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Global practical tracking with prescribed transient performance for inherently nonlinear systems with extremely severe uncertainties

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Abstract This paper considers the global practical tracking for a class of uncertain nonlinear systems. Remarkably, the systems under investigation admit rather inherent nonlinearities, and especially allow arguably the most severe uncertainties: unknown control directions and non-parametric uncertainties. Despite this, a refined tracking objective, rather than a reduced one, is sought. That is, not only pre-specified arbitrary tracking accuracy is guaranteed, but also certain prescribed transient performance (e.g., arrival time and maximum overshoot) is ensured to better meet real applications. To solve the problem, a new tracking scheme is established, crucially introducing delicate time-varying gains to counteract the severe uncertainties and guarantee the prescribed performance. It is shown that the designed controller renders the tracking error to forever evolve within a prescribed performance funnel, through which the desired tracking objective is accomplished for the systems. Particularly, by subtly specifying the funnel, global fixed-time practical tracking (i.e., that with prescribed arrival time) and semiglobal practical tracking with prescribed maximal overshoot can be achieved for the systems. Moreover, the tracking scheme remains valid in the presence of rather less-restrictive unmodeled dynamics.

Keywords nonlinear systems, practical tracking, prescribed transient performance, unknown control directions, non-parametric uncertainties, unmodeled dynamics

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1 Introduction

Tracking is one dominant objective of control design for nonlinear systems, which is usually to drive the system output to track a prescribed reference signal asymptotically or with a prescribed accuracy, accordingly called asymptotic tracking or practical tracking. Although practical tracking has a relatively conservative objective, it requires rather milder restrictions on the systems and the reference signals than asymptotic tracking, and particularly, is adequate for many real applications. Therefore, practical tracking has been an appealing research topic (see, e.g., [1–15] and references therein). However, in many of the related studies, e.g., [1–9], only the basic steady-state tracking performance (i.e., the ultimate convergence of the tracking error) is achieved, which is sometimes far from enough for real applications. Preferably, certain prescribed transient performance, e.g., maximum overshoot or arrival time, should be

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jointly taken into account, to guarantee the system efficiency and reliability. Unfortunately, this is indeed challenging particularly because of severe limitations by the system uncertainties and nonlinearities.

In this paper, we consider the global practical tracking with prescribed transient performance for a class of uncertain nonlinear systems in the following form:

$$\begin{cases} \dot{x}_i = g_i(t,x)x_{i+1}^{p_i} + f_i(t,x,u), \quad i = 1,\ldots,n-1, \\ \dot{x}_n = g_n(t,x)u^{p_n} + f_n(t,x,u), \\ y = x_1, \end{cases} \tag{1}$$

where $x = [x_1,\ldots,x_n]^{\mathrm{T}} \in \mathbb{R}^n$ is the system state vector with the initial value $x(t_0) = x_0$; $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are the control input and system output, respectively; $p_i \in \mathbb{R}_{\text{odd}}^{\geqslant 1} = \{q \in \mathbb{R} \,|\, q \geqslant 1$ and q is a ratio of odd integers$\}$, $i = 1,\ldots,n$; $g_i : [t_0,+\infty) \times \mathbb{R}^n \to \mathbb{R}$, $i = 1,\ldots,n$ and $f_i : [t_0,+\infty) \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $i = 1,\ldots,n$ are unknown continuous functions, called control coefficients and nonlinearities of the system, respectively, and the signs of $g_i(\cdot)$, $i = 1,\ldots,n$ are called control directions of the system. System (1) is the generalization of strict-feedback nonlinear system, and allows more inherent nonlinearities which make the system more difficult to be controlled. Indeed, system (1) is closely related with more actual plants, e.g., underactuated mechanical systems with weak coupling [16] and boiler system in thermal power plants [17].

During the past two decades, a variety of control design methods have been developed for system (1) (see, e.g., [2,3,6,8,17–20] and references therein). Particularly, inspired by [19], Ref. [20] introduced the method of adding a power integrator, based on which [2] proposed a scheme of practical tracking for system (1). Subsequently, Ref. [3] considered the case with serious parametric uncertainties in the system and the reference signal to be tracked. As further development, Refs. [8,21] investigated the case simultaneously with unknown control directions, that is, the sign of each $g_i(\cdot)$ is unknown. However, Refs. [3,8,21] do not allow non-parametric uncertainties in the system, that is, $g_i(\cdot)$ and $f_i(\cdot)$ are required to be dominated by unknown constants and known functions. Remark that [4,10,22,23] and [24] respectively proposed time-varying schemes of practical tracking and stabilization for systems with non-parametric uncertainties. But Refs. [4, 22, 24] are inapplicable to systems of form (1) with $p_i > 1$, and Ref. [4] requires the system nonlinearities to merely depend on the system output while [22] needs known control directions. Refs. [10, 23] are restricted to pure-feedback systems of distinct nonlinear structure, and particularly, Ref. [10] excludes the cases with unknown control directions. Moreover, Refs. [5, 13, 14] gave adaptive tracking schemes allowing the presence of non-parametric uncertainties, but handling the uncertainties is based on the approximation method (neural networks or fuzzy system).

Besides somewhat limitations on handling uncertainties, the tracking schemes in [3, 8, 21] only guarantee pre-specified tracking accuracy, but are incapable of prescribing the arrival time or the maximum overshoot of the tracking error. Here we specially mention [4, 10, 11, 22, 23], where prescribed transient performance is jointly involved. Specifically, Ref. [11] established a switching scheme of practical tracking, which can guarantee the tracking error to reach pre-specified accuracy after a prescribed period of time with prescribed maximum overshoot, but is concerned with linear systems. In [4, 10, 22, 23], the time-varying schemes can ensure the tracking error to evolve within a prescribed funnel, but as stated previously, somewhat serious restrictions are made on the system nonlinearities or uncertainties. Moreover, by the barrier Lyapunov function, [13, 14] gave tracking schemes with ensuring prescribed bound for the system states, but do not allow the presence of unknown control directions.

This paper is devoted to developing a scheme of practical tacking with prescribed transient performance for system (1) allowing extremely severe uncertainties: unknown control directions and serious non-parametric uncertainties. Motivated by the funnel control method in [4, 25], delicate time-varying gains are introduced to effectively counteract the severe system uncertainties, and to ensure the prescribed performance. Based on this and combining the method of adding a power integrator and Nussbaum-type gain technique, a time-varying state-feedback controller is constructed, such that all the closed-loop signals are bounded, and the tracking error forever evolves within a prescribed performance funnel, through which pre-specified ultimate convergence and prescribed transient performance are guaranteed for the tracking error. Particularly, by suitably specifying the funnel, global fixed-time practical tracking (i.e., that with prescribed arrival time) and semiglobal practical tracking with prescribed maximum overshoot can be achieved for the system. Furthermore, the designed controller is shown still valid for the more general systems additionally containing rather less-restrictive unmodeled dynamics.

The remainder of the paper is organized as follows. Section 2 provides the rigorous formulation of the control problem. Section 3 is to establish the tracking scheme. Section 4 summarizes the main results of this paper. Section 5 extends the scheme to the case with unmodeled dynamics. Section 6 gives two simulation examples, and Section 7 addresses some concluding remarks.

2 Formulation of control problem

Throughout this paper, the following notation is adopted. Let $\mathbb{Z}_+$ denote the set of all positive integers, $\mathbb{R}_+$ denote the set of all nonnegative real numbers, $\mathbb{R}_{\geqslant t_0}$ denote the set of all real numbers not less than t_0, and $\mathbb{R}^n$ denote the real n-dimensional space. For any vector $x \in \mathbb{R}^n$, let x_i denote its i-th element, and $x_{[i]} = [x_1, \ldots, x_i]^{\mathrm{T}}$. For any $\phi : \mathbb{R}_{\geqslant t_0} \to \mathbb{R}$, let $\operatorname{ess\,sup}_{t\geqslant t_0}\phi(t)$ denote its essential supremum, that is, $\operatorname{ess\,sup}_{t\geqslant t_0}\phi(t) = \sup\{b \,|\, |\phi(t)| \leqslant b \text{ for almost all } t \geqslant t_0\}$. Let $\mathcal{W}^{1,\infty}(\mathbb{R}_{\geqslant t_0}, \mathbb{R})$ denote the set of locally absolutely continuous functions $\psi : \mathbb{R}_{\geqslant t_0} \to \mathbb{R}$ with $\operatorname{ess\,sup}_{t\geqslant t_0}(|\psi(t)| + |\dot{\psi}(t)|) < +\infty$.

This paper is to consider the practical tracking of system (1) under the following rather general assumptions on the system itself and the reference signal y_r to be tracked by system output y.

Assumption 1. The signs of $g_i(t,x)$, $i = 1, \ldots, n$ are unknown but remain unchanged. Moreover, there exist unknown smooth positive functions $\underline{g}_i(x_{[i]})$ and $\bar{g}_i(x_{[i]})$, $i = 1, \ldots, n$, such that

$$\underline{g}_i(x_{[i]}) \leqslant |g_i(t,x)| \leqslant \bar{g}_i(x_{[i]}), \quad i = 1, \ldots, n.$$

Assumption 2. There exist unknown continuous nonnegative functions $\bar{f}_i(x_{[i]})$, $i = 1, \ldots, n$, and unknown constants $q_i \in [0, p_i)$, $i = 1, \ldots, n$, such that

$$|f_i(t,x,u)| \leqslant (1 + |x_{i+1}|^{q_i})\bar{f}_i(x_{[i]}), \quad i = 1, \ldots, n, \tag{2}$$

where $x_{n+1} = u$.

Assumption 3. The reference signal y_r belongs to $\mathcal{W}^{1,\infty}$ $(\mathbb{R}_{\geqslant t_0}, \mathbb{R})$. Moreover, there exists an unknown positive constant M such that

$$\operatorname{ess\,sup}_{t\geqslant t_0}\big(|y_r(t)| + |\dot{y}_r(t)|\big) \leqslant M.$$

Assumption 1 shows that rather serious unknowns are admitted in the control coefficients of system (1). Specifically, the control coefficients are of unknown sign, which implies unknown control directions. Meanwhile, the unknown $\underline{g}_i(\cdot)$ and $\bar{g}_i(\cdot)$ mean allowing the presence of fairly serious non-parametric uncertainties, obviously as well as strong nonlinearities. Remark that the tracking schemes in [2,3,5,7,10,13,14,22] are inapplicable to the cases of unknown-sign control coefficients, and those in [1–3,7,8,13,14,21] do not allow non-parametric uncertainties in the control coefficients.

Assumption 2 shows that the nonlinearities of system (1) contain serious non-parametric uncertainties. Specifically, in each nonlinearity $f_i(\cdot)$, the structure with respect to $x_{[i]}$ and the power of x_{i+1} are both unknown, reflected by the unknown $\bar{f}_i(\cdot)$ and q_i. This makes the system nonlinearities far more general than those in [1, 3, 8, 21] with only parametric uncertainties. Remark that, although [4, 5, 10, 13, 14] allow non-parametric uncertainties in the system nonlinearities, [4] requires the system nonlinearities to inherently only depend on the system output, and [5,10,13,14,22] exclude the cases with unknown control directions. Moreover, it is worth noting that $q_i \in [0, p_i)$ in Assumption 2 is necessary for achieving global practical tracking of system (1), which has been shown by a counterexample in Remark 4.3 of [2].

Assumptions 1 and 2 do not provide any information available for feedback, although a series of estimations are involved. Indeed, the two assumptions just characterize some essential properties that the unknown functions $g_i(t,x)$ and $f_i(t,x,u)$ should satisfy, and the controller to be sought must be independent of the unknown $\underline{g}_i(x_{[i]})$, $\bar{g}_i(x_{[i]})$, $\bar{f}_i(x_{[i]})$ and q_i. Moreover, Assumptions 1 and 2 make system

(1) cover many actual plants, e.g., the reduced-order model for a two degrees of freedom underactuated, weakly coupled, unstable mechanical system (see [16]), pendulum model (see [4]) and single-link manipulator with including direct current motor dynamics (see [13]).

Like [3, 4, 8], Assumption 3 shows that rather coarse information is needed on the reference signal y_r: (1) y_r is known at present but not in advance; (2) the derivative of y_r is not available, unlike that in [2, 22]; (3) y_r and its derivative are essentially bounded but belong to an unknown constant interval for almost all time. A real example is that, in the process of missile interception, the position of enemy missile, i.e., the actual trajectory to be tracked by ours, can be measured at present by radar but not in advance. However, the reference signal y_r merely brings parametric uncertainties which can be incorporated into the non-parametric uncertainties in the later performance analysis.

Detailedly, the expected tracking for system (1) under Assumptions 1–3 is to design a time-varying state-feedback controller

$$u = u(t, x, y_r),$$

such that, for any initial value, the solutions of the resulting closed-loop system are bounded on $\mathbb{R}_{\geqslant t_0}$, and the tracking error $e(t) = y(t) - y_r(t)$ evolves within the following funnel (as those in [4, 12, 25]):

$$\mathcal{F}_{\psi_\lambda} := \{(t, e) \in \mathbb{R}_{\geqslant t_0} \times \mathbb{R} \,|\, \psi_\lambda(t)|e| < 1\},$$

where λ is a prescribed positive constant representing the ultimate tracking accuracy, and $\psi_\lambda \in \mathcal{S}$ with

$$\mathcal{S} \triangleq \left\{ \psi \in \mathcal{W}^{1,\infty}(\mathbb{R}_{\geqslant t_0}, \mathbb{R}) \,\middle|\, \begin{array}{l} \psi(t_0) = 0, \\ \psi(t) > 0, \ \forall t > t_0, \\ \liminf_{t\to+\infty} \psi(t) \geqslant \frac{2}{\lambda} \end{array} \right\}.$$

The funnel $\mathcal{F}_{\psi_\lambda}$ renders the described tracking to own prescribed steady-state and transient behaviors. (i) Practical tracking (or called λ-tracking). For any prescribed $\lambda > 0$, combining $\psi_\lambda(t)|e(t)| < 1$, $\forall t \geqslant t_0$ with $\liminf_{t\to+\infty} \psi_\lambda(t) > \frac{2}{\lambda}$ implies $\limsup_{t\to+\infty} |e(t)| \leqslant \frac{1}{\liminf_{t\to+\infty} \psi_\lambda(t)} \leqslant \frac{\lambda}{2}$. Hence, there is $T_\lambda \geqslant t_0$, such that $\sup_{t \geqslant T_\lambda} |e(t)| < \lambda$. (ii) Tracking error boundary. The funnel $\mathcal{F}_{\psi_\lambda}$ ensures that the tracking error $e(t)$ always satisfies $|e(t)| \leqslant 1/\psi_\lambda(t)$. Thus, there is a prescribed boundary for error $e(t)$.

3 Tracking control design

This section is devoted to designing the desired tracking controller for system (1) under Assumptions 1–3. Moreover, a characterization via candidate Lyapunov functions is given to the evolution of the resulting closed-loop system, for the validity analysis in Section 4.

For clarity, we would like to first illustrate the motivation of our controller design through the case with dimension $n = 2$.

Step 1. Let $\xi_1 = x_1 - y_r$, and introduce the time-varying gain $r_1 = \frac{1}{1-(\varphi_1(t)\xi_1)^2}$ with $\varphi_1 = \psi_\lambda$, which is to counteract the non-parametric uncertainties and ensure the prescribed performance. Then, motivated by [8, 21], we take $(x_2^{p_1} + x_2)^{\frac{1}{p_1}}$ as the control input of ξ_1-subsystem, and correspondingly, design the following virtual control law with no information of $g_1(t, x)$ and $f_1(t, x, u)$:

$$\alpha_1(r_1, \xi_1) = (k_1 N(r_1)\xi_1)^{\frac{1}{p_1}}, \quad k_1 > 0,$$

where $N(\cdot)$, a Nussbaum-type function (see (6) below), is introduced to deal with the unknown control direction of the ξ_1-subsystem. Furthermore, defining $\xi_2 = x_2^{p_1} + x_2 - \alpha_1^{p_1}(r_1, \xi_1)$ yields

$$\begin{cases} \dot{\xi}_1 = g_1(t, x)\big(\xi_2 + \alpha_1^{p_1}(r_1, \xi_1)\big) + f_1(t, x, u) - g_1(t, x)x_2 - \dot{y}_r, \\ \dot{\xi}_2 = \big(p_1 x_2^{p_1-1} + 1\big)\big(g_2(t, x)u^{p_2} + f_2(t, x, u)\big) - \dfrac{\mathrm{d}\alpha_1^{p_1}}{\mathrm{d}t}. \end{cases}$$

Step 2. Owing to the presence of serious non-parametric uncertainties, $\frac{\mathrm{d}\alpha_1^{p_1}}{\mathrm{d}t}$, as well as $g_2(t,x)$ and $f_2(t,x,u)$, cannot provide any information available for feedback design of the ξ_2-subsystem. Thus, similar to Step 1, we design the following controller for the ξ_2-subsystem:

$$u = (k_2 N(r_2)\xi_2)^{\frac{1}{p_2}}, \quad k_2 > 0,$$

where $r_2 = \frac{1}{1-(\varphi_2(t)\xi_2)^2}$ with $\varphi_2 \in \mathcal{S}_0$ (see (5) below).

Motivated by the above two-step design, we design the following controller for system (1):

$$u = \alpha_n(r_n, \xi_n), \tag{3}$$

which is generated from the recursive procedure (from 1 to n, with the initial assignment $\xi_1 = x_1 - y_r$)

$$\begin{cases} r_i := r_i(t,\xi_i) = \dfrac{1}{1-(\varphi_i(t)\xi_i)^2}, \\ \alpha_i(r_i,\xi_i) = \big(k_i N(r_i)\xi_i\big)^{\frac{1}{p_i}}, \\ \xi_{i+1} = x_{i+1}^{p_i} + x_{i+1} - \alpha_i^{p_i}(r_i,\xi_i), \end{cases} \tag{4}$$

where k_i's are positive constants, $\varphi_1 = \psi_\lambda$ and $\varphi_i \in \mathcal{S}_0$, $i = 2,\ldots,n$ with

$$\mathcal{S}_0 \triangleq \left\{ \psi \in \mathcal{W}^{1,\infty}(\mathbb{R}_{\geqslant t_0},\mathbb{R}) \;\middle|\; \begin{array}{l} \psi(t_0) = 0, \\ \psi(t) > 0,\ \forall t > t_0, \\ \liminf_{t\to+\infty} \psi(t) > 0 \end{array} \right\}, \tag{5}$$

and $N : \mathbb{R}_+ \to \mathbb{R}$ is chosen as a smooth Nussbaum-type function satisfying

$$\begin{cases} \limsup\limits_{s\to+\infty} N(s) = +\infty, \\ \liminf\limits_{s\to+\infty} N(s) = -\infty, \end{cases} \tag{6}$$

such as $s \mapsto \ln(1+s^2)\sin(s)$, $s \mapsto s\cos(s)$ and $s \mapsto e^s \sin(s)$.

It is worth pointing out that, unlike ξ_1, only boundedness is required for ξ_i, $i = 2,\ldots,n$. Thus, for $i = 2,\ldots,n$, the less restrictive φ_i (belonging to $\mathcal{S}_0$, rather than $\mathcal{S}_\lambda$) is chosen in the definition of r_i.

Remark 1. In each step i, we take $(x_{i+1}^{p_i} + x_{i+1})^{\frac{1}{p_i}}$, rather than x_{i+1}, as the virtual control, for which the virtual control law $\alpha_i(r_i,\xi_i)$ is designed. Moreover, each intermediate variable ξ_{i+1} is essentially to reflect the error between the virtual control $(x_{i+1}^{p_i} + x_{i+1})^{\frac{1}{p_i}}$ and the virtual control law $\alpha_i(r_i,\xi_i)$, which is rather crucial for the validity of our scheme.

Remark 2. The time-varying gains r_i, $i = 1,\ldots,n$ are pivotal in handling the severe uncertainties and achieving the prescribed performance. This is mainly because that each r_i can rapidly grow to a sufficiently large value when the system state gets sufficiently close to the boundary of the funnel $\{(t,x) \in \mathbb{R}_{\geqslant t_0} \times \mathbb{R}^n \mid \varphi_i(t)|\xi_i(t,x_{[i]})| < 1\}$, but remains bounded on $\mathbb{R}_{\geqslant t_0}$. See the proofs of Lemma 2 and Theorem 1 later for details.

Remark 3. Combining the Nussbaum-type function $N(\cdot)$ and gains r_i, $i = 1,\ldots,n$ can effectively deal with the unknown control directions. It is worth pointing out that, when the control directions of the system are known, the Nussbaum-type function becomes unnecessary and could be replaced by "$s \mapsto -\mathrm{sign}(g_i(t,x))s$" in each α_i (the corresponding validity analysis can be implemented quite similar to that in Section 4 later).

The validity analysis of the designed controller is somewhat complex, and hence, will be rigorously implemented in Section 4. As necessary technical preparation, we now provide a critical lemma which characterizes the dynamic behavior of each ξ_i via a candidate Lyapunov function.

Lemma 1. Let $V_i(\xi_i) = \frac{\xi_i^2}{2}$, $i = 1, \dots, n$ be candidate Lyapunov functions. Then, along the system trajectories, it holds that

$$\begin{cases} \dot{V}_1(\xi_1) \leqslant k_1 \hat{g}_1(t,x) N(r_1)\xi_1^2 + \rho_1(x_1, r_{[0]}, \xi_{[2]}), \\ \dot{V}_i(\xi_i) \leqslant k_i(p_{i-1}x_i^{p_{i-1}-1} + 1)\hat{g}_i(t,x)N(r_i)\xi_i^2 + \rho_i(x_{[i]}, r_{[i-1]}, \xi_{[i+1]}), \quad i = 2, \dots, n, \end{cases} \tag{7}$$

where $r_{[0]} = 0$, $\rho_i(\cdot)$ is an unknown continuous nonnegative function, and $\hat{g}_i(t,x)$ is an unknown function defined as

$$\hat{g}_i(t,x) = \begin{cases} \dfrac{7}{4} g_i(t,x), & \text{if } g_i(t,x)\xi_i x_{i+1} \geqslant 0, \\ \dfrac{1}{4} g_i(t,x), & \text{else.} \end{cases}$$

Remark 4. Rigorously speaking, in this section, the estimate of each $\dot{V}_i(\xi_i)$ should be for almost all t in the existence intervals of the system solutions, because the involved $\dot{\varphi}_i(t)$ and $\dot{y}_r(t)$ are defined and bounded for almost all $t \geqslant t_0$. However, to avoid tediousness, this would not be mentioned in the present section if no confusion.

Before proving Lemma 1, we give Proposition 1 on $f_i(\cdot)$ in (1) and $\alpha_i(\cdot)$ defined by (4).

Proposition 1. Along the system trajectories, it holds that, for each $i = 1, \dots, n$,

$$\begin{cases} |f_i(t,x,u)| \leqslant \dfrac{1}{4}|g_i(t,x)(x_{i+1}^{p_i} + x_{i+1})| + \phi_i(x_{[i]}), \\ \left|\dfrac{\mathrm{d}\alpha_i^{p_i}}{\mathrm{d}t}\right| \leqslant \beta_i(x_{[i+1]}, r_{[i]}, \xi_{[i]}), \end{cases} \tag{8}$$

where $\phi_i(\cdot)$ and $\beta_i(\cdot)$ are unknown nonnegative continuous functions.

Proof. To start with, let us prove the first inequality of (8). By the Young's inequality and Assumption 1, we deduce

$$|x_{i+1}|^{q_i}\bar{f}_i(x_{[i]}) \leqslant \frac{1}{4}|g_i(t,x)x_{i+1}^{p_i}| + \frac{p_i - q_i}{p_i}\left(\frac{4q_i}{p_i}\right)^{\frac{q_i}{p_i-q_i}}\left(\frac{\bar{f}_i^{p_i}(x_{[i]})}{\underline{g}_i^{q_i}(x_{[i]})}\right)^{\frac{1}{p_i-q_i}}, \quad i = 1, \dots, n.$$

Substituting this into (2), and noting $|x_{i+1}^{p_i} + x_{i+1}| \geqslant |x_{i+1}^{p_i}|$, the first inequality can be obtained.

We now prove the second inequality of (8) by induction on i. Noting $\dot{\xi}_1 = g_1(t,x)x_2^{p_1} + f_1(t,x,u) - \dot{y}_r$, and using Assumptions 1–3 can yield $|\dot{\xi}_1| \leqslant \beta_{1,1}(x_{[2]})$ with an unknown continuous nonnegative function $\beta_{1,1}(\cdot)$. Furthermore, noting $\dot{r}_1 = 2r_1^2\big(\varphi_1^2\xi_1\dot{\xi}_1 + \varphi_1\dot{\varphi}_1\xi_1^2\big)$, and by the boundedness of φ_1 and $\dot{\varphi}_1$, one can conclude $|\dot{r}_1| \leqslant \beta_{1,2}(x_{[2]}, r_1, \xi_1)$ with an unknown continuous nonnegative function $\beta_{1,2}(\cdot)$. Substituting the estimations of $\dot{\xi}_1$ and $\dot{r}_1$ into $\frac{\mathrm{d}\alpha_1^{p_1}}{\mathrm{d}t} = \frac{\partial\alpha_1^{p_1}}{\partial r_1}\dot{r}_1 + \frac{\partial\alpha_1^{p_1}}{\partial\xi_1}\dot{\xi}_1$ can yield the estimation of $\frac{\mathrm{d}\alpha_1^{p_1}}{\mathrm{d}t}$ in (8). This establishes the base case $i = 1$ of the induction.

Let us establish the inductive step of the induction. Suppose for any $i = 2, \dots, n$ that the estimation of $\frac{\mathrm{d}\alpha_{i-1}^{p_{i-1}}}{\mathrm{d}t}$ in (8) holds. Then, noting $\dot{\xi}_i = (p_{i-1}x_i^{p_{i-1}} + 1)\big(g_i(t,x)x_{i+1}^{p_i} + f_i(t,x,u)\big) - \frac{\mathrm{d}\alpha_{i-1}^{p_{i-1}}}{\mathrm{d}t}$, and using Assumptions 1 and 2, one can obtain $|\dot{\xi}_i| \leqslant \beta_{i,1}(x_{[i+1]}, r_{[i-1]}, \xi_{[i-1]})$ with an unknown continuous nonnegative function $\beta_{i,1}(\cdot)$. Furthermore, from $\dot{r}_i = 2r_i^2(\varphi_i^2\xi_i\dot{\xi}_i + \varphi_i\dot{\varphi}_i\xi_i^2)$ and the boundedness of φ_i and $\dot{\varphi}_i$, we see that $|\dot{r}_i| \leqslant \beta_{i,2}(x_{[i+1]}, r_{[i]}, \xi_{[i]})$ with an unknown continuous nonnegative function $\beta_{i,2}(\cdot)$. Substituting the estimations of $\dot{\xi}_i$ and $\dot{r}_i$ into $\frac{\mathrm{d}\alpha_i^{p_i}}{\mathrm{d}t} = \frac{\partial\alpha_i^{p_i}}{\partial r_i}\dot{r}_i + \frac{\partial\alpha_i^{p_i}}{\partial\xi_i}\dot{\xi}_i$, one can obtain the estimation of $\frac{\mathrm{d}\alpha_i^{p_i}}{\mathrm{d}t}$ in (8). Therefore, (8) holds for any $i = 1, \dots, n$ by induction.

Proof of Lemma 1. The following proceeds for arbitrary fixed $i = 1, \dots, n$. By the definitions of $\hat{g}_i(\cdot)$ and $\alpha_i(\cdot)$, and noting that p_i is odd, we see that

$$\begin{aligned} \xi_i g_i(t,x)(x_{i+1}^{p_i} + x_{i+1}) &= \hat{g}_i(t,x)\xi_i(x_{i+1}^{p_i} + x_{i+1}) - \frac{3}{4}|g_i(t,x)\xi_i(x_{i+1}^{p_i} + x_{i+1})| \\ &= k_i\hat{g}_i(t,x)N(r_i)\xi_{i+1}^2 + \hat{g}_i(t,x)\xi_i\xi_{i+1} - \frac{3}{4}|g_i(t,x)\xi_i(x_{i+1}^{p_i} + x_{i+1})|. \end{aligned} \tag{9}$$

Noting $|x_{i+1}| \leqslant \frac{1}{2}|x_{i+1}^{p_i} + x_{i+1}| + 1$, and by Assumption 1, we have

$$|g_i(t,x)x_{i+1}| \leqslant \frac{1}{2}|g_i(t,x)(x_{i+1}^{p_i} + x_{i+1})| + \bar{g}_i(x_{[i]}).$$

Then, by (8), there exists an unknown continuous nonnegative function $\gamma_i(x_{[i]}, \xi_i)$ such that

$$|\xi_i(f_i(t,x,u) - g_i(t,x)x_{i+1})| \leqslant \frac{3}{4}|g_i(t,x)\xi_i(x_{i+1}^{p_i} + x_{i+1})| + \gamma_i(x_{[i]}, \xi_i). \tag{10}$$

Moreover, from Assumption 3 and (8), it follows that

$$\begin{cases} |\xi_1 \dot{y}_r| \leqslant |\xi_1| M =: |\xi_1|\beta_0, \\ \left|\xi_i \dfrac{\mathrm{d}\alpha_{i-1}^{p_{i-1}}}{\mathrm{d}t}\right| \leqslant |\xi_i|\beta_{i-1}(x_{[i]}, r_{[i-1]}, \xi_{[i-1]}), \quad i = 2, \ldots, n. \end{cases} \tag{11}$$

Substituting (9)–(11) into

$$\dot{V}_1(\xi_1) = \xi_1((g_1(t,x)x_2^{p_2} + f_2(t,x,u)) - \dot{y}_r)$$

or

$$\dot{V}_i(\xi_i) = \xi_i \left(\left(p_{i-1}x_i^{p_{i-1}-1} + 1\right)(g_i(t,x)x_{i+1}^{p_i} + f_i(t,x,u)) - \frac{\mathrm{d}\alpha_{i-1}^{p_{i-1}}}{\mathrm{d}t} \right), \quad i = 2, \ldots, n,$$

and using Assumption 1, we can verify (7) through simple calculations by letting $\rho_i(x_{[i]}, r_{[i-1]}, \xi_{[i+1]}) = \gamma_i(\cdot) + |\xi_i|\beta_{i-1}(\cdot) + \frac{7}{4}(p_{i-1}x_i^{p_{i-1}-1} + 1)\bar{g}_i(x_{[i]})|\xi_i\xi_{i+1}|$. The proof is completed.

Noting $\xi_{n+1} = \alpha_n(r_n, \xi_n)$, we see that $\rho_n(x, r_{[n-1]}, \xi_{[n+1]})$ depends on r_n, which means that one cannot prove the boundedness of r_n via the estimation of $\dot{V}_n(\xi_n)$ in (7). For this, we substitute $\xi_{n+1} = (k_n N(r_n)\xi_n)^{\frac{1}{p_n}}$ into the second term "$\hat{g}_n(t,x)\xi_n\xi_{n+1}$" on the right-hand side of (9), instead of estimating it as $\frac{7}{4}\bar{g}_n(x)|\xi_n\xi_{n+1}|$. Hence, we can obtain another estimation of $\dot{V}_n$ (unlike that in (7)):

$$\begin{aligned} \dot{V}_n(\xi_n) \leqslant{} & k_n(p_{n-1}x_n^{p_{n-1}-1} + 1)\hat{g}_n(t,x)N(r_n)\xi_n^2 + (p_{n-1}x_n^{p_{n-1}-1} + 1)\hat{g}_n(t,x)(k_n N(r_n))^{\frac{1}{p_n}}\xi_n^{\frac{p_n+1}{p_n}} \\ & + \gamma_n(x, \xi_n) + |\xi_n|\beta_{n-1}(x, r_{[n-1]}, \xi_{[n-1]}). \end{aligned} \tag{12}$$

4 Main results

This section collects the main results on the practical tracking, particularly to analyze the boundedness and practical tracking for system (1) with (3).

To analyze the existence of solutions of the closed-loop system, we give the following statement: ξ_i and r_i given above are continuous functions of $(t, x_{[i-1]}, x_i)$ respectively defined on $\mathcal{D}_{i-1} \times \mathbb{R}$ and $\mathcal{D}_i$, for any $i = 1, \ldots, n$, where

$$\begin{cases} \mathcal{D}_0 = \mathbb{R}_{\geqslant t_0}, \\ \mathcal{D}_i = \{(t, x_{[i-1]}, x_i) \in \mathcal{D}_{i-1} \times \mathbb{R} \mid \varphi_i(t)|\xi_i(t, x_{[i]})| < 1\}, \quad i = 1, \ldots, n, \end{cases}$$

and $\xi_i(t, x_{[i]})$ is exactly the above variable ξ_i. Let us next show this by induction. Note that y_r is a continuous function of time, and hence $\xi_1 = x_1 - y_r$ is a continuous function of (t, x_1) on $\mathcal{D}_0 \times \mathbb{R}$. Furthermore, by the definition of r_1, we see that r_1, as a function of (t, ξ_1), can be taken as a continuous function of (t, x_1) defined on $\mathcal{D}_1$. Suppose for any $i = 1, \ldots, n-1$ that ξ_j and r_j are continuous functions of $(t, x_{[j]})$ respectively defined on $\mathcal{D}_{j-1} \times \mathbb{R}$ and $\mathcal{D}_j$, for each $j = 1, \ldots, i$. Then, by the definition of ξ_{i+1}, we see that ξ_{i+1}, as a continuous function of (x_{i+1}, r_i, ξ_i), can be regarded as a continuous function of $(t, x_{[i+1]})$ defined on $\mathcal{D}_i \times \mathbb{R}$. Furthermore, noting the definition of r_{i+1}, we know that r_{i+1}, as a function of (t, ξ_{i+1}), can be taken as a continuous function of $(t, x_{[i+1]})$ defined on $\mathcal{D}_{i+1}$. The statement is thus shown.

Clearly, $(t_0, x_0) \in \mathcal{D}_n$ for any $x_0 \in \mathbb{R}^n$ because $\varphi_i(t_0) = 0$, $i = 1, \ldots, n$. By the continuity of $\xi_i(t, x_{[i]})$ and $\varphi_i(t)$, there exists a constant $\delta > 0$ such that $\mathcal{U} \subset \mathcal{D}_n$ with $\mathcal{U} := \{(t, x) \in \mathbb{R}_{\geqslant t_0} \times \mathbb{R}^n \mid |t - t_0| \leqslant \delta, \|x - x_0\| \leqslant \delta\}$. Moreover, by the above statement and the definition of $\alpha_n(r_n, \xi_n)$, one can take $u(\cdot)$ as a continuous function of (t, x) defined on $\mathcal{D}_n$. Therefore, by Theorem 2.1 in [26], for any initial value $x_0 \in \mathbb{R}^n$, the continuous closed-loop system has a solution $x(t)$ such that $(t, x(t)) \in \mathcal{U} \subset \mathcal{D}_n$ for all $t \in [t_0, t_0 + \delta)$. Furthermore, by Theorem 3.1 in [26], the δ can be enlarged to obtain the maximal interval of existence $[t_0, t_e)$ of solution $x(t)$ on $\mathcal{D}_n$, where $t_0 < t_e \leqslant +\infty$, and particularly, $t_e < +\infty$ implies $\lim_{t\to t_e}(\|x(t)\| + \sum_{i=1}^{n} \frac{1}{1-|\varphi_i(t)\xi_i(t,x_{[i]}(t))|}) = +\infty$.

Here, the uniqueness of solutions is not involved for the closed-loop system, owing to the lack of local Lipschitzness in x for the vector field. Note that $u^{p_n} = \alpha_n^{p_n}(r_n, \xi_n)$, and $\alpha_n^{p_n}(r_n(t, \xi_n(t, x)), \xi_n(t, x))$ is locally Lipschitz in x on $\mathcal{D}_n$ (which can be shown similar to the proof of Proposition 2.6 in [23]). Thus, if $g_i(\cdot)$ and $f_i(\cdot)$ are strengthened to be locally Lipschitz in x and independent of u, the uniqueness of solutions would hold for the closed-loop system.

The analysis below proceeds for a solution $x(t)$. Because the solution has no any specificality, the properties to be derived actually hold for all the solutions. For convenience, write $\xi_i(t, x_{[i]}(t))$ and $r_i(t, \xi_i(t))$ as $\xi_i(t)$ and $r_i(t)$, respectively.

Lemma 2. Suppose that $\xi_{[n]}(t)$ is bounded on $[t_0, t_e)$. Then there are the following implications:

$$\sup_{t\in[t_0,t_e)} (\|x_{[i]}(t)\| + \|r_{[i-1]}(t)\|) < +\infty \Longrightarrow \sup_{t\in[t_0,t_e)} r_i(t) < +\infty, \quad i = 1, \ldots, n. \tag{13}$$

Proof. Only the case $i = n$ is considered via the estimation of $\dot{V}_n(\xi_n)$ given by (12), because other cases are rather similar via the estimations of $\dot{V}_i(\xi_i)$ in Lemma 1.

Suppose that $\sup_{t\in[t_0,t_e)}(\|x(t)\| + \|r_{[n-1]}(t)\|) < +\infty$, and for contradiction, that $r_n(t)$ is unbounded on $[t_0, t_e)$. Then, noting $r_n(t) \geqslant 1$, $\forall t \in [t_0, t_e)$, it holds that $\limsup_{t\to t_e} r_n(t) = +\infty$. Furthermore, there exist time sequences $\{\tau_m\}$, $\{\sigma_m\}$ and $\{\varsigma_m\}$ such that, for any $m \in \mathbb{Z}_+$,

$$\begin{aligned}
\tau_m &= \inf\{t \in [t_0, t_e) \,|\, r_n(t) = \delta_{m+1}\},\\
\sigma_m &= \sup\{t \in [t_0, \tau_m) \,|\, N(r_n(t)) = N(\delta_m)\},\\
\varsigma_m &= \sup\{t \in [t_0, \tau_m) \,|\, r_n(t) = \delta_m\},
\end{aligned}$$

where $\{\delta_m\}$ is a strictly increasing and unbounded sequence (with $\delta_1 \geqslant 2$) rendering $\{|N(\delta_m)|\}$ strictly increasing and unbounded, and

$$\begin{cases} N(\delta_m) < 0, \ \forall m \in \mathbb{Z}_+ & \text{if } \mathrm{sign}(g_n) = 1,\\ N(\delta_m) > 0, \ \forall m \in \mathbb{Z}_+ & \text{if } \mathrm{sign}(g_n) = -1. \end{cases} \tag{14}$$

It is worth pointing out that such $\{\delta_m\}$ must exist because $N(\cdot)$ satisfies (6).

For any $m \in \mathbb{Z}_+$, it holds that $\tau_m > \sigma_m \geqslant \varsigma_m$ and $r_n(\tau_m) > r_n(\sigma_m)$, because $N(r_n(\tau_m)) \neq N(\delta_m)$ and $N(r_n(\sigma_m)) = N(r_n(\varsigma_m)) = N(\delta_m)$. Furthermore, for any $m \in \mathbb{Z}_+$, we derive that $r_n(t) \geqslant \delta_m$, $\forall t \in [\sigma_m, \tau_m]$, which, together with the definition of r_n, yields

$$(\varphi_n(t)\xi_n(t))^2 \geqslant 1 - \frac{1}{\delta_m} \geqslant \frac{1}{2}, \quad \forall t \in [\sigma_m, \tau_m]. \tag{15}$$

Moreover, by (14) and the definitions of τ_m and σ_m, it holds that

$$\begin{cases} N(r_n(t)) < 0 \text{ and } N(r_n(t)) \in [N(\delta_{m+1}), N(\delta_m)], \ \forall t \in [\sigma_m, \tau_m], \ \forall m \in \mathbb{Z}_+ & \text{if } \mathrm{sign}(g_n) = 1,\\ N(r_n(t)) > 0 \text{ and } N(r_n(t)) \in [N(\delta_m), N(\delta_{m+1})], \ \forall t \in [\sigma_m, \tau_m], \ \forall m \in \mathbb{Z}_+ & \text{if } \mathrm{sign}(g_n) = -1, \end{cases} \tag{16}$$

which, together with Assumption 1 and the definition of $\hat{g}_n(t, x)$, implies that, for any $m \in \mathbb{Z}_+$,

$$\hat{g}_n(t, x(t))N(r_n(t)) \leqslant -\frac{1}{4}\theta_{g_n}|N(\delta_m)|, \quad \forall t \in [\sigma_m, \tau_m], \tag{17}$$

where $\theta_{g_n} = \inf_{t\in[t_0,t_e)} \underline{g}_n(x(t)) > 0$.

By (12) and the boundedness of $x(t)$, $r_{[n-1]}(t)$, $\xi_n(t)$, $\varphi_n(t)$ and $\dot{\varphi}_n(t)$, there exists a constant $d > 0$ such that, for almost all $t \in [t_0, t_e)$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\varphi_n(t)\xi_n(t))^2 &= 2\varphi_n^2(t)\dot{V}_n(\xi_n(t)) + 2\varphi_n(t)\dot{\varphi}_n(t)\xi_n^2(t)\\ &\leqslant 2k_n(p_{n-1}x_n^{p_{n-1}-1}(t)+1)\varphi_n^2(t)\xi_n^2(t)\hat{g}_n(t,x(t))N(r_n(t))\\ &\quad + 2(p_{n-1}x_n^{p_{n-1}-1}(t)+1)\varphi_n^2(t)\xi_n^{\frac{p_n+1}{p_n}}(t)\hat{g}_n(t,x(t))(k_nN(r_n(t)))^{\frac{1}{p_n}} + d. \qquad (18)\end{aligned}$$

Choose m^* sufficiently large such that $|N(\delta_{m^*})| > \frac{2d\delta_1}{k_n\theta_{g_n}(\delta_1-1)}$. Then, by (15)–(18), and noting that p_n is odd and $p_{n-1}x_n^{p_{n-1}-1} + 1 \geqslant 1$, we obtain that, for almost all $t \in [\sigma_{m^*}, \tau_{m^*}]$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\varphi_n(t)\xi_n(t))^2 &\leqslant -\frac{1}{2}k_n\theta_{g_n}|N(\delta_{m^*})|(p_{n-1}x_n^{p_{n-1}-1}(t)+1)\varphi_n^2(t)\xi_n^2(t)\\ &\quad - 2(k_n|N(\delta_{m^*})|)^{\frac{1}{p_n}}(p_{n-1}x_n^{p_{n-1}-1}(t)+1)\varphi_n^2(t)\xi_n^{\frac{p_n+1}{p_n}}(t)|\hat{g}_n(t,x(t))| + d\\ &\leqslant -\frac{1}{4}k_n\theta_{g_n}|N(\delta_{m^*})|(p_{n-1}x_n^{p_{n-1}-1}(t)+1) + d\\ &< 0,\end{aligned}$$

which implies $(\varphi_n(\tau_{m^*})\xi_n(\tau_{m^*}))^2 < (\varphi_n(\sigma_{m^*})\xi_n(\sigma_{m^*}))^2$. This, together with $r_n(\sigma_{m^*}) < r_n(\tau_{m^*})$, leads to the following contradiction:

$$0 < r_n(\sigma_{m^*}) - r_n(\tau_{m^*}) = \frac{1}{1-(\varphi_n(\sigma_{m^*})\xi_n(\sigma_{m^*}))^2} - \frac{1}{1-(\varphi_n(\tau_{m^*})\xi_n(\tau_{m^*}))^2} < 0.$$

Therefore, $r_n(t)$ is bounded on $[t_0, t_e)$, and implication (13) is proved. This completes the proof.

Remark 5. In the boundedness analysis of each r_i, the key lies in counteracting the serious non-parametric uncertainties. Specifically, the boundedness of $x_{[i]}(t)$, $r_{[i-1]}(t)$ and $\xi_{[i+1]}(t)$ means that the non-parametric uncertainties involved in the estimate of $\dot{V}_i(\xi_i)$ can be bounded by unknown constants along the system solutions. Then, when the system states sufficiently get close to the boundary of the funnel $\mathcal{F}_{\varphi_i} = \{(t,x) \in \mathbb{R}_{\geqslant t_0} \times \mathbb{R}^n | \varphi_i(t)|\xi_i(t,x_{[i]})| < 1\}$, gain r_i would become sufficiently large, and play a compensation role to capture the unknown constant bounds of the non-parametric uncertainties, which can prevent the system states from contacting the boundary of funnel $\mathcal{F}_{\varphi_i}$. Consequently, the system states is rendered to forever evolve within the funnel $\mathcal{F}_{\varphi_i}$, and the boundedness of r_i is guaranteed.

We are now ready to establish Theorem 1 on practical tracking with prescribed transient performance of system (1).

Theorem 1. For system (1) under Assumptions 1–3, controller (3) with (4) guarantees that, for any initial value $x_0 \in \mathbb{R}^n$, all the signals of the closed-loop system (i.e., system state x, controller u and gain r_i) are bounded on $\mathbb{R}_{\geqslant t_0}$, and furthermore, $\sup_{t\geqslant t_0}\varphi_\lambda(t)|y(t) - y_r(t)| < 1$, that is, the tracking error evolves within the funnel $\mathcal{F}_{\psi_\lambda}$.

Proof. For any initial value $x_0 \in \mathbb{R}^n$, the closed-loop system has a solution $x(t)$, with the maximal interval of existence $[t_0, t_e)$, and for any $i = 1, \ldots, n$,

$$\varphi_i(t)|\xi_i(t)| < 1, \quad \forall t \in [t_0, t_e), \qquad (19)$$

where t_e, as above, is the maximal time of existence of solution $x(t)$.

We first show that $\xi_{[n]}(t)$ is bounded on $[t_0, t_e)$. In fact, by $\varphi_i(t) > 0$, $\forall t > t_0$ and $\liminf_{t\to+\infty}\varphi_i(t) > 0$, we know for any $i = 1, \ldots, n$ that, whether $t_e = +\infty$, it always holds $\varphi_i(t) > \varepsilon_i$, $\forall t \in [T_i, t_e)$ for some $\varepsilon_i > 0$ and $T_i \in (t_0, t_e)$, which, together with (19), yields $|\xi_i(t)| < 1/\varphi_i(t) < 1/\varepsilon_i$, $\forall t \in [T_i, t_e)$. The boundedness of $\xi_i(t)$ is thus derived by the continuity of $\xi_i(t)$.

Next, we prove by induction that $x_i(t)$ and $r_i(t)$, $i = 1, \ldots, n$ are bounded on $[t_0, t_e)$. From the boundedness of $\xi_1(t)$ and $y_r(t)$, it follows that $x_1(t) = \xi_1(t) + y_r(t)$ is bounded on $[t_0, t_e)$. Hence, by (13), we deduce that $r_1(t)$ is bounded on $[t_0, t_e)$. Suppose for any $l = 1, \ldots, n-1$ that $x_{[l]}(t)$ and $r_{[l]}(t)$ are bounded on $[t_0, t_e)$. Then, by the boundedness of $\xi_{[l+1]}(t)$, and noting $x_{l+1}^{p_l} + x_{l+1} = \xi_{l+1} + \alpha_l^{p_l}(r_l, \xi_l)$, we get that $x_{l+1}(t)$ is bounded on $[t_0, t_e)$. Furthermore, invoking (13) yields that $r_{l+1}(t)$ is bounded on $[t_0, t_e)$. Therefore, the boundedness of $x(t)$ and $r_{[n]}(t)$ is obtained.

By the boundedness of $x(t)$ and $r_{[n]}(t)$, and the definition of each $r_i(t)$, we deduce

$$\lim_{t\to t_e}\left(\|x(t)\| + \sum_{i=1}^{n}\frac{1}{1-|\varphi_i(t)\xi_i(t)|}\right) < +\infty,$$

which, together with the definition of t_e, concludes $t_e = +\infty$. Furthermore, combining with the definition of $u(r_n, \xi_n)$, we see that $u(r_n(t), \xi_n(t))$ is bounded on $\mathbb{R}_{\geqslant t_0}$. In addition, by the boundedness of $r_1(t)$, there exists $\varepsilon > 1$ such that

$$\frac{1}{1-(\varphi_1(t)\xi_1(t))^2} = r_1(t) \leqslant \varepsilon, \quad \forall t \geqslant t_0,$$

which implies

$$\psi_\lambda(t)|y(t) - y_r(t)| = \varphi_1(t)|\xi_1(t)| \leqslant \sqrt{1-\frac{1}{\varepsilon}}, \quad \forall t \geqslant t_0.$$

This completes the proof.

In terms of the above scheme, certain prescribed tracking performance can be ensured for system (1) by delicately specifying the performance funnel. Particularly, we can achieve the below two interesting tracking objectives for system (1), which are impossible for the design schemes in [2, 3, 7, 8, 21]. One objective is to steer the tracking error to pre-specified accuracy no later than a prescribed time.

Corollary 1. Under Assumptions 1–3, global fixed-time practical tracking can be solved for system (1) via a time-varying state-feedback controller of form (3) with (4). That is, for any prescribed $\lambda > 0$ and $T \geqslant t_0$, there is a controller of form (3) with (4) such that, for any initial value $x_0 \in \mathbb{R}^n$, all the signals of the closed-loop system are bounded on $\mathbb{R}_{\geqslant t_0}$, and $\sup_{t\geqslant T}|e(t)| \leqslant \lambda$.

Proof. Given $\lambda > 0$ and $T > t_0$, we choose $\psi_\lambda(t) = \frac{2}{\lambda}\min\{\frac{t-t_0}{T-t_0}, 1\}$. Then, in terms of the design procedure in Section 3, a desirable controller of form (3) with (4) is constructed for system (1). Furthermore, by Theorem 1, it is deduced that, for any initial value $x_0 \in \mathbb{R}^n$, all the signals of the resulting closed-loop system are bounded on $\mathbb{R}_{\geqslant t_0}$, and $\sup_{t\geqslant t_0}\psi_\lambda(t)|e(t)| < 1$, which implies that $|e(t)| \leqslant 1/\psi_\lambda(t) \leqslant \lambda/2$ for all $t \geqslant T$.

The other tracking objective is to guarantee the maximum overshoot of the tracking error less than a prescribed level. For this, we define

$$\mathcal{S}' \triangleq \left\{\psi \in \mathcal{W}^{1,\infty}(\mathbb{R}_{\geqslant t_0}, \mathbb{R}) \,\middle|\, \inf_{t\geqslant t_0}\psi(t) \geqslant \frac{1}{\varepsilon}, \liminf_{t\to+\infty}\psi(t) \geqslant \frac{2}{\lambda}\right\},$$

where λ and ε are prescribed positive constants representing the ultimate accuracy and the maximum overshoot of the tracking error, respectively. It can be directly checked that, if $\psi_\lambda \in \mathcal{S}$ is replaced by $\psi_{\lambda,\varepsilon} \in \mathcal{S}'$, the above control design and analysis is still feasible for any initial value with $\psi_{\lambda,\varepsilon}(t_0)|e(t_0)| < 1$. Recognizing this, we establish Corollary 2.

Corollary 2. Under Assumptions 1–3, semiglobal practical tracking with prescribed maximum overshoot is solvable for system (1) via a time-varying state-feedback controller of form (3) with (4). That is, for given $\varepsilon > \delta > \lambda > 0$, there is a controller of form (3) with (4) such that, for any initial value $x_0 \in \mathbb{R}^n$ with $|e(t_0)| \leqslant \delta$, all the signals of the resulting closed-loop system are bounded on $\mathbb{R}_{\geqslant t_0}$, and $\sup_{t\geqslant T}|e(t)| \leqslant \lambda$ for some finite $T \geqslant t_0$, with $\sup_{t\geqslant t_0}|e(t)| \leqslant \varepsilon$.

Proof. For given $\varepsilon > \delta > \lambda > 0$, we choose a $\psi_{\lambda,\varepsilon} \in \mathcal{S}'$ with $\psi_{\lambda,\varepsilon}(t_0) < 1/\delta$, such as $\psi_{\lambda,\varepsilon}(t) = \frac{2}{\lambda+(2\varepsilon-\lambda)e^{t_0-t}}$ or $\psi_{\lambda,\varepsilon}(t) = \frac{2(t-t_0)+1}{\varepsilon+\lambda(t-t_0)}$. Then, in terms of the design procedure in Section 3 with replacing $\psi_\lambda(t)$ by $\psi_{\lambda,\varepsilon}(t)$, a desirable controller of form (3) with (4) is constructed for system (1). Furthermore, by $\psi_{\lambda,\varepsilon}(t_0) < 1/\delta$ and the same analysis as that in Theorem 1, it can be shown that, for any initial value $x_0 \in \mathbb{R}^n$ with $|e(t_0)| \leqslant \delta$, all the signals of the resulting closed-loop system are bounded on $\mathbb{R}_{\geqslant t_0}$, and $\sup_{t\geqslant t_0}\psi_{\lambda,\varepsilon}(t)|e(t)| < 1$. From this, it follows that $\sup_{t\geqslant t_0}|e(t)| \leqslant \frac{1}{\inf_{t\geqslant t_0}\psi_{\lambda,\varepsilon}(t)} \leqslant \varepsilon$, and that $\limsup_{t\to+\infty}|e(t)| \leqslant \frac{1}{\liminf_{t\to+\infty}\psi_{\lambda,\varepsilon}(t)} \leqslant \frac{\lambda}{2}$, which implies $\sup_{t\geqslant T}|e(t)| < \lambda$ for some $T \geqslant t_0$.

5 Further discussions

In this section, we demonstrate the applicability of the tracking scheme to more general systems additionally with less-restrictive unmodeled dynamics. For this, consider the practical tracking for the following uncertain nonlinear system with unmodeled dynamics (described by z-subsystem):

$$\begin{cases} \dot{z} = f_0(t, z, x_1), \\ \dot{x}_i = g_i(t, z, x)x_{i+1}^{p_i} + f_i(t, z, x, u), \ i = 1, \dots, n-1, \\ \dot{x}_n = g_n(t, z, x)u^{p_n} + f_n(t, z, x, u), \\ y = x_1, \end{cases} \tag{20}$$

where $z \in \mathbb{R}^m$ and $x = [x_1, \dots, x_n]^{\mathrm{T}} \in \mathbb{R}^n$ are the system state vectors with the initial values $z(t_0) = z_0$ and $x(t_0) = x_0$; $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are the control input and system output, respectively; $p_i \in \mathbb{R}_{\mathrm{odd}}^{\geqslant 1}$, $i = 1, \dots, n$; $g_i : \mathbb{R}_{\geqslant t_0} \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, n$, $f_0 : \mathbb{R}_{\geqslant t_0} \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ and $f_i : \mathbb{R}_{\geqslant t_0} \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $i = 1, \dots, n$ are unknown continuous functions. Moreover, suppose that z is unmeasurable, and $g_i(\cdot)$ and $f_i(\cdot)$ obey Assumptions 4–6.

Assumption 4. Dynamics $\dot{z} = f_0(z, x_1)$ is bounded-input-bounded-state (BIBS) stable with x_1 as input. That is, if $x_1(t)$ is bounded on some interval, then $z(t)$ is bounded on the same interval.

Assumption 5. The signs of $g_i(t, z, x)$, $i = 1, \dots, n$ are unknown but remain unchanged. Moreover, there exist unknown smooth positive functions $\underline{g}_i(z, x_{[i]})$ and $\bar{g}_i(z, x_{[i]})$, $i = 1, \dots, n$, such that

$$\underline{g}_i(z, x_{[i]}) \leqslant |g_i(t, z, x)| \leqslant \bar{g}_i(z, x_{[i]}), \quad i = 1, \dots, n.$$

Assumption 6. There exist unknown continuous nonnegative functions $\bar{f}_i(z, x_{[i]})$, $i = 1, \dots, n$, and unknown constants $q_i \in [0, p_i)$, $i = 1, \dots, n$, such that

$$|f_i(t, z, x, u)| \leqslant (1 + |x_{i+1}|^{q_i})\bar{f}_i(z, x_{[i]}), \quad i = 1, \dots, n,$$

where $x_{n+1} = u$.

Indeed, Assumption 4 is a rather weak hypothesis on unmodeled dynamics, whereas most of the related studies (e.g., [3, 27]) require the conditions of input-to-state practical stability type. Besides allowing the severe uncertainties as stated in Section 2, Assumptions 5 and 6 mean that the control coefficients and the system nonlinearities inherently depend on the unmeasurable state z, and particularly, the amplitudes of the control coefficients need not be lower bounded by positive continuous functions independent of z as in [3]. For example, $g_i(t, z, x) = \frac{1}{1+\|z\|^2}$ is covered by Assumption 5, but excluded by [3].

We are now ready to show the validity of controller (3) with (4) for system (20). Let $V_i(\xi_i) = \frac{\xi_i^2}{2}$, $i = 1, \dots, n$. Then, along the trajectories of the resulting closed-loop system, it can be verified as in Lemma 1 and (12) that

$$\begin{cases} \dot{V}_1(\xi_1) \leqslant k_1\hat{g}_1(t, z, x)N(r_1)\xi_1^2 + \phi_1(z, x_1, \xi_{[2]}), \\ \dot{V}_i(\xi_i) \leqslant k_i(p_{i-1}x_i^{p_{i-1}-1} + 1)\hat{g}_i(t, z, x)N(r_i)\xi_i^2 + \phi_i(z, x_{[i]}, r_{[i-1]}, \xi_{[i+1]}), \ i = 2, \dots, n-1, \\ \dot{V}_n(\xi_n) \leqslant k_n(p_{n-1}x_n^{p_{n-1}-1} + 1)\hat{g}_n(t, z, x)N(r_n)\xi_n^2 + \phi_n(z, x, r_{[n-1]}, \xi_{[n]}) \\ \qquad\qquad + (p_{n-1}x_n^{p_{n-1}-1} + 1)\hat{g}_n(t, z, x)(k_nN(r_n))^{\frac{1}{p_n}}\xi_n^{\frac{p_n+1}{p_n}}, \end{cases} \tag{21}$$

where $\hat{g}_i(t, z, x)$ is an unknown function defined similar to $\hat{g}_i(t, x)$ in (7), and $\phi_i(\cdot)$ is an unknown continuous nonnegative function. Based on this, Theorem 2 can be established.

Theorem 2. For system (20) under Assumptions 3–6, controller (3) with (4) guarantees that, for any initial value $z_0 \in \mathbb{R}^m$ and $x_0 \in \mathbb{R}^n$, all the signals of the closed-loop system (i.e., system states z and x, controller u and gain r_i) are bounded on $\mathbb{R}_{\geqslant t_0}$, and the tracking error evolves within the funnel $\mathcal{F}_{\psi_\lambda}$.

Proof. Like in Section 4, for any initial value $z_0 \in \mathbb{R}^m$ and $x_0 \in \mathbb{R}^n$, the closed-loop system has a solution $(z(t), x(t))$, with the maximal interval of existence $[t_0, t_e)$, and for any $i = 1, \ldots, n$,

$$\varphi_i(t)|\xi_i(t)| < 1, \quad \forall t \in [t_0, t_e). \tag{22}$$

As in the proof of Theorem 1, the boundedness of $\xi_{[i]}(t)$ on $[t_0, t_e)$ can be derived based on (22). Then, by (21) and $|g_i(t, z, x)| \geqslant \underline{g}_i(z, x_{[i]})$, $i = 1, \ldots, n$ (in Assumption 5), the following relation can be verified quite similar to the proof of Lemma 2:

$$\sup_{t\in[t_0,t_e)} (\|z(t)\| + \|x_{[i]}(t)\| + \|r_{[i-1]}(t)\|) < +\infty \Longrightarrow \sup_{t\in[t_0,t_e)} r_i(t) < +\infty, \quad i = 1, \ldots, n.$$

Moreover, by Assumption 3 and the boundedness of $\xi_1(t)$, we see that $x_1(t) = \xi_1(t) + y_r(t)$ is bounded on $[t_0, t_e)$, which, together with Assumption 4, implies that $z(t)$ is bounded on $[t_0, t_e)$. Therefore, in terms of the proof of Theorem 1, the boundedness of $x_i(t)$ and $r_i(t)$, $i = 1, \ldots, n$ on $[t_0, t_e)$ can be recursively derived, from which, we can arrive at $t_e = +\infty$ and $\sup_{t\geqslant t_0} \varphi_\lambda(t)|y(t) - y_r(t)| < 1$. This completes the proof.

6 Simulation examples

In this section, two examples are given to further illustrate the effectiveness of the tracking scheme established. Specifically, Example 1, as a theoretical one, involves not only non-parametric uncertainties but also unknown control directions, and achieves global fixed-time practical tracking. Example 2, as a practical one, concentrates on the achievement of practical tracking with prescribed maximum overshoot, and additionally demonstrates the practicability of the tracking scheme.

Example 1. Consider the global practical tracking for the following 3-dimensional nonlinear system:

$$\begin{cases} \dot{z} = f_0(t, z, x_1), \\ \dot{x}_1 = g_1(t, z, x)x_2^3 + f_1(t, z, x, u), \\ \dot{x}_2 = g_2(t, z, x)u + f_2(t, z, x, u), \\ y = x_1, \end{cases} \tag{23}$$

where $f_0(t, z, x_1)$ and $g_i(t, z, x)$, $f_i(t, z, x, u)$, $i = 1, 2$ satisfy Assumptions 4–6, respectively.

The tracking schemes in [2,3,8,21] inherently depend on the parametrization of $g_i(t, x)$ and $f_i(t, x, u)$, and moreover, cannot tune the arrive time of the tracking error. We now design a tracking controller allowing non-parametric uncertainties, unknown control directions and unmodeled dynamics, to guarantee the tracking error to reach pre-specified accuracy before a prescribed time.

In what follows, let $t_0 = 0$, $y_r(t) = \sin(t)$, and $\psi_\lambda(t) = \min\{t, 10\}$ with the ultimate accuracy $\lambda = 0.2$ and the arrival time $T = 10$ s. According to the design procedure in Section 3, we obtain the following controller for system (23):

$$u = 5r_2 \cos(\pi r_2)\xi_2,$$

where ξ_2 and r_2 are given by

$$\begin{cases} \xi_1 = x_1 - y_r, \\ \xi_2 = x_2^3 + x_2 - 15\xi_1 r_1 \cos(\pi r_1), \\ r_i = \dfrac{1}{1 - (\varphi_i(t)\xi_i)^2}, \quad i = 1, 2, \end{cases}$$

with $\varphi_1(t) = \psi_\lambda(t)$ and $\varphi_2(t) = \min\{2t, 4\}$.

Let $g_1(t, z, x) = 5(e^z + \frac{x_1^2}{1+x_2^2} + 3)$, $g_2(t, z, x) = 2(x_2^2 + 5)$, $f_0(t, z, x_1) = -z + x_1^2$, $f_1(t, z, x, u) = x_2^2 \sin x_1$, $f_2(t, z, x, u) = z^3 - 10 \sin x_2$, and the initial condition be $z(0) = 1$ and $x(0) = [3, -2]^{\mathrm{T}}$. Using MATLAB, Figure 1 is obtained to exhibit the trajectories of the tracking error e and all the signals of the closed-loop system (to clearly show the transient behavior, the logarithmic X-coordinate has been adopted before

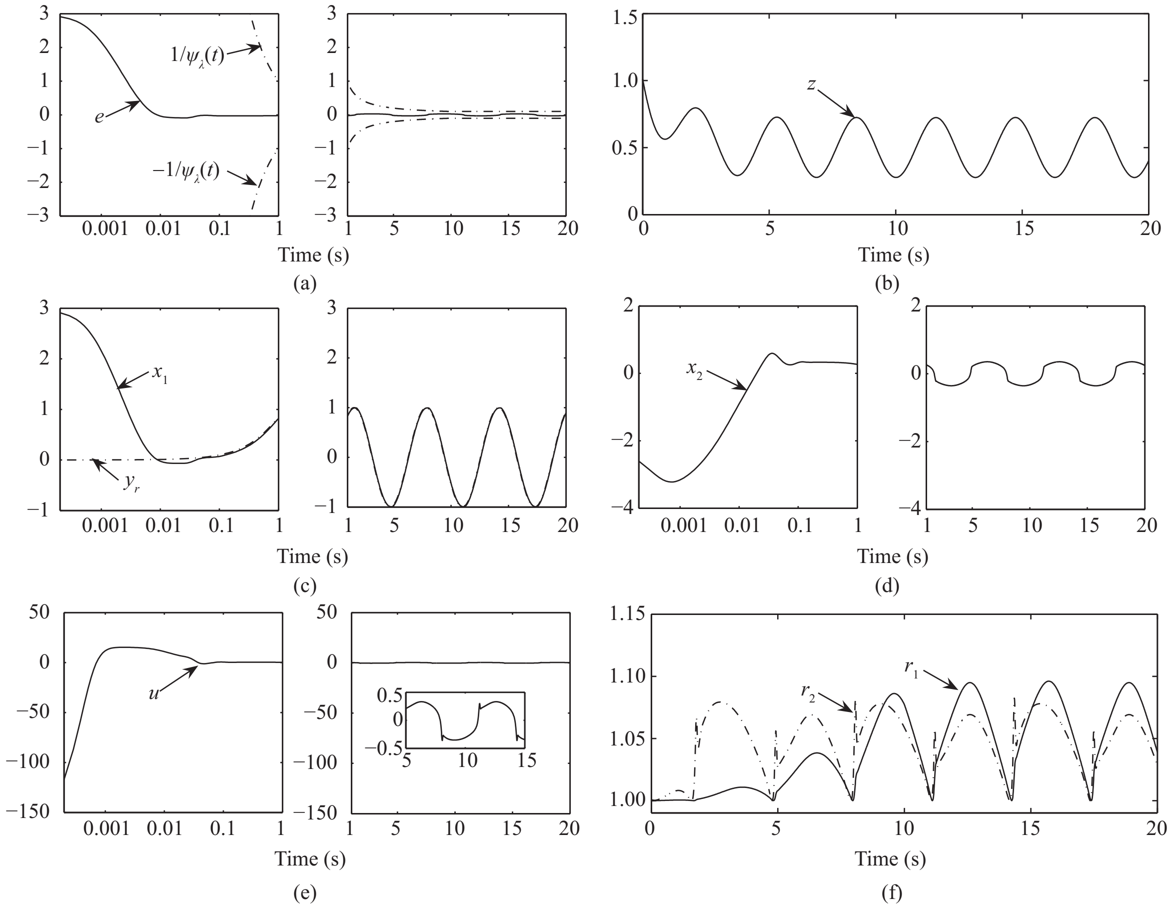

Figure 1 The evolution of system (23). The trajectory of (a) error e, (b) state z, (c) state x_1, (d) state x_2, (e) control u, and (f) gains r_1 and r_2.

1 s in the figures of e, x_1, x_2 and u). It can be seen that the tracking error evolves within the funnel $\mathcal{F}_{\psi_\lambda} = \{(t, e) \in \mathbb{R}_+ \times \mathbb{R} \mid \psi_\lambda(t)|e| < 1\}$, which means $|e(t)| < 0.2$ after $T = 10$ s.

Example 2. Consider the practical tracking of the following uncertain nonlinear system, which describes the reduced-order model for a two degrees of freedom underactuated, weakly coupled, unstable mechanical system [2,3,16]:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = \dfrac{k_s}{ml}(x_3 - l \sin x_1 \sqrt[3]{\cos x_1})^3 + \dfrac{g}{l} \sin x_1, \\ \dot{x}_3 = u, \\ y = x_1, \end{cases} \tag{24}$$

where g is the gravitational acceleration, and l, k_s and m are unknown physical parameters.

The tracking schemes in [2,3,8,21] only guarantee the ultimate convergence of the tracking error, but are not concerned with its overshoot. However, in terms of the scheme in Section 3, we can design a concise controller to achieve practical tracking with prescribed maximum overshoot for system (24).

In what follows, let $t_0 = 0$, $y_r(t) = 0.1 \sin(t)$, and $\psi_{\lambda,\varepsilon}(t) = \frac{2t+1}{1+0.1t}$ with the ultimate accuracy $\lambda = 0.1$ and the maximum overshoot $\varepsilon = 1$. According to the design procedure in Section 3, the following controller is obtained for system (24):

$$u = 20r_3 \sin(\pi r_3)\xi_3,$$

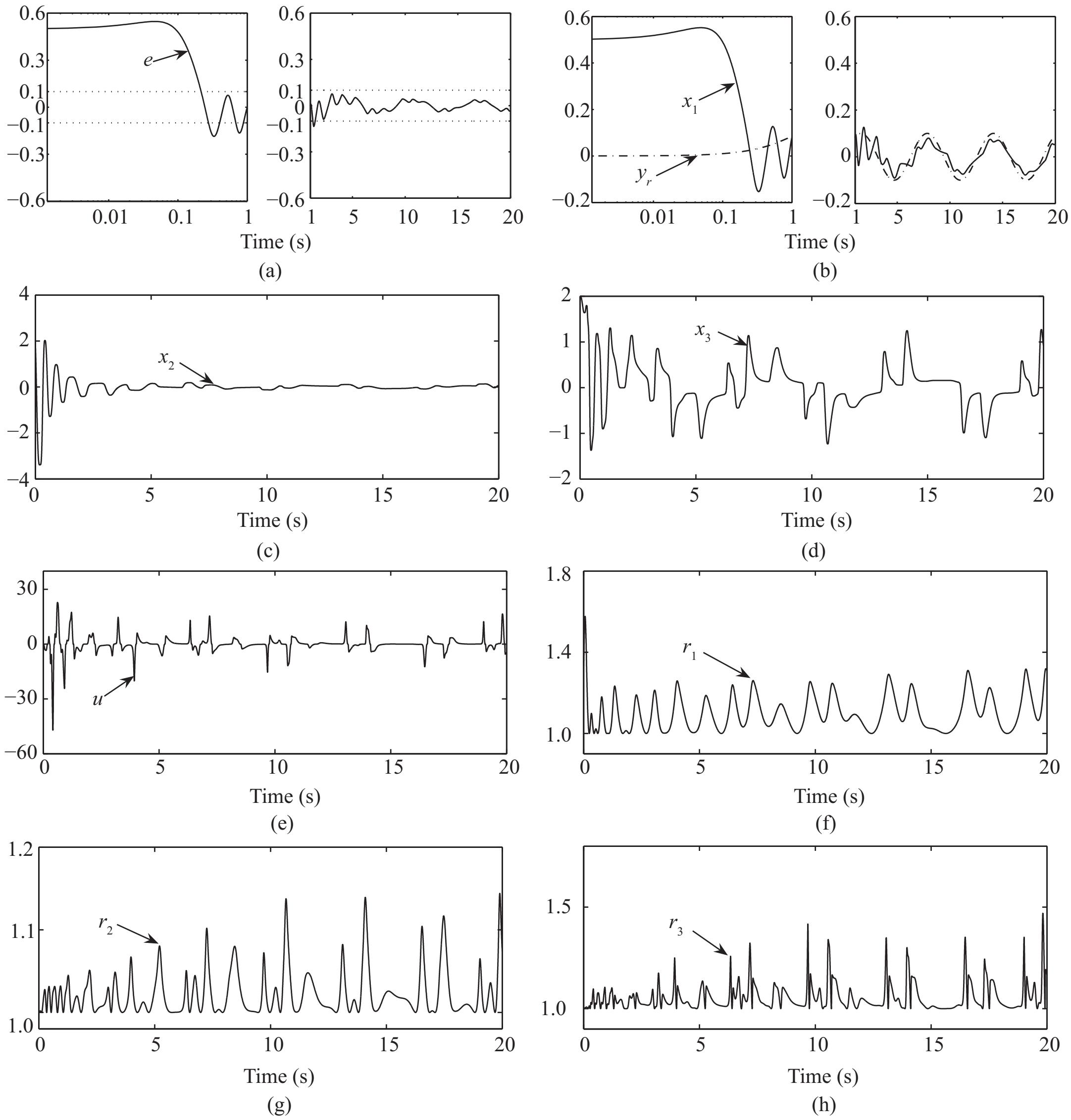

Figure 2 The evolution of system (24). The trajectory of (a) error e, (b) state x_1, (c) state x_2, (d) state x_3, (e) control u, (f) gain r_1, (g) gain r_2, and (h) gain r_3.

where ξ_3 and r_3 are given by

$$\begin{cases} \xi_1 = x_1 - y_r, \\ \xi_2 = 2x_2 - 10\xi_1 r_1 \sin(\pi r_1), \\ \xi_3 = x_3^3 + x_3 - 20\xi_2 r_2 \sin(\pi r_2), \\ r_i = \dfrac{1}{1-(\varphi_i(t)\xi_i)^2}, \quad i = 1,2,3, \end{cases}$$

with $\varphi_1(t) = \psi_\lambda(t)$ and $\varphi_i(t) = \min\{0.1t, 1\}$, $i = 2,3$.

Let $l = 9.8$, $\frac{k_s}{ml} = 2$ and the initial condition be $x(0) = [0.5, 2, 2]^{\mathrm{T}}$. Then, the evolution of the closed-loop system is shown in Figure 2 (to clearly show the transient behavior, the logarithmic X-coordinate has been adopted before 1 s in the figures of e and x_1). Particularly, it is demonstrated that the tracking error satisfies $|e(t)| < 0.1$ after 5 s, and its overshoot remains lower than the level $\varepsilon = 1$.

Moreover, Figure 2 illustrates the presence of oscillation in the evolution, which mainly arises from the oscillation of the reference signal y_r and the introduction of Nussbaum-type function (i.e., $s \mapsto s\sin(\pi s)$) in the controller. Noting the positive control directions of system (24) and by Remark 3, "$s \mapsto -s$", instead of "$s \mapsto s\sin(\pi s)$", can be adopted in the controller to reduce the oscillation in the evolution while achieving the desired tracking.

7 Concluding remarks

In this paper, a time-varying scheme of practical tracking with prescribed transient performance has been established for inherently nonlinear systems with unknown control directions and non-parametric uncertainties. Despite the presence of serious uncertainties and inherent nonlinearities in the systems, the tracking objective is refined to render the tracking error to forever evolve within a prescribed funnel, through which prescribed transient performance is guaranteed. However, the tracking scheme we proposed requires the boundedness of $\psi_\lambda(t)$, which makes it impossible that the asymptotic tracking by the funnel $\mathcal{F}_{\psi_\lambda}$ is already defined. Although in [28], asymptotic tracking with prescribed performance was considered by the funnel control method, the tracking scheme is restricted to linear systems. Therefore, it is very interesting to explore the feasibility of asymptotic tracking with prescribed performance for uncertain nonlinear systems and the design scheme of the tracking controller if feasible.

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