqslant \kappa_1\lambda_{P_1}\int_0^L\xi^{\mathrm{T}}\xi\mathrm{e}^{-\mu x}\mathrm{d}x+\kappa_2\lambda_{P_2}\eta^{\mathrm{T}}\eta+\bar{\lambda}\|(\delta,\epsilon)\|_X\\ &= \tau_{31}+\tau_{32}+\bar{\lambda}\|(\delta,\epsilon)\|_X,\end{aligned} \tag{16}$$

with $\tau_{31}\triangleq\kappa_1\lambda_{P_1}\int_0^L\xi^{\mathrm{T}}\xi\mathrm{e}^{-\mu x}\mathrm{d}x$, $\tau_{32}\triangleq\kappa_2\lambda_{P_2}\eta^{\mathrm{T}}\eta$, and $\bar{\lambda}=\max\{\frac{\lambda_{P_1}}{\kappa_1},\frac{\lambda_{P_2}}{\kappa_2}\}$.

By grouping the terms V_1 and τ_{32}, and using the boundary condition (3) of hyperbolic PDEs, we obtain

$$\begin{aligned}V_1+\tau_{32} &= -\left[\xi^{\mathrm{T}}\Lambda P_1\xi \mathrm{e}^{-\mu x}\right]_0^L+\eta^{\mathrm{T}}(A^{\mathrm{T}}P_2+P_2A)\eta+2\xi^{\mathrm{T}}(L,t)K^{\mathrm{T}}B^{\mathrm{T}}P_2\eta+\mu\eta^{\mathrm{T}}P_2\eta+\kappa_2\lambda_{P_2}\eta^{\mathrm{T}}\eta\\ &= \xi^{\mathrm{T}}(0,t)\Lambda P_1\xi(0,t)-\xi^{\mathrm{T}}(L,t)\Lambda P_1\xi(L,t)\mathrm{e}^{-\mu L}\\ &\quad +\eta^{\mathrm{T}}(A^{\mathrm{T}}P_2+P_2A)\eta+2\xi^{\mathrm{T}}(L,t)K^{\mathrm{T}}B^{\mathrm{T}}P_2\eta+\mu\eta^{\mathrm{T}}P_2\eta+\kappa_2\lambda_{P_2}\eta^{\mathrm{T}}\eta\\ &= \begin{bmatrix}\eta\\ \xi(L,t)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}A^{\mathrm{T}}P_2+P_2A+\Lambda P_1+\kappa_2\lambda_{P_2}I_n+\mu P_2 & P_2BK\\ K^{\mathrm{T}}B^{\mathrm{T}}P_2 & -\mathrm{e}^{-\mu L}\Lambda P_1\end{bmatrix}\begin{bmatrix}\eta\\ \xi(L,t)\end{bmatrix}.\end{aligned} \tag{17}$$

It is noteworthy that the condition (12) implies that $V_1+\tau_{32}$ is always negative or zero. Meanwhile, grouping the terms V_2 and τ_{31}, and using the condition (13), we can prove that there exists a small enough real positive $\nu>0$ such that

$$\begin{aligned}V_2+\tau_{31} &= \int_0^L \xi^{\mathrm{T}}[-\mu\Lambda P_1+M^{\mathrm{T}}(x)P_1+P_1M(x)]\xi\mathrm{e}^{-\mu x}\mathrm{d}x-\mu\eta^{\mathrm{T}}P_2\eta+\kappa_1\lambda_{P_1}\int_0^L\xi^{\mathrm{T}}\xi\mathrm{e}^{-\mu x}\mathrm{d}x\\ &\leqslant -\nu\int_0^L\xi^{\mathrm{T}}P_1\xi\mathrm{e}^{-\mu x}\mathrm{d}x-\mu\eta^{\mathrm{T}}P_2\eta\\ &\leqslant -\lambda V(\xi,\eta),\end{aligned} \tag{18}$$

with $\lambda=\min\{\nu,\mu\}$. Combing (16)–(18) yields

$$\dot{V}\leqslant -\lambda V(\xi,\eta)+\bar{\lambda}\|(\delta,\epsilon)\|_X. \tag{19}$$

Following directly from the definition of candidate V and the straightforward estimation, for all solutions $(\xi,\eta)\in X$, there exists a constant $\beta>0$ satisfying

$$\frac{1}{\beta}\mathrm{e}^{-\mu L}\|(\xi,\eta)\|_X \leqslant V(\xi,\eta)\leqslant \beta\|(\xi,\eta)\|_X. \tag{20}$$

Therefore, according to Definition 2, V of (14) is an ISS-Lyapunov function for the coupled hyperbolic PDE-ODE system (1)–(6) in the norm X; hence, the system (1)–(6) is input-to-state stable to the disturbance $(\delta,\epsilon)\in X$.

This concludes the proof of Theorem 1.

In the following, a boundary controller design (6) for the coupled system (1)–(5) is easily found by utilizing the result of Theorem 1.

Corollary 1. Consider the coupled system (1)–(5). If there exist a real matrix $G\in\mathbb{R}^{n\times n}$, real numbers $\mu>0$, $\kappa_i>0$, and diagonal positive definite matrices $P_i>0$, $i=1,2$, such that the following matrix inequalities

$$\begin{bmatrix} A^{\mathrm{T}}P_2+P_2A+\Lambda P_1+\kappa_2\lambda_{P_2}I_n+\mu P_2 & G \\ G^{\mathrm{T}} & -\mathrm{e}^{-\mu L}\Lambda P_1\end{bmatrix}\leqslant 0, \tag{21}$$

$$-\mu\Lambda P_1+M^{\mathrm{T}}(x)P_1+P_1M(x)+\kappa_1\lambda_{P_1}I_n<0 \tag{22}$$

hold, then Eq. (6) with the feedback gain $K=B^{+}P_2^{-1}G$ is an ISS control law of the system (1)–(5), where B^{+} is the Moore-Penrose pseudoinverse of B.

Proof. The result directly follows the result of Theorem 1 by applying $G=P_2BK$.

4 Computational aspects

The balance term $M(x)$ involves the spatial variable $x\in[0,L]$, which leads to inequality constraints (22) of Corollary 1 becoming infinite. In this section, we use the polyhedron approximating techniques to obtain the numerical computational conditions.

We first divide the spatial domain $[0,L]$ into N isometric subspaces $\{x_k, k\in\mathcal{N}, x_0=0, x_N=L\}$ with

$$x_k-x_{k-1}=\frac{L}{N},\quad \mathcal{N}=\{0,1,\ldots,N\}. \tag{23}$$

Subsequently, we obtain a sequence of sample matrices $M(x_k)=(m_{ij}(x_k))\in\mathbb{R}^{n\times n}$, for $k\in\mathcal{N}$.

Let $D(x)=M(x)-M(x_k)$, as $x_{k-1}<x\leqslant x_k$, for all $x\in[0,L]$. Because $M(x)=(m_{ij}(x))\in L_\infty(0,L)$, there exist two real numbers $\underline{d}_{ij}$, $\overline{d}_{ij}$, such that $\underline{d}_{ij}\leqslant m_{ij}(x)-m_{ij}(x_k)\leqslant\overline{d}_{ij}$, for every entry $m_{ij}(x)$, $i,j=1,\ldots,n$. The convex hull $\mathcal{D}$ is a set:

$$\mathcal{D}\triangleq\left\{\sum_{s=1}^{2n^2}\alpha_sD_s,\quad 0\leqslant\alpha_s\leqslant 1\right\}, \tag{24}$$

with the vertex matrices $D_s=(d_{lk}^{(s)})$, for all $s\in\mathcal{S}=\{1,\ldots,2n^2\}$, defined as

$$d_{lk}^{(s)}=\begin{cases}\overline{d}_{ij}, & \text{as } s=1,\ldots,n^2;\ l=i,k=j,\\ \underline{d}_{ij}, & \text{as } s=n^2+1,\ldots,2n^2;\ l=i,k=j,\\ 0, & \text{otherwise.}\end{cases} \tag{25}$$

It is easily shown that the matrix $D(x)\in\mathcal{D}$, $x\in[0,L]$, for some α_s with $\alpha_s+\alpha_{s+n^2}=1$, $s=1,\ldots,n^2$. Subsequently, the stability of $M(x)$ is guaranteed by the absolute stability of the sample matrices $M(x_k)$, $k\in\mathcal{N}$, and the convex hull $\mathcal{D}$.

Lemma 1. Given $M(x) \in L_\infty(0,L)^{n\times n}$, if there exist a constant $\nu > 0$ and a common diagonal positive definite matrix $P_1 > 0$ such that

$$-\mu\Lambda P_1 + M^{\mathrm{T}}(x_k)P_1 + P_1M(x_k) + \kappa_1\lambda_{P_1}I_n < -\nu, \tag{26}$$

$$D_s^{\mathrm{T}}P_1 + P_1D_s \leqslant \frac{\nu}{n^2} \tag{27}$$

hold, for all $k \in \mathcal{N}$, $s \in \mathcal{S}$, then condition (13) of Theorem 1 is satisfied for all $x \in [0,L]$.

Proof. Because $D(x)$ lies in the convex hull $\mathcal{D}$, for all $x \in [0,L]$, i.e., $D(x) \in \mathcal{D}$, we have

$$D^{\mathrm{T}}(x)P_1 + P_1D(x) \leqslant \sum_{s=1}^{2n^2} \alpha_s(D_s^{\mathrm{T}}P_1 + P_1D_s) \tag{28}$$

for all vertical matrices D_s, $s \in \mathcal{S}$. Thus, using (26) and (28), it follows directly that

$$\begin{aligned}
&-\mu\Lambda P_1 + M^{\mathrm{T}}(x_k)P_1 + P_1M(x_k) + \kappa_1\lambda_{P_1}I_n\\
&\quad = -\mu\Lambda P_1 + M^{\mathrm{T}}(x_k)P_1 + P_1M(x_k) + D(x)^{\mathrm{T}}P + P_1D(x) + \kappa_1\lambda_{P_1}I_n\\
&\quad < -\nu + \sum_{s=1}^{2n^2}\alpha_s\frac{\nu}{n^2}\\
&\quad = 0.
\end{aligned} \tag{29}$$

This concludes the proof of Lemma 1.

Remark 2. The constructed convex hull $\mathcal{D}$ of (24) includes at most $2n^2$ vertices. Meanwhile, the relaxation structure $M(x)$, as the source term of the hyperbolic balance laws, is mostly marginally stable in practical physical models. It often includes several identical or zero entries in $M(x)$, which leads to the number of vertical matrices D_s that is significantly smaller than $2n^2$.

Remark 3. As the division number N of the convex hull $\mathcal{D}$ increases, $\underline{d}_{ij}$ and $\overline{d}_{ij}$ might reduce to zero. Subsequently, an iterative procedure with $N \to N+1$ could be designed to finally obtain a feasible solution of the inequalities (26) and (27).

Remark 4. The sufficient condition (21) of Corollary 1, and the conditions (26) and (27) of Lemma 1 are nonlinear with respect to the unknown variables μ, κ_i and P_i, $i = 1,2$. However, because μ, κ_i are scalar variables, one may combine a linear search algorithm with the semi-definite programming technologies to solve (21), (26) and (27).

The following result is thus obtained as a corollary of Theorem 1.

Corollary 2. Consider the coupled system (1)–(5). If there exist a real matrix $G \in \mathbb{R}^{n\times n}$, real numbers $\nu > 0$, $\kappa_i > 0$, and two diagonal positive definite matrices $P_i > 0$, $i = 1,2$, such that the following matrix inequalities

$$\begin{bmatrix} A^{\mathrm{T}}P_2 + P_2A + \Lambda P_1 + \kappa_2\lambda_{P_2}I_n + \mu P_2 & G \\ G^{\mathrm{T}} & -\mathrm{e}^{-\mu L}\Lambda P_1 \end{bmatrix} \leqslant 0, \tag{30}$$

$$-\mu\Lambda P_1 + M^{\mathrm{T}}(x_k)P_1 + P_1M(x_k) + \kappa_1\lambda_{P_1}I_n < -\nu, \tag{31}$$

$$D_s^{\mathrm{T}}P_1 + P_1D_s \leqslant \frac{\nu}{n^2} \tag{32}$$

hold, for all $k \in \mathcal{N}$, $s \in \mathcal{S}$, then Eq. (6) with the feedback gain $K = B^{+}P_2^{-1}G$ is an ISS control law.

Proof. Following the results of Lemma 1, the condition (22) of Corollary 1 is satisfied under the new inequalities (31)–(32), for all $x \in [0,L]$.

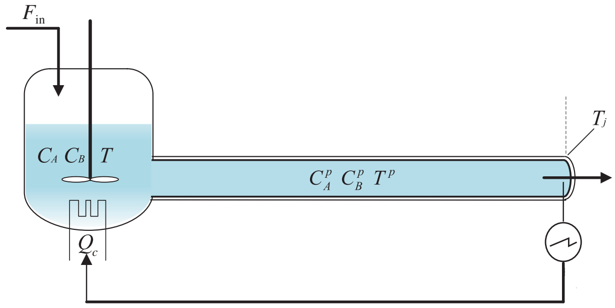

Figure 1 (Color online) Coupled hyperbolic PFR-CSTR model.

5 Application to the hyperbolic PFR-CSTR model

5.1 The coupled hyperbolic PFR-CSTR model

The distributed chemical processes are often coupled with the lumped parameter processes, which are typically modeled as a combination of PDE and ODE models. In this subsection, we consider the PFR-CSTR configuration, shown in Figure 1, as a coupled hyperbolic PDE-ODE system. The exothermic reactions occur in both the CSTR and PFR, in which component A is the reactant and B is the desired product. The liquid flows directly into the PFR after a quick reaction in the CSTR. Here, we consider the following chemical reaction:

$$A \to bB,$$

where $b > 0$ denotes the stoichiometric coefficient of the reaction and we set $b = 1$ for simplicity.

The objective is to control the components' concentration C_A, C_B, C_A^p and C_B^p, and the temperatures T, T^p of the CSTR and PFR, using the cooling rate Q_c as the manipulated variable. It is noteworthy that Q_c acts directly on the CSTR and indirectly on the PFR through the CSTR.

With the assumption of a constant fluid velocity in the PFR with respect to the spatial coordinate, constant physical properties, and incompressible fluid [8, 25], the mathematical model of the system is given as

$$\frac{\partial C_A^p}{\partial t} + v\frac{\partial C_A^p}{\partial x} = -k\mathrm{e}^{\frac{-E}{RT^p}} C_A^p, \tag{33}$$

$$\frac{\partial C_B^p}{\partial t} + v\frac{\partial C_B^p}{\partial x} = k\mathrm{e}^{\frac{-E}{RT^p}} C_A^p, \tag{34}$$

$$\frac{\partial T^p}{\partial t} + v\frac{\partial T^p}{\partial x} = \frac{k}{\rho c_p}\mathrm{e}^{\frac{-E}{RT^p}} C_A^p(-\Delta H) - \beta\left(T^p - T_j\right), \tag{35}$$

$$\frac{\mathrm{d}C_A}{\mathrm{d}t} = \frac{F_{\rm in}}{V_c}(C_A^{\rm in} - C_A) - k\mathrm{e}^{\frac{-E}{RT}} C_A, \tag{36}$$

$$\frac{\mathrm{d}C_B}{\mathrm{d}t} = -\frac{F_{\rm in}}{V_c}C_B + k\mathrm{e}^{\frac{-E}{RT}} C_A, \tag{37}$$

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{1}{\rho c_p}k\mathrm{e}^{\frac{-E}{RT}} C_A(-\Delta H) + \frac{F_{\rm in}}{V_c}(T_{\rm in} - T) + \frac{Q_c}{\rho c_p V_c}, \tag{38}$$

where v is the fluid velocity in the PFR given by $v = \frac{F_{\rm in}}{V_p}$; $F_{\rm in}$ is the inlet flow rate; k is pre-exponential constant; E is the activation energy; R is the universal gas constant; V_c and V_p are the volumes of the CSTR and the PFR, respectively; ΔH is the heat of reaction; ρ and c_p are the average fluid density and specific heat, respectively; β is the heat transfer coefficient, given by $\beta = \frac{4h}{\rho c_p d}$, where h and d are the wall heat transfer coefficient and the reactor diameter, respectively; T_j is the coolant temperature of the PFR's jacket. Some parameter values used in the simulation are given in Table 1.

Table 1 PFR-CSTR model parameters

Process parameter	Notation	Value
Kinetic constant	k	225.225×10^6 s^{-1}
Activation energy/universal gas constant	E/R	9758.3 K
Steady inlet flow rate	F_{in}	0.0041 m^3/s
Steady cooling rate	Q_{ss}	-1.36 kJ/s
Inlet reactant concentration	C_A^{in}	3 kmol/m^3
Inlet temperature	T_{in}	429 K
Heat of reaction for reactions 1	ΔH	-4200 kJ/kmol
Volume of the CSTR	V_c	0.01 m^3
Volume of the PFR	V_p	0.022 m^3
Average fluid density	ρ	934.2 kg/m^3
Specific heat	c_p	3.01 kJ/kgK
Heat transfer coefficient	β	0.2 s^{-1}

At the left boundary of the PFR, $x = 0$, we have

$$\begin{cases} C_A^p(0,t) = C_A, \\ C_B^p(0,t) = C_B, \\ T^p(0,t) = T. \end{cases} \tag{39}$$

We consider the following initial conditions

$$\begin{cases} C_A(0) = C_A^{\text{in}}, \\ C_B(0) = 0, \\ T(0) = T_{\text{in}}, \end{cases} \tag{40}$$

and

$$\begin{cases} C_A^p(x,0) = C_{A,0}^p(x), \\ C_B^p(x,0) = 0, \\ T^p(x,0) = T_0^p(x). \end{cases} \tag{41}$$

5.2 The linearized hyperbolic PFR-CSTR model

Let $\xi = [C_A^p, C_B^p, T^p]^{\text{T}}$, and $\eta = [C_A, C_B, T]^{\text{T}}$. No explicit solution exists for the hyperbolic PDE-ODE system (33)–(38). To better understand the dynamics of the PFR-CSTR model, we linearize the model around the steady states $(\xi_{\text{ss}}(x), \eta_{\text{ss}})$, given as

$$\xi_{\text{ss}}(x) = \Big[C_{A\text{ss}}^p(x) \; C_{B\text{ss}}^p(x) \; T_{\text{ss}}^P(x) \Big]^{\text{T}}, \tag{42}$$

$$\eta_{\text{ss}} = \Big[C_{A\text{ss}} \; C_{B\text{ss}} \; T_{\text{ss}} \Big]^{\text{T}}. \tag{43}$$

The equilibrium temperature in the PFR is assumed to be consistent with the steady-state temperature in the CSTR, that is $T_{\text{ss}}^P(x) = T_{\text{ss}}, 0 < x < L$. Subsequently, it can be easily shown that the equilibria concentrations are given by

$$C_{A\text{ss}}^p(x) = C_{A\text{ss}} \mathrm{e}^{-\alpha x}, \;\; 0 < x < L, \tag{44}$$

$$C_{B\text{ss}}^p(x) = C_{A\text{ss}}(1 - \mathrm{e}^{-\alpha x}) + C_{B\text{ss}}, \;\; 0 < x < L, \tag{45}$$

where α is the positive constant

$$\alpha = \frac{k}{v} \mathrm{e}^{-\frac{E}{RT_{\text{ss}}}} > 0, \tag{46}$$

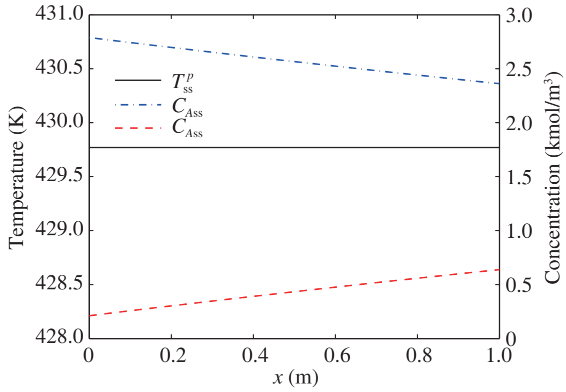

Figure 2 (Color online) Steady states in the PFR.

and we can also obtain the equilibrium condition of T_j, which is given by

$$T_{jss}(x) = T_{\rm ss} + \frac{\Delta H dv}{4h}\alpha C_{A{\rm ss}}{\rm e}^{-\alpha x}. \tag{47}$$

As the jacket temperature T_j is held at its stead state T_{jss}, we can cause $T_{\rm ss}^P(x)$ to be constant. The small deviations from the nominal profile are defined as

$$\tilde{\xi} = \xi - \xi_{\rm ss}(x), \tag{48}$$

$$\tilde{\eta} = \eta - \eta_{\rm ss}. \tag{49}$$

The model equations (33)–(38) are linearized around the steady state to yield the linear system (1)–(5) on $L_2(0,1)^3 \oplus \mathbb{R}^3$. Here, $M(x) \in L_\infty(0,1)^{3\times 3}$ and A are the Jacobian matrices of (33)–(35), with respect to C_A^p, C_B^p, T^p, and (36)–(38) with C_A, C_B, and T, evaluated at the steady states, respectively; $\Lambda = -\frac{F_{\rm in}}{V_p}{\rm diag}(1,1,1)$; $\delta(x,t)$ and $\varepsilon(t)$ are the higher-order terms of the nonlinear PDE and ODE, where

$$M(x) = {\rm e}^{\frac{-E}{RT_{\rm ss}}}\begin{bmatrix} -k & 0 & -k\frac{E}{RT_{\rm ss}^2}C_{A{\rm ss}}^p(x) \\ k & 0 & k\frac{E}{RT_{\rm ss}^2}C_{A{\rm ss}}^p(x) \\ \frac{-\Delta Hk}{\rho c_p} & 0 & \frac{-\Delta Hk}{\rho c_p}\frac{E}{RT_{\rm ss}^2}C_{A{\rm ss}}^p(x) - \frac{4h}{\rho c_p d}{\rm e}^{\frac{E}{RT_{\rm ss}}} \end{bmatrix}, \tag{50}$$

$$A = {\rm e}^{\frac{-E}{RT_{\rm ss}}}\begin{bmatrix} -\frac{F_{\rm in}}{V_c}{\rm e}^{\frac{E}{RT_{\rm ss}}} - k & 0 & -k\frac{E}{RT_{\rm ss}^2}C_{A{\rm ss}} \\ k & -\frac{F_{\rm in}}{V_c}{\rm e}^{\frac{E}{RT_{\rm ss}}} & k\frac{E}{RT_{\rm ss}^2}C_{A{\rm ss}} \\ \frac{-\Delta Hk}{\rho c_p} & 0 & \frac{-\Delta Hk}{\rho c_p}\frac{E}{RT_{\rm ss}^2}C_{A{\rm ss}} - \frac{F_{\rm in}}{V_c}{\rm e}^{\frac{E}{RT_{\rm ss}}} \end{bmatrix}, \tag{51}$$

$$B = \begin{bmatrix} 0 & 0 & \frac{1}{\rho c_p V_c} \end{bmatrix}^{\rm T}. \tag{52}$$

5.3 Simulations

Let the length of the PFR be $L = 1$ m. By using the parameters values given in Table 1, we can obtain the steady states $(\xi_{\rm ss}(x), \eta_{\rm ss})$ depicted in Figure 2. In this case, the system matrices $M(x)$, Λ, A, and B are given as

$$M(x) = \begin{bmatrix} -3.10 & 0 & -0.46{\rm e}^{-\alpha x} \\ 3.10 & 0 & 0.46{\rm e}^{-\alpha x} \\ 4.63 & 0 & 0.68{\rm e}^{-\alpha x} - 20 \end{bmatrix} \times 10^{-2}, \quad \Lambda = \begin{bmatrix} 0.1864 & 0 & 0 \\ 0 & 0.1864 & 0 \\ 0 & 0 & 0.1864 \end{bmatrix},$$

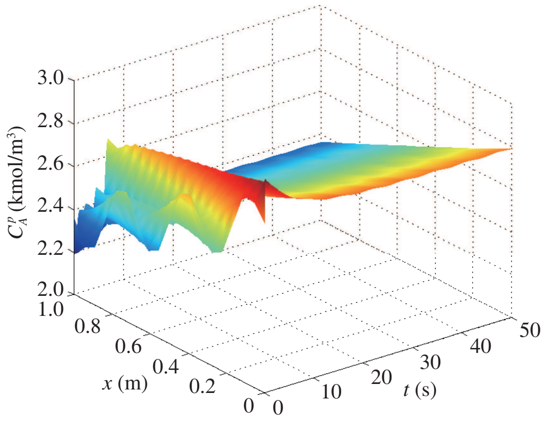

Figure 3 (Color online) Time evolution of the concentration C_A^p in the PFR.

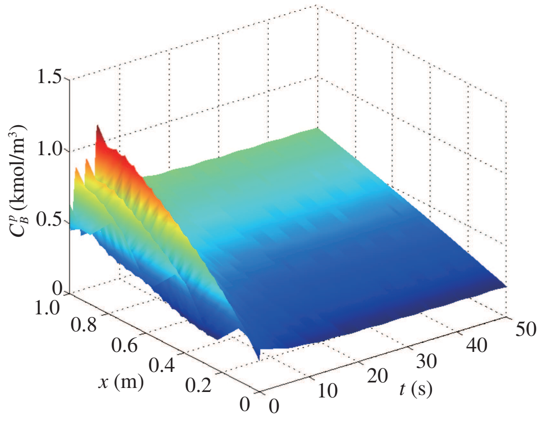

Figure 4 (Color online) Time evolution of the concentration C_B^p in the PFR.

$$A = \begin{bmatrix} -44.10 & 0 & -0.46 \\ 3.10 & -41 & 0.46 \\ 4.63 & 0 & -40.32 \end{bmatrix} \times 10^{-2}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 3.56 \end{bmatrix} \times 10^{-2}.$$

Using $N = 10$, and solving the conditions (30)–(32) of Corollary 2, we obtain $\mu = 0.3$, and $\kappa_1 = 4.400 \times 10^{-3}$, $\kappa_2 = 4.679 \times 10^{-5}$, and

$$G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.8257 & 1.6468 & 0.7824 \end{bmatrix} \times 10^3,$$

$$P_1 = \begin{bmatrix} 1.6044 & 0 & 0 \\ 0 & 2.0462 & 0 \\ 0 & 0 & 0.8257 \end{bmatrix} \times 10^3, \quad P_2 = \begin{bmatrix} 1.6467 & 0 & 0 \\ 0 & 0.7824 & 0 \\ 0 & 0 & 60.7943 \end{bmatrix} \times 10^3.$$

Subsequently, we obtain the boundary feedback gain

$$K = \begin{bmatrix} 0.3815 & 0.7609 & 0.3615 \end{bmatrix}.$$

To numerically compute the solutions of the system (33)–(38), we discretize them using a two-step variant of the Lax-Wendroff method in [26]. We select the following initial deviations

$$\begin{cases} \tilde{C}_A^p(x) = 0.05 C_A^{\text{in}} \sin(6\pi x), \\ \tilde{C}_B^p(x) = 0.05 \sin(6\pi x), \\ \tilde{T}^p(x) = 0.07 T_{\text{in}} \sin(6\pi x), \end{cases} \tag{53}$$

and the disturbances

$$\begin{cases} \delta(x,t) = 2.4 \times 10^{-3} \sin(xt), \\ \varepsilon(t) = 6.8 \times 10^{-3} \sin(t). \end{cases} \tag{54}$$

Figures 3–5 show the time evolutions of the concentrations C_A^p, C_B^p, and the temperature T^p in the PFR, respectively. We observed that C_A^p, C_B^p, T^p converge to the bounded domains of their steady-states $C_{A\text{ss}}^p(x)$, $C_{B\text{ss}}^p(x)$, $T_{\text{ss}}^p(x)$, respectively, as time progresses, as expected from Theorem 1. Figure 6 depicts the concentrations of C_A C_B, and T in the CSTR that enters some bounded domains ultimately. The variations in the boundary control Q_c are also depicted in Figure 7.

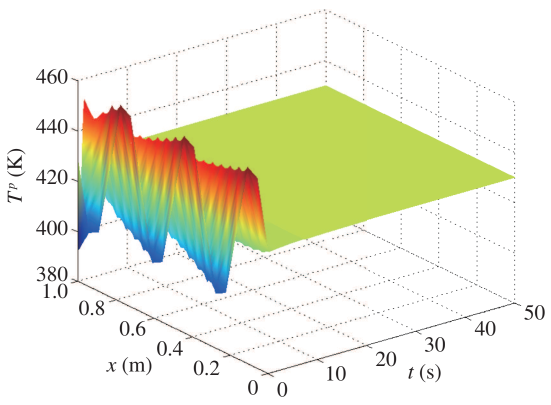

Figure 5 (Color online) Time evolution of the temperature T^p in the PFR.

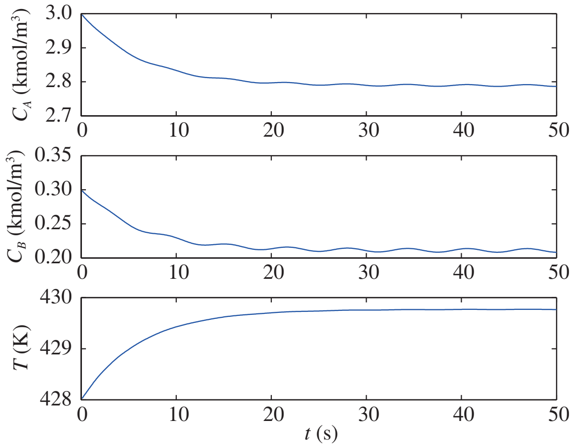

Figure 6 (Color online) Time evolutions of the concentration C_A, C_B, and the temperature T in the CSTR.

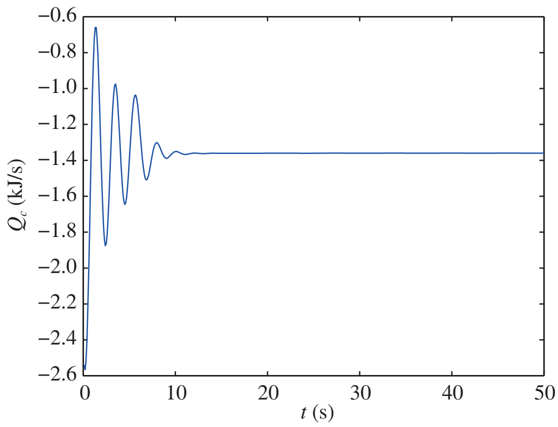

Figure 7 (Color online) Time evolution of the control input Q_c.

6 Conclusion

This paper described the boundary ISS for a class of coupled hyperbolic PDE-ODE systems. Some sufficient conditions in terms of matrix inequalities are provided using the strict Lyapunov function. The boundary feedback control is applied to stabilize a coupled hyperbolic PFR-CSTR model. This study leaves many open questions. It is natural to extend the theoretical results, such as Theorem 1 and Corollary 2 to more general nonlinear PDE-ODE systems and consider the proportional-integral boundary control.

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