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Efficient quantum state transmission via perfect quantum network coding

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Abstract Quantum network coding with the assistance of auxiliary resources can achieve perfect transmission of the quantum state. This paper suggests a novel perfect network coding scheme to efficiently solve the quantum k-pair problem, in which only a few assisting resources are introduced. Specifically, only one pair of maximally entangled state needs to be pre-shared between two intermediate nodes, and only $O(k)$ of classical information is transmitted though the network. Moreover, the classical communication used in our protocol does not cause transmission congestion, providing better adaptability to large-scale quantum k-pair networks. Through relevant analyses and comparisons, we demonstrate that our proposed scheme saves resources and has good application value, thereby showing its high efficiency. Furthermore, the proposed scheme achieves 1-max flow quantum communication, and the achievable rate region result is extended from its counterpart over the butterfly network.

Keywords perfect quantum network coding, quantum k-pair problem, efficient quantum state transmission, communication efficiency, achievable rate region

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1 Introduction

In 2000, classical network coding (CNC) was first proposed by Ahlswede et al. [1]. CNC enables the multicast transmission rate to achieve the upper limit of the network capacity, which is determined by the max-flow min-cut theorem. In other words, the network coding technology can significantly improve communication efficiency as it can achieve maximum flow network communication and save bandwidth-resource consumption. Currently, significant advances have been made in the theories [2] and applications [3–5] of CNC. These have a wide range of research meanings in some network structures [6–8]. A simple and nice example is the butterfly network, as illustrated in Figure 1, which is a type of abstract model used to study the bottleneck problem of data transmission. The edge (n_1, n_2) represents the bottleneck channel. Owing to the network's directional property and limited capacity of one bit on every edge, the node n_1 cannot transmit information to both t_1 and t_2 for a one-shot case. However, CNC solves this problem by allowing the nodes n_1, t_1 and t_2 to perform XOR operations, and the nodes s_1, s_2 and n_2

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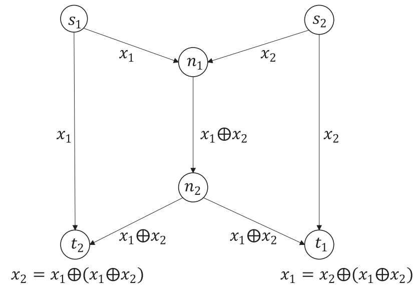

Figure 1 Classical network coding (CNC) over the butterfly network.

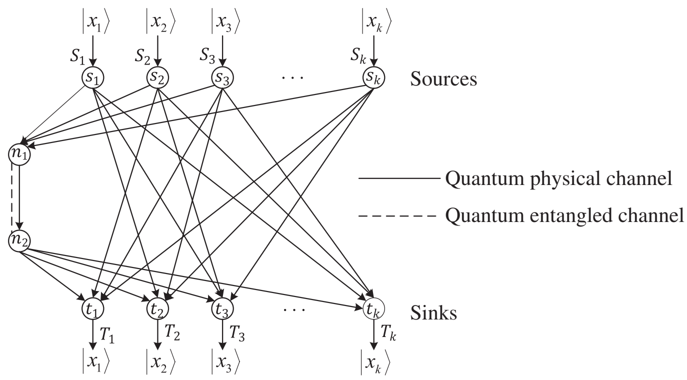

Figure 2 Quantum k-pair network $\mathcal{N}$ as an extension of the butterfly network. Each source node s_i is connected to n_1 and all receivers except t_i; each sink node t_i is connected to n_2 and all senders except s_i.

to perform duplication operations. Thus, two bits of classical information x_1 and x_2 are simultaneously transmitted from s_1, s_2, respectively, to t_1, t_2, gaining a larger network throughput as compared with that obtained via routing [9].

As regards the quantum networks the feasibility and construction have been fully verified both theoretically [10–18] and experimentally [19]. To better transmit quantum information via a quantum network, related quantum network communication technologies must be explored. In 2007, longing for the corresponding quantum technology to improve the communication efficiency of a quantum network, Hayashi et al. [20] first proposed quantum network coding (QNC) . However, multicasting an unknown quantum state is infeasible in principle owing to the quantum no-cloning theorem. Consequently, the feasibility of QNC for multi-unicast sessions was explored instead. Perfect quantum network coding (PQNC) was proved to be impossible without extra resources assisting the quantum setting where each quantum channel has a capacity of one. However, approximate transmission based on QNC can be achieved, and the upper bound of fidelity is 0.983. Accordingly, researchers are attempting to find appropriate assisting resources in hopes of unveiling a PQNC solution to the quantum multi-unicast problem [21,22], especially for the quantum k-pair problem that will be described later in Subsection 2.1, where k represents the number of source-sink pairs. Note that according to [23], a multi-unicast network is a network for which every source message is emitted by exactly one source node and is demanded by exactly one receiver node. Therefore, multi-unicast networks contain communications between collections of pairs of nodes. Thus, it can be observed that the quantum k-pair network $\mathcal{N}$ illustrated in Figure 2 is a kind of typical multi-unicast network, and the butterfly network illustrated in Figure 1 can be considered a special case of the quantum k-pair network when $k = 2$.

In general, representative resources in quantum information processing mainly include prior entanglement and classical communication. An entanglement resource is a key quantum resource used for various quantum communication tasks such as quantum key distribution [24], quantum correlation generation [25], teleportation [26], remote state preparation [27], and quantum-state sharing [28]. Therefore, it is believed that with the assistance of prior entanglement, new breakthroughs in QNC [29–39] can be achieved. In 2007, Hayashi [29] first introduced prior entanglement into QNC, where two maximal entangled states are pre-shared between two source nodes over the butterfly network. It was proved that perfect quantum state transmission can be achieved by QNC with an assisting entanglement resource, using classical coding operations under the same network model. Ma et al. [30] substituted the non-maximal quantum state with the maximal one, and found the probability of obtaining the expected state to be less than 1. Satoh et al. [31, 32] and Li et al. [39] considered PQNC under a scenario, where adjacent nodes share one EPR-pair in quantum repeater networks. Zhang et al. [33] studied an application in which the EPR-pair is shared between the two side nodes, and the side channels are replaced by the hidden ones. Mahdian et al. [34] acquired the perfect k-pair QNC using k maximal states shared among the k source nodes in a solid-state system base. Li et al. [35] analyzed the solvability of several typical quantum multi-unicast networks using global entangled two-dimensional (2D) and three-dimensional (3D) cluster states. Wang et al. [36] achieved nearly perfect quantum k-pair transmission by means of an extended photonic entanglement with multiple degrees of freedom, such that the classical unsolvable network becomes quantum solvable. As can be seen, the pair number and the location of the entangled states play an important role in the network coding solution to a quantum network problem. Thus, it is reasonable to consider these factors when analyzing communication efficiency.

Besides prior entanglement, classical communication is often used in the quantum information theory as another assisting resource, e.g., entanglement distillation and dilution [40]. In recent years, several QNC schemes assisted by classical communication [41–46] have also been investigated for quantum communication over the multi-unicast network. In 2009 and 2011, Kobayashi et al. [41, 42] explored two perfect k-pair QNC schemes with the assistance of classical communication, based on classical linear and nonlinear network coding, respectively. A specific amount of classical communication required in terms of the network size was provided in their research. Following this, to reduce the cost of classical communication, Li et al. [43] constructed a PQNC solution independent of CNC, thereby avoiding the classical transmission bottleneck problem and lowering the communication complexity. Furthermore, Yang et al. [44] studied the quantum multi-unicast problem in quantum-walk architecture, thus providing a new approach to study quantum network communication. Remarkably, the cost of classical communication obtained with classical assistance was explored, similar to the above protocols, revealing that a decreased cost leads to improved communication efficiency. This study emphasizes on the proposal of an efficient PQNC solution to the quantum k-pair problem, over a k-pair network $\mathcal{N}$ extended from the butterfly network. An explicit protocol is proposed herein, in which prior entanglement and classical communication are introduced, and specified coding operations are performed at the corresponding nodes. In addition, to raise the communication efficiency, we attempt to reduce the number of extra assisting resources, if possible. In the proposed protocol, only one pair of the maximally entangled state is pre-shared between the two intermediate nodes, which sufficiently reduces the number of entanglement resources. Only $O(k)$ of classical information needs to be transmitted during decoding, which reduces classical communication by a considerable degree and prevents classical information congestion in the network. These numerical and practical advantages show the protocol's high communication efficiency. In addition, the achievable rate region [47, 48] is also analyzed, and it is proved that the 1-max flow quantum communication of value k over the network $\mathcal{N}$ can be attained.

2 Preliminaries

2.1 Quantum k-pair problem

Let us consider a directed acyclic graph (DAG) $G = (V, E)$, where V is the set of nodes and E is the set of edges that connect pairs of nodes in V. K pairs of nodes $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$ constitute a subset of V. Next, a directed acyclic network (DAN) $\mathcal{N}$ is deemed a quantum network if it comprises a DAG G and the edge quantum capacity function $c : E \rightarrow \mathbb{Z}^+$. The quantum k-pair network $\mathcal{N}$ explored herein is shown in Figure 2. In this figure, arrows indicate the transmission direction of the quantum information. The solid line represents the quantum physical channel, while the dotted line represents the quantum entangled channel. Similar to the case of the classical network, in this quantum network, we attempt to solve the quantum k-pair problem using QNC. Let $\mathcal{H}=\mathbb{C}^d$ denote a Hilbert space. According to the task of the quantum k-pair problem, one needs to transmit a quantum state $|\Psi\rangle\in\mathcal{H}^{\otimes k}$ supported on the source nodes $s_1,\ldots,s_k$ (in this order) to the sink nodes $t_1,\ldots,t_k$ (in this order) through $\mathcal{N}$, under the condition that $c(e)\equiv 1, e\in E$, i.e., each edge of $\mathcal{N}$ can transmit no more than one qudit state over $\mathcal{H}$. For $i\in\{1,2,\ldots,k\}$, each quantum register S_i is possessed by the source node s_i, while the quantum register T_i is possessed by the sink node t_i. A PQNC solution over $\mathcal{H}$ to the quantum k-pair problem refers to the corresponding protocol, which contains certain quantum operations for all nodes in V that enable the above transmission to be accomplished perfectly.

2.2 Quantum operators and quantum measurement

2.2.1 *Quantum operators*

Firstly, let us introduce the controlled-X operation on a d-dimension quantum system $\mathcal{H}=\mathbb{C}^d$, which is written as follows:

$$\widetilde{\mathrm{CX}}_{A\to B}:=\sum_{i=0}^{d-1}|i\rangle\langle i|_A\otimes X_B^i, \tag{1}$$

where $X|i\rangle=|i\oplus 1 \bmod d\rangle$ is an analogue of the unitary Pauli operator σ_x on qubits. Then, the controlled-X operation over the control-target parties of one-to-many/many-to-one can be implemented as

$$\bigotimes_{i=1}^{k}\widetilde{\mathrm{CX}}_{A\to B_i}|y\rangle_A|z_1\rangle_{B_1}\otimes\cdots\otimes\rangle_{B_k}=|y\rangle_A\bigotimes_{i=1}^{k}|z_i\oplus y\rangle_{B_i}; \tag{2}$$

$$\bigotimes_{i=1}^{k}\widetilde{\mathrm{CX}}_{A_i\to B}|y_1\rangle_{A_1}\otimes\cdots\otimes|y_k\rangle_{A_k}|z\rangle_B=\bigotimes_{i=1}^{k}|y_i\rangle_{A_i}\left|\bigoplus_{i=1}^{k}y_i\oplus z\right\rangle_B. \tag{3}$$

Here $|y\rangle,|y_i\rangle,|z\rangle,|z_i\rangle$ $(i\in\{1,2,\ldots,k\})$ are obtained from the basis states of $\mathcal{H}=\mathbb{C}^d$ and " $\oplus$ " means addition modulo d.

In addition, another quantum operation named the controlled-R operation in the d-dimension quantum system $\mathcal{H}=\mathbb{C}^d$ is also presented here, defined as

$$\widetilde{\mathrm{CR}}_{A\to B}:=\sum_{i=0}^{d-1}|i\rangle\langle i|_A\otimes R_B^i, \tag{4}$$

where $R|i\rangle=|i-1 \bmod d\rangle$ is the reverse transformation of X on qudits. Analogous to the controlled-X operation, the controlled-R operation over the control-target parties of one-to-many/many-to-one can be implemented as well.

2.2.2 *Quantum measurement*

Quantum Fourier transform $\mathcal{F}$ is a unitary transformation that transforms the computing basis states $\{|k\rangle\}_{k\in\mathbb{Z}_d}$ to the Fourier basis as follows:

$$|w_k\rangle=\mathcal{F}|k\rangle=\frac{1}{\sqrt{d}}\sum_{l=0}^{d-1}\mathrm{e}^{2\pi\iota kl/d}|l\rangle, \tag{5}$$

where $\iota^2=-1$, $\mathcal{F}=\frac{1}{\sqrt{d}}\sum_{l,k=0}^{d-1}\mathrm{e}^{2\pi\iota kl/d}|l\rangle\langle k|$. Thus similar to [41–43, 45], the basis states $\{|w_k\rangle\}_{k\in\mathbb{Z}_d}$ are called the quantum Fourier basis, and the quantum measurement in the Fourier basis is called the quantum Fourier measurement. Quantum Fourier measurement is usually used to measure a single-particle state over $\mathcal{H}=\mathbb{C}^d$.

Furthermore, another quantum measurement called the Bell measurement is also be performed in our scheme. In the quantum system $\mathcal{H} = \mathbb{C}^d$, the Bell states (EPR pair) are represented as follows:

$$|\phi(u_1, u_2)\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \mathrm{e}^{2\pi\iota j u_1/d} |j, j \oplus u_2\rangle, \quad u_1, u_2 \in \mathbb{Z}_d. \tag{6}$$

Hence, the basis states $\{|\phi(u_1, u_2)\rangle\}_{u_1,u_2\in\mathbb{Z}_d}$ are called the Bell basis, and the quantum measurement in the Bell basis is called the Bell measurement. In particular, when $u_1 = u_2 = 0$, the Bell state $|\phi_{00}\rangle$ is expressed as $|\phi_{00}\rangle = \frac{1}{\sqrt{d}} \sum_{m=0}^{d-1} |m, m\rangle$, which can be used as the entanglement resource that is introduced later. Bell measurement is typically used to jointly measure a two-particle state over $\mathcal{H} = \mathbb{C}^d$.

Generally, in QNC schemes, the quantum Fourier measurement and the Bell measurement are used to remove the unnecessary particles from the entire quantum system during decoding process so as to obtain the final desired quantum states.

3 A novel PQNC protocol for the quantum k-pair problem

Assume that the source state of the entire system is of the form

$$|\Psi\rangle_S^{(k)} = \sum_{x_1,\ldots,x_k\in\mathbb{Z}_d} \alpha_{x_1,\ldots,x_k} |x_1\rangle_{S_1} \otimes \cdots \otimes |x_k\rangle_{S_k}, \tag{7}$$

where the $\alpha_{x_1,\ldots,x_k}$ are complex numbers such that $\sum_{x_1,\ldots,x_k\in\mathbb{Z}_d} |\alpha_{x_1,\ldots,x_k}|^2 = 1$. For $i \in \{1, 2, \ldots, k\}$, each quantum register S_i is received by the source node s_i from its incoming virtual edge. The task is to transmit k qudits from s_i to t_i. Prior to proposing our protocol, we assume that the two intermediate nodes n_1, n_2 share a pair of the maximally entangled state $|\phi_{00}\rangle$ by introducing ancillary registers on N_1 and N_2, respectively. Then, the initial state prior to transmission can be written as

$$|\Psi\rangle_0^{(k)} = \frac{1}{\sqrt{d}} \sum_{x_1,\ldots,x_k\in\mathbb{Z}_d} \sum_{m=0}^{d-1} \alpha_{x_1,\ldots,x_k} |m, m\rangle_{N_1,N_2} \bigotimes_{i=1}^{k} |x_i\rangle_{S_i}. \tag{8}$$

Herein, we propose a novel PQNC protocol in which an additional entanglement resource is involved to solve the quantum k-pair problem. The protocol comprises two processes, i.e., encoding and decoding. The encoding objective is to mutually entangle the particles in the ancillary registers by applying relevant quantum operators on them. Contrarily, the decoding objective is to disentangle the particles owned by the sink nodes from the others and ensure that the ultimate quantum state on the sink nodes is identical to that on the source state.

3.1 Protocol specification

Process I: encoding. As the network $\mathcal{N}$ is a directed acyclic graph, all the nodes have topological orders. Next, we entangle the registers node by node in the following topological order.

(S1) Registers at the source nodes are entangled into the whole quantum system using the quantum operator $\widetilde{\mathrm{CX}}$. k quantum registers R_{ij} $(j \in \{1, 2, \ldots, k\})$, each initialized to $|0_{\mathcal{H}}\rangle$, are introduced at each source node s_i; then, the operator $\widetilde{\mathrm{CX}}_{S_i\to R_{ii}}$ is applied to the registers S_i and R_{ii}, and the operator $\widetilde{\mathrm{CR}}_{S_i\to R_{ij}}$ is applied to the registers S_i and R_{ij} $(j \in \{1, 2, \ldots, k\}, j \neq i)$. Thus, the quantum state becomes

$$|\Psi\rangle_1^{(k)} = \frac{1}{\sqrt{d}} \sum_{x_1,\ldots,x_k\in\mathbb{Z}_d} \sum_{m=0}^{d-1} \alpha_{x_1,\ldots,x_k} |m, m\rangle_{N_1,N_2} \bigotimes_{i,j=1,j\neq i}^{k} |x_i\rangle_{S_i} |x_i\rangle_{R_{ii}} |-x_i\rangle_{R_{ij}}. \tag{9}$$

Then, the quantum registers R_{ii} are sent from each node s_i to n_1, $k-1$ registers R_{ij} $(j \neq i)$ are sent to the sink nodes t_j $(j \in \{1, 2, \ldots, k\})$, and the registers S_i are maintained at the node s_i. Meanwhile, the ancillary register R_b initialized to $|0_{\mathcal{H}}\rangle$ is introduced at the node n_1.

(S2) Registers at the intermediate node n_1 are entangled into the whole quantum system using the quantum operator $\widetilde{\text{CX}}$. Applying $\widetilde{\text{CX}}_{R_{ii}\to N_1}$ on the registers R_{ii} $(i=1,2,\ldots,k)$ and N_1, followed by the application of $\widetilde{\text{CX}}_{N_1\to R_b}$ on the registers N_1 and R_b at the intermediate node n_1, we obtain the following quantum state:

$$|\Psi\rangle_2^{(k)} = \frac{1}{\sqrt{d}} \sum_{x_1,\ldots,x_k\in\mathbb{Z}_d} \sum_{m=0}^{d-1} \alpha_{x_1,\ldots,x_k} |\overline{\boldsymbol{X}}\rangle_{N_1,R_b,N_2} \bigotimes_{i,j=1,j\neq i}^{k} |x_i\rangle_{S_i} |x_i\rangle_{R_{ii}} |-x_i\rangle_{R_{ij}}, \tag{10}$$

where $|\overline{\boldsymbol{X}}\rangle_{N_1,R_b,N_2} = |\bigoplus_{i=1}^k x_i \oplus m, \bigoplus_{i=1}^k x_i \oplus m, m\rangle_{N_1,R_b,N_2}$. Then, the quantum register R_b is sent from the node n_1 to n_2; the registers R_{ii} $(i=1,2,\ldots,k)$ and N_1 are kept at n_1.

(S3) Registers at the intermediate node n_2 are entangled into the whole quantum system using the quantum operators $\widetilde{\text{CX}}$ and $\widetilde{\text{CR}}$. At the intermediate node n_2, k quantum registers $r_i (i=1,2,\ldots,k)$, each initialized to $|0_{\mathcal{H}}\rangle$, are introduced; then the quantum operator $\widetilde{\text{CX}}_{R_b\to r_i}$ is applied to the registers R_b and r_i, and $\widetilde{\text{CR}}_{N_2\to r_i}$ is applied to the registers N_2 and r_i. Thus, the quantum state becomes

$$|\Psi\rangle_3^{(k)} = \frac{1}{\sqrt{d}} \sum_{x_1,\ldots,x_k\in\mathbb{Z}_d} \sum_{m=0}^{d-1} \alpha_{x_1,\ldots,x_k} |\overline{\boldsymbol{X}}\rangle_{N_1,R_b,N_2} \bigotimes_{i,j=1,j\neq i}^{k} \left|\bigoplus_{i=1}^{k} x_i\right\rangle_{r_i} |x_i\rangle_{S_i} |x_i\rangle_{R_{ii}} |-x_i\rangle_{R_{ij}}. \tag{11}$$

Then, k quantum registers r_i $(i=1,2,\ldots,k)$ are transmitted from the node n_2 to the sink nodes t_i respectively, and the registers R_b, N_2 are maintained at n_2.

(S4) Registers at the sink nodes are entangled into the whole quantum system using the quantum operators $\widetilde{\text{CX}}$ and $\widetilde{\text{CR}}$. For each sink node t_i, the quantum register T_i initialized to $|0_{\mathcal{H}}\rangle$ is introduced. Remembering that t_i has received $k-1$ registers R_{ji} $(j\in\{1,2,\ldots,k\}, j\neq i)$ in (S1) and register r_i in (S2), the quantum operator $\widetilde{\text{CX}}_{r_i\to T_i}$ is applied to r_i and T_i, $\widetilde{\text{CX}}_{R_{ji}\to T_i}$ is applied to R_{ji} $(j\neq i)$ and T_i. Hence, the resulting state becomes

$$|\Psi\rangle_4^{(k)} = \frac{1}{\sqrt{d}} \sum_{x_1,\ldots,x_k\in\mathbb{Z}_d} \sum_{m=0}^{d-1} \alpha_{x_1,\ldots,x_k} |\overline{\boldsymbol{X}}\rangle_{N_1,R_b,N_2} \bigotimes_{i,j=1,j\neq i}^{k} \left|\bigoplus_{i=1}^{k} x_i\right\rangle_{r_i} |x_i\rangle_{T_i} |x_i\rangle_{S_i} |x_i\rangle_{R_{ii}} |-x_i\rangle_{R_{ij}}. \tag{12}$$

Thus far, all introduced ancillary registers have been mutually entangled, which denotes the end of the encoding process. The following is the decoding process.

Process II: decoding. We remove the registers from the node n_2 to the source nodes in a reverse topological order, and finally to the sink nodes.

(T1) Registers at the intermediate node n_2 are removed from the whole quantum system via qudit teleportation. Considering the owned registers R_b, N_2, the intermediate node n_2 performs the quantum operation $\widetilde{\text{CX}}_{R_b\to N_2}$, followed by the Bell measurement on the two qudits, providing the measurement result u_1u_2. Because

$$|j, j\oplus u_2\rangle = \frac{1}{\sqrt{d}} \sum_{u_1=0}^{d-1} \mathrm{e}^{-2\pi\iota j u_1/d} |\phi(u_1,u_2)\rangle, \tag{13}$$

we obtain the quantum state:

$$\begin{aligned} |\Psi\rangle_5^{(k)} = \frac{1}{\sqrt{d}} \sum_{x_1,\ldots,x_k\in\mathbb{Z}_d} \alpha_{x_1,\ldots,x_k} \mathrm{e}^{-2\pi\iota(\bigoplus_{i=1}^k x_i\oplus u_2)u_1/d} \left|\bigoplus_{i=1}^{k} x_i \oplus u_2\right\rangle_{N_1} \\ \bigotimes_{i,j=1,j\neq i}^{k} \left|\bigoplus_{i=1}^{k} x_i\right\rangle_{r_i} |x_i\rangle_{T_i} |x_i\rangle_{S_i} |x_i\rangle_{R_{ii}} |-x_i\rangle_{R_{ij}}. \end{aligned} \tag{14}$$

Then, $2\lceil\log d\rceil$ bits of classical information u_1u_2 are transmitted from the node n_2 to n_1 through the bottleneck channel.

(T2) Registers at the intermediate node n_1 are removed from the whole quantum system using Fourier measurement. Upon receiving the information u_1u_2, the node n_1 applies the quantum unitary operator on its register N_1, mapping the state $|x\rangle$ to $\mathrm{e}^{2\pi\iota u_1 x/d}|x-u_2\rangle$ for each $x \in \mathbb{Z}_d$. Thus, the phase resulting from the Bell measurement in (T1) is corrected. Next, quantum Fourier measurement is performed on N_1, providing the measurement result l. Because

$$|l\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \mathrm{e}^{-2\pi\iota kl/d}|k\rangle, \tag{15}$$

we obtain the quantum state:

$$|\Psi\rangle_{5'}^{(k)} = \frac{1}{\sqrt{d}} \sum_{x_1,\ldots,x_k\in\mathbb{Z}_d} \alpha_{x_1,\ldots,x_k} \mathrm{e}^{-2\pi\iota(\bigoplus_{i=1}^{k} x_i)l/d} \bigotimes_{i,j=1,j\neq i}^{k} \left|\bigoplus_{i=1}^{k} x_i\right\rangle_{r_i} |x_i\rangle_{T_i}|x_i\rangle_{S_i}|x_i\rangle_{R_{ii}}|-x_i\rangle_{R_{ij}}. \tag{16}$$

The phase introduced is corrected as followings: the node n_1 applies the unitary operator on its registers R_{ii} $(i=1,2,\ldots,k)$, mapping the state $|x_1,\ldots,x_k\rangle$ to the state $\mathrm{e}^{2\pi\iota l(\bigoplus_{i=1}^{k} x_i)/d}|x_1,\ldots,x_k\rangle$ for any $x_i \in \mathbb{Z}_d$. Consequently, the state then becomes

$$|\Psi\rangle_{5''}^{(k)} = \frac{1}{\sqrt{d}} \sum_{x_1,\ldots,x_k\in\mathbb{Z}_d} \alpha_{x_1,\ldots,x_k} \bigotimes_{i,j=1,j\neq i}^{k} \left|\bigoplus_{i=1}^{k} x_i\right\rangle_{r_i} |x_i\rangle_{T_i}|x_i\rangle_{S_i}|x_i\rangle_{R_{ii}}|-x_i\rangle_{R_{ij}}. \tag{17}$$

Next, quantum Fourier measurement is performed on each register R_{ii}, producing the measurement result h_i $(i=1,2,\ldots,k)$. The resulting state is

$$|\Psi\rangle_{6}^{(k)} = \frac{1}{\sqrt{d}} \sum_{x_1,\ldots,x_k\in\mathbb{Z}_d} \alpha_{x_1,\ldots,x_k} \prod_{i=1}^{k} \mathrm{e}^{-2\pi\iota h_i x_i/d} \bigotimes_{i,j=1,j\neq i}^{k} \left|\bigoplus_{i=1}^{k} x_i\right\rangle_{r_i} |x_i\rangle_{T_i}|x_i\rangle_{S_i}|-x_i\rangle_{R_{ij}}. \tag{18}$$

Following this, h_i is transmitted from the node n_1 to s_i, and thus, $k\lceil\log d\rceil$ bits of classical information are transmitted in this step.

(T3) Registers at the source nodes are removed from the whole quantum system via Fourier measurement. At each source node s_i, quantum unitary operator mapping of the state $|x_i\rangle$ to $\mathrm{e}^{2\pi\iota h_i x_i/d}|x_i\rangle$ for each $x_i \in \mathbb{Z}_d$, is performed on the register S_i to correct the phase. Following this, quantum Fourier measurement is applied to S_i to disentangle it from the other registers, which returns the measurement result g_i. As result, the quantum state becomes

$$|\Psi\rangle_{7}^{(k)} = \frac{1}{\sqrt{d}} \sum_{x_1,\ldots,x_k\in\mathbb{Z}_d} \alpha_{x_1,\ldots,x_k} \prod_{i=1}^{k} \mathrm{e}^{-2\pi\iota g_i x_i/d} \bigotimes_{i,j=1,j\neq i}^{k} \left|\bigoplus_{i=1}^{k} x_i\right\rangle_{r_i} |x_i\rangle_{T_i}|-x_i\rangle_{R_{ij}}. \tag{19}$$

Then, g_i is transmitted from the node s_i to t_{i-1} respectively, and thus $k\lceil\log d\rceil$ bits of classical information are transmitted in this step.

(T4) Registers at the sink nodes are removed from the whole quantum system via Fourier measurement. Upon receiving g_{i+1}, the sink node t_i computes $-g_{i+1}$ and then corrects the phase by performing the quantum unitary operator mapping on its own register $R_{i+1,i}$, wherein the state $|-x_{i+1}\rangle$ is mapped to $\mathrm{e}^{2\pi\iota g_{i+1}x_{i+1}/d}|-x_{i+1}\rangle$ for each $-x_{i+1} \in \mathbb{Z}_d$. Hereafter, quantum Fourier measurements are applied to the k registers respectively, namely r_i and R_{ji} $(j \neq i)$, thereby producing the measurement results p_i and q_{ji} $(j \neq i)$, respectively. The state then becomes

$$|\Psi\rangle_{8}^{(k)} = \frac{1}{\sqrt{d}} \sum_{x_1,\ldots,x_k\in\mathbb{Z}_d} \alpha_{x_1,\ldots,x_k} \prod_{i=1}^{k} \mathrm{e}^{-2\pi\iota(p_i \bigoplus_{j=1,j\neq i}^{k} q_{ji})x_i/d} \bigotimes_{i=1}^{k} |x_i\rangle_{T_i}. \tag{20}$$

To correct the phase produced by the measurements, each t_i then applies the unitary operator on its register T_i, mapping the state $|x_i\rangle$ to the state $\mathrm{e}^{2\pi\iota(p_i \bigoplus_{j=1,j\neq i}^{k} q_{ji})x_i/d}|x_i\rangle$ for any $x_i \in \mathbb{Z}_d$. Now, the final quantum state becomes the desired state, as follows:

$$|\Psi\rangle_{9}^{(k)} = \sum_{x_1,\ldots,x_k\in\mathbb{Z}_d} \alpha_{x_1,\ldots,x_k}|x_1\rangle_{T_1} \otimes \cdots \otimes |x_k\rangle_{T_k}. \tag{21}$$

Thus far, the state of the quantum system over every source node is perfectly transmitted to the corresponding sink node; i.e., the PQNC scheme for the quantum k-pair problem is achievable using the presented protocol. The total amount of classical communication in our protocol is $(2k+2)\lceil \log d \rceil$ bits.

In essence, the entire abovementioned scheme can be understood using Schmidt decomposition [40]. Initially, the k qudits respectively possessed by the k source nodes s_i $(i \in \{1, 2, \ldots, k\})$ can be deemed a whole quantum state in the superposition as (7), where the superposed coefficients are unknown. Then, the encoding process is considered to convert this superposed state into the multi-particle state in the format of the Schmidt decomposition by gradually performing the concerned quantum operations; the Schmidt co-efficients thereby become the original superposed coefficients. Subsequently, based on the algebraic invariance properties of the Schmidt decomposition, the decoding process is considered to remove the unnecessary particles from the entire quantum system by gradually performing the concerned quantum measurements and the appropriate unitary transformations; finally, the superposed coefficients of the desired quantum state remained are the same as the original Schmidt co-efficients.

3.2 Example: the butterfly network over $\mathcal{H} = \mathbb{C}^2$

This subsection illustrates the techniques developed in the previous descriptions using the example of the butterfly network shown in Figure 1 over Hilbert space $\mathcal{H} = \mathbb{C}^2$. Thus, the quantum K-pair problem here is reduced to the quantum 2-pair problem, and the task is to transmit quantum states supported on the source nodes s_1, s_2 (in this order) to the sink nodes t_1, t_2 (in this order). This can be achieved perfectly, i.e., with the fidelity of one, using the proposed protocol. Now the maximally entangled pair pre-shared between the two intermediate nodes n_1 and n_2 is $|\phi\rangle^{(2)}_{N_1N_2} = \frac{1}{\sqrt{2}}(|00\rangle + b_1|11\rangle)_{N_1N_2}$. Meanwhile, the source state that needs to be transmitted is written as $|\Psi\rangle^{(2)}_S = (\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle)_{S_1S_2}$. Thus, the quantum state of the whole system is

$$|\Psi\rangle^{(2)}_0 = \frac{1}{\sqrt{2}}(\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|00\rangle + \alpha_{11}|00\rangle)_{S_1S_2} \otimes (|00\rangle + |11\rangle)_{N_1N_2}. \tag{22}$$

Hereafter, we proceed to the processes of encoding and decoding over the whole network. For the encoding process, the quantum states of the four steps (S1)–(S4) are detailed as follows:

$$\begin{aligned}|\Psi\rangle^{(2)}_1 = \frac{1}{\sqrt{2}}(&\alpha_{00}|000\rangle_{S_1R_{11}R_{12}}|000\rangle_{S_2R_{21}R_{22}} + \alpha_{01}|000\rangle_{S_1R_{11}R_{12}}|111\rangle_{S_2R_{21}R_{22}} \\ &+ \alpha_{10}|111\rangle_{S_1R_{11}R_{12}}|000\rangle_{S_2R_{21}R_{22}} + \alpha_{11}|111\rangle_{S_1R_{11}R_{12}}|111\rangle_{S_2R_{21}R_{22}})(|00\rangle + |11\rangle)_{N_1N_2},\end{aligned} \tag{23}$$

$$\begin{aligned}|\Psi\rangle^{(2)}_2 = \frac{1}{\sqrt{2}}[&\alpha_{00}|000\rangle_{S_1R_{11}R_{12}}|000\rangle_{S_2R_{21}R_{22}}(|000\rangle + |111\rangle)_{N_1R_bN_2} \\ &+ \alpha_{01}|000\rangle_{S_1R_{11}R_{12}}|111\rangle_{S_2R_{21}R_{22}}(|110\rangle + |001\rangle)_{N_1R_bN_2} \\ &+ \alpha_{10}|111\rangle_{S_1R_{11}R_{12}}|000\rangle_{S_2R_{21}R_{22}}(|110\rangle + |001\rangle)_{N_1R_bN_2} \\ &+ \alpha_{11}|111\rangle_{S_1R_{11}R_{12}}|111\rangle_{S_2R_{21}R_{22}}(|000\rangle + |111\rangle)_{N_1R_bN_2}],\end{aligned} \tag{24}$$

$$\begin{aligned}|\Psi\rangle^{(2)}_3 = \frac{1}{\sqrt{2}}[&\alpha_{00}|000\rangle_{S_1R_{11}R_{12}}|000\rangle_{S_2R_{21}R_{22}}(|000\rangle + |111\rangle)_{N_1R_bN_2}|00\rangle_{r_1r_2} \\ &+ \alpha_{01}|000\rangle_{S_1R_{11}R_{12}}|111\rangle_{S_2R_{21}R_{22}}(|110\rangle + |001\rangle)_{N_1R_bN_2}|11\rangle_{r_1r_2} \\ &+ \alpha_{10}|111\rangle_{S_1R_{11}R_{12}}|000\rangle_{S_2R_{21}R_{22}}(|110\rangle + |001\rangle)_{N_1R_bN_2}|11\rangle_{r_1r_2} \\ &+ \alpha_{11}|111\rangle_{S_1R_{11}R_{12}}|111\rangle_{S_2R_{21}R_{22}}(|000\rangle + |111\rangle)_{N_1R_bN_2}|00\rangle_{r_1r_2}],\end{aligned} \tag{25}$$

$$\begin{aligned}|\Psi\rangle^{(2)}_4 = \frac{1}{\sqrt{2}}[&\alpha_{00}|000\rangle_{S_1R_{11}R_{12}}|000\rangle_{S_2R_{21}R_{22}}(|000\rangle + |111\rangle)_{N_1R_bN_2}|00\rangle_{r_1r_2}|00\rangle_{T_1T_2} \\ &+ \alpha_{01}|000\rangle_{S_1R_{11}R_{12}}|111\rangle_{S_2R_{21}R_{22}}(|110\rangle + |001\rangle)_{N_1R_bN_2}|11\rangle_{r_1r_2}|01\rangle_{T_1T_2} \\ &+ \alpha_{10}|111\rangle_{S_1R_{11}R_{12}}|000\rangle_{S_2R_{21}R_{22}}(|110\rangle + |001\rangle)_{N_1R_bN_2}|11\rangle_{r_1r_2}|10\rangle_{T_1T_2} \\ &+ \alpha_{11}|111\rangle_{S_1R_{11}R_{12}}|111\rangle_{S_2R_{21}R_{22}}(|000\rangle + |111\rangle)_{N_1R_bN_2}|00\rangle_{r_1r_2}|11\rangle_{T_1T_2}].\end{aligned} \tag{26}$$

For the decoding process, the registers R_b, N_2 are performed the quantum operation $\widetilde{\mathrm{CX}}_{R_b\to N_2}$, and are then measured on the basis of Bell measurement in (T1), generating four possible outcomes: $|\beta_{00}\rangle, |\beta_{01}\rangle$, $|\beta_{10}\rangle, |\beta_{11}\rangle$. In particular, the four measurement outcomes corresponding to the remaining system states are as follows:

$$\begin{aligned}|\beta_{00}\rangle : |\Psi\rangle_5^{(2)} = \frac{1}{2}[&\alpha_{00}|000\rangle_{S_1R_{11}R_{12}}|000\rangle_{S_2R_{21}R_{22}}|0\rangle_{N_1}|00\rangle_{r_1r_2}|00\rangle_{T_1T_2}\\ &+\alpha_{01}|000\rangle_{S_1R_{11}R_{12}}|111\rangle_{S_2R_{21}R_{22}}|1\rangle_{N_1}|11\rangle_{r_1r_2}|01\rangle_{T_1T_2}\\ &+\alpha_{10}|111\rangle_{S_1R_{11}R_{12}}|000\rangle_{S_2R_{21}R_{22}}|1\rangle_{N_1}|11\rangle_{r_1r_2}|10\rangle_{T_1T_2}\\ &+\alpha_{11}|111\rangle_{S_1R_{11}R_{12}}|111\rangle_{S_2R_{21}R_{22}}|0\rangle_{N_1}|00\rangle_{r_1r_2}|11\rangle_{T_1T_2}],\end{aligned} \tag{27}$$

$$\begin{aligned}|\beta_{10}\rangle : |\Psi\rangle_5^{(2)} = \frac{1}{2}[&\alpha_{00}|000\rangle_{S_1R_{11}R_{12}}|000\rangle_{S_2R_{21}R_{22}}|0\rangle_{N_1}|00\rangle_{r_1r_2}|00\rangle_{T_1T_2}\\ &-\alpha_{01}|000\rangle_{S_1R_{11}R_{12}}|111\rangle_{S_2R_{21}R_{22}}|1\rangle_{N_1}|11\rangle_{r_1r_2}|01\rangle_{T_1T_2}\\ &-\alpha_{10}|111\rangle_{S_1R_{11}R_{12}}|000\rangle_{S_2R_{21}R_{22}}|1\rangle_{N_1}|11\rangle_{r_1r_2}|10\rangle_{T_1T_2}\\ &+\alpha_{11}|111\rangle_{S_1R_{11}R_{12}}|111\rangle_{S_2R_{21}R_{22}}|0\rangle_{N_1}|00\rangle_{r_1r_2}|11\rangle_{T_1T_2}],\end{aligned} \tag{28}$$

$$\begin{aligned}|\beta_{01}\rangle : |\Psi\rangle_5^{(2)} = \frac{1}{2}[&\alpha_{00}|000\rangle_{S_1R_{11}R_{12}}|000\rangle_{S_2R_{21}R_{22}}|1\rangle_{N_1}|00\rangle_{r_1r_2}|00\rangle_{T_1T_2}\\ &+\alpha_{01}|000\rangle_{S_1R_{11}R_{12}}|111\rangle_{S_2R_{21}R_{22}}|0\rangle_{N_1}|11\rangle_{r_1r_2}|01\rangle_{T_1T_2}\\ &+\alpha_{10}|111\rangle_{S_1R_{11}R_{12}}|000\rangle_{S_2R_{21}R_{22}}|0\rangle_{N_1}|11\rangle_{r_1r_2}|10\rangle_{T_1T_2}\\ &+\alpha_{11}|111\rangle_{S_1R_{11}R_{12}}|111\rangle_{S_2R_{21}R_{22}}|1\rangle_{N_1}|00\rangle_{r_1r_2}|11\rangle_{T_1T_2}],\end{aligned} \tag{29}$$

$$\begin{aligned}|\beta_{11}\rangle : |\Psi\rangle_5^{(2)} = \frac{1}{2}[&-\alpha_{00}|000\rangle_{S_1R_{11}R_{12}}|000\rangle_{S_2R_{21}R_{22}}|1\rangle_{N_1}|00\rangle_{r_1r_2}|00\rangle_{T_1T_2}\\ &+\alpha_{01}|000\rangle_{S_1R_{11}R_{12}}|111\rangle_{S_2R_{21}R_{22}}|0\rangle_{N_1}|11\rangle_{r_1r_2}|01\rangle_{T_1T_2}\\ &+\alpha_{10}|111\rangle_{S_1R_{11}R_{12}}|000\rangle_{S_2R_{21}R_{22}}|0\rangle_{N_1}|11\rangle_{r_1r_2}|10\rangle_{T_1T_2}\\ &-\alpha_{11}|111\rangle_{S_1R_{11}R_{12}}|111\rangle_{S_2R_{21}R_{22}}|1\rangle_{N_1}|00\rangle_{r_1r_2}|11\rangle_{T_1T_2}].\end{aligned} \tag{30}$$

As the global phases $\frac{1}{2}$ can be ignored during computation, the next three decoding steps (T2)–(T4) result in the following quantum states:

$$\begin{aligned}|\Psi\rangle_6^{(2)} = {}&\alpha_{00}|00\rangle_{S_1R_{12}}|00\rangle_{S_2R_{21}}|00\rangle_{r_1r_2}|00\rangle_{T_1T_2} - \alpha_{01}|00\rangle_{S_1R_{12}}|11\rangle_{S_2R_{21}}|11\rangle_{r_1r_2}|01\rangle_{T_1T_2}\\ &-\alpha_{10}|11\rangle_{S_1R_{12}}|00\rangle_{S_2R_{21}}|11\rangle_{r_1r_2}|10\rangle_{T_1T_2} + \alpha_{11}|11\rangle_{S_1R_{12}}|11\rangle_{S_2R_{21}}|00\rangle_{r_1r_2}|11\rangle_{T_1T_2},\end{aligned} \tag{31}$$

$$\begin{aligned}|\Psi\rangle_7^{(2)} = {}&\alpha_{00}|0\rangle_{R_{12}}|0\rangle_{R_{21}}|00\rangle_{r_1r_2}|00\rangle_{T_1T_2} - \alpha_{01}|0\rangle_{R_{12}}|1\rangle_{R_{21}}|11\rangle_{r_1r_2}|01\rangle_{T_1T_2}\\ &-\alpha_{10}|1\rangle_{R_{12}}|0\rangle_{R_{21}}|11\rangle_{r_1r_2}|10\rangle_{T_1T_2} + \alpha_{11}|1\rangle_{R_{12}}|1\rangle_{R_{21}}|00\rangle_{r_1r_2}|11\rangle_{T_1T_2},\end{aligned} \tag{32}$$

$$|\Psi\rangle_8^{(2)} = \alpha_{00}|00\rangle_{T_1T_2} - \alpha_{01}|01\rangle_{T_1T_2} - \alpha_{10}|10\rangle_{T_1T_2} + \alpha_{11}|11\rangle_{T_1T_2}. \tag{33}$$

Finally, the phases of all items caused by the measurements are corrected via appropriate unitary operations at the two target nodes, and then the final quantum state becomes the desired state $|\Psi\rangle_9^{(2)} = \alpha_{00}|00\rangle_{T_1T_2} + \alpha_{01}|01\rangle_{T_1T_2} + \alpha_{10}|10\rangle_{T_1T_2} + \alpha_{11}|11\rangle_{T_1T_2}$. Thus, the PQNC scheme for the quantum 2-pair problem is achievable under the condition that a maximally entangled state is pre-shared between the two intermediate nodes over the butterfly network.

3.3 Pauli errors in the initial entangled state

Here, we consider how a mixed entangled pair affects the protocol regarding the maximally entangled pair. Still, we consider the butterfly network over $\mathcal{H} = \mathbb{C}^2$ as an example. Now, errors that occur in both particles during the pre-sharing of the entangled pair can be reduced to an error on any one particle. Hence, let us assume that the initial maximally entangled pair has an error on either of the two particles. Let us investigate three types of errors, namely the Z error, the X error, and general Pauli error, presenting the error models [40] as

$$\varepsilon_Z(p)\rho = (1-p)\rho + pZ\rho Z, \tag{34}$$

$$\varepsilon_X(p)\rho = (1-p)\rho + pZ\rho Z, \tag{35}$$

$$\varepsilon_G(p)\rho = (1-p)\rho + \frac{p}{4}\sum_{i=0}^{3}\sigma_i\rho\sigma_i, \tag{36}$$

where $\sigma_i, i \in \{0,1,2,3\}$ denote I, X, Y and Z, respectively, ρ represents the initial quantum state and p is the error parameter. Thus, for the initial maximally entangled state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$ pre-shared, the mixed entanglement caused by the Z error is a probabilistic mixture of $|\Phi^+\rangle$ and $|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle-|11\rangle)$. Reviewing all the PQNC steps, if $|\Phi^+\rangle$ is replaced by $|\Phi^-\rangle$, the resulting quantum state is different, which is reflected in the decoding step (T1): when the measurement result of the registers R_b, N_2 is $|\beta_{00}\rangle(|\beta_{10}\rangle)$, $|\Psi\rangle_5^{(2)}$ is identical to that in (27) (or (28)); when the measurement result of the registers R_b, N_2 is $|\beta_{01}\rangle(|\beta_{11}\rangle)$, $|\Psi\rangle_5^{(2)}$ is identical to that in (29) (or (30)), apart from the global phase of $-\frac{1}{2}$, which can be ignored during computation as well. Then, the resulting quantum states in the following decoding steps remain identical with (31)–(33). That is, $|\Phi^-\rangle$ can also be used to provide the PQNC solution to the quantum 2-pair problem. Therefore, the mixed entangled pair of the Z error enables the successful execution of the protocol, and the PQNC solution can thus be achieved.

However, for the X error, the mixed entanglement is a probabilistic mixture of $|\Phi^+\rangle$ and $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle+|10\rangle)$. If $|\Phi^+\rangle$ is replaced by $|\Psi^+\rangle$, the decoding steps will raise errors and be unable to proceed. Eventually, we can obtain the desired quantum state only with the fidelity of $1-p$; and then, when p is small enough, our protocol can resist the X error. Similarly, for the general Pauli error, we can obtain the desired quantum state with the fidelity of $1-\frac{p}{2}$; when p is small enough, our protocol can resist the general Pauli error. These can be further verified on the quantum k-pair network $\mathcal{N}$ over $\mathcal{H} = \mathbb{C}^d$. Therefore, we conclude that the proposed protocol can resist the Z error for the mixed entangled pair pre-shared. Furthermore, when p is small enough, the X error and the general Pauli error can also be resisted.

4 Protocol analysis

4.1 Communication efficiency

As we introduce the ancillary assisting resources, namely prior entanglement and classical communication, the proposed QNC protocol can ensure that the quantum states successfully transmit with the fidelity of one. By analyzing these two factors that affect the communication efficiency through some comparisons, the advantages of our protocol can be revealed clearly. Notably, the quantum k-pair network model $\mathcal{N}$ explored in this paper is the generation of butterfly network with two pairs of source-sink nodes. Therefore, the QNC schemes over the butterfly network model [29–33], along with the ones over the k-pair network model $\mathcal{N}$ [34, 41–43], will be used together for relevant comparisons with our protocol. For clarity, Tables 1 and 2 show the comparison results of different QNC schemes under the two network models, respectively.

Now, let us firstly analyze the entanglement resource introduced in the QNCSs used for the comparisons. It can easily been seen that, for the pair number of quantum entanglement, only one pair of the entangled state is pre-shared between the two intermediate nodes in our protocol, which is less than the number of the entangled pairs introduced in the schemes (except for [36]) under the most basic network model—butterfly network listed in Table 1. The same benefit is also evident in some schemes under the k-pair network $\mathcal{N}$ listed in Table 2. For the scheme in [36], the pair number of the quantum entanglement is 1, which is the same as that of our scheme. Thus, when comparing with the other displayed QNCSs,

Table 1 Comparison results of different quantum network coding schemes (QNCSs) under butterfly network

QNCSs	Prior entanglement		Classical communication
	Location	Pair number	Total amount (bits)
Hayashi [29]	Source nodes	2	10
Ma et al. [30]	Source nodes	2	13
Satoh et al. [31, 32]	Neighbor nodes	7	10
Zhang et al. [33]	Side nodes	2	10
Wang et al. [36]	Intermediate nodes	1	8
Ours	Intermediate nodes	1	6

Table 2 Comparison results of different quantum network coding schemes (QNCSs) under the k-pair network $\mathcal{N}$

QNCSs	Prior entanglement		Classical communication
	Location	Pair number	Total amount (bits)
Kobayashi et al. [41]	–	–	$kM\|V\|\lceil\log d\rceil$
Kobayashi et al. [42]	–	–	$2\|E\|\lceil\log d\rceil$
Mahdian et al. [34]	Source nodes	k	$2(k^2+k-1)\lceil\log d\rceil$
Li et al. [43]	–	–	$(k^2+k-1)\lceil\log d\rceil$
Wang et al. [36]	Intermediate nodes	1	$(3k+2)\lceil\log d\rceil$
Ours	Intermediate nodes	1	$2(k+1)\lceil\log d\rceil$

our protocol has distinct advantages in terms of the amount of the entangled state introduced, thereby reducing the consumption of the entanglement resource.

Subsequently, the classical communication introduced in the QNCSs is compared and analyzed. The total amount of classical information transmitted over the quantum network, which refers to the end-to-end communication, can be the quantitative indicator to evaluate the cost of the classical communication. It is observed that the amount of the classical communication cost in [41] becomes $2k^2(k+1)\lceil\log d\rceil$ bits under the quantum k-pair network model as $M = k, |V| = 2k+2$. Similarly, the one in [42] becomes $2(k^2+k+1)\lceil\log d\rceil$ bits as $|E| = k^2+k+1$. Thus, it appears that the previous schemes [34, 41–43] cost either $O(k^3)$ or $O(k^2)$ of the classical communication. In contrast, our protocol costs only $2(k+1)\lceil\log d\rceil$ (i.e., $O(k)$) of the classical communication, which is also less than that of the scheme in [36]. These comparisons show that the amount of the classical communication costs in our protocol is the least among all the listed schemes. Moreover, to avoid the congestion, Kobayashi et al. [42] brought to attention that the transmission of classical information through the bottleneck channel depends on the CNC technology. However, the classical communication evolved in our protocol has no chance to block the bottleneck channel as only $2\lceil\log d\rceil$ bits are transmitted in its reverse direction. Thus, our protocol does not need to rely on the CNC, which greatly decreases the communication complexity and would be better suited for a large-scale quantum network. Hence, our protocol has an absolute advantage over other schemes in terms of classical communication, owing to minimum communication cost and independence from the CNC.

Above all, our protocol obtains the PQNC solution to the quantum k-pair problem in the case wherein only one pair of the entangled state and only $O(k)$ of the classical communication (without transmission congestion) are consumed. Consequently, our protocol has the advantages of high communication efficiency over other schemes.

4.2 Achievable rate region

It is known that in an asymptotic scenario [47], the rate k-tuple $(r_1, \ldots, r_k)$ is achievable in a quantum network $\mathcal{N}$, if there exists a protocol $\mathcal{P}_n$ that uses the network n times, resulting in $n(r_i - \delta_n)$ qudits transmission from each s_i to t_i with a fidelity of $1 - \epsilon_n$, where $\delta_n, \epsilon_n \to 0$ as $n \to 0$. In this case, the network $\mathcal{N}$ is said to have an F-flow of value $\sum_i r_i^{(n)}$ $(F = 1-\epsilon_n)$, and the F-max flow is the maximization of the F-flow over all achievable rate vectors. In practice, we can deduce a theorem as following:

Theorem 1. For all $k \geqslant 2$, there exists a protocol $\mathcal{P}_n$ that provides a PQNC solution to the quantum

k-pair problem for the network $\mathcal{N}$, such that the 1-max flow of value k over the quantum k-pair network $\mathcal{N}$ can be achieved. That is, the achievable rate region is $\{(r_1,\ldots,r_k)|\sum_{i=1}^k r_i \leqslant k\}$.

Proof. Let us introduce the definition of the communication rate between s_i and t_i in n network uses, i.e.,

$$r_i^{(n)} = \frac{1}{n}\log|\mathcal{H}_i|, \tag{37}$$

where $\mathcal{H}_i$ denotes the Hilbert space of the transmitted quantum state owned by s_i, and $|\cdot|$ denotes the dimension of the Hilbert space. In addition, an edge capacity constraint [49], i.e.,

$$\log|\mathcal{H}_{(u,v)}| \leqslant n \cdot c((u,v)) \tag{38}$$

exists when the quantum state is transmitted with the fidelity of one over the edge $(u,v) \in E$ in n uses.

Based on our protocol $\mathcal{P}_1$ presented in Section 3, the perfect transmission of the quantum state over the quantum k-pair network $\mathcal{N}$ is achieved by QNC in one use of $\mathcal{N}$, which means that the 1-flow value reaches,

$$\sum_{i=1}^k r_i^{(1)} = \sum_{i=1}^k \log|\mathcal{H}_i| \leqslant \sum_{i=1}^k c((u,v)) = \sum_{i=1}^k 1 = k, \tag{39}$$

under the condition that the capacity $c((u,v))$ of each edge (u,v) always remains equal to 1 according to the quantum k-pair network model.

In fact, the 1-max flow is the supremum of 1-flow over all achievable rate; hence, 1-max flow of value k is achievable through our PQNC protocol. Therefore, we can prove that the achievable rate region for quantum communication on the k-pair network $\mathcal{N}$ is

$$\left\{(r_1,\ldots,r_k)\middle|\sum_{i=1}^k r_i \leqslant k\right\}, \tag{40}$$

which extends the corresponding result over the butterfly network [47].

5 Conclusion

In summary, we proposed a novel protocol based on PQNC for a network $\mathcal{N}$, which can efficiently solve the quantum k-pair problem. First, only one pair of the entangled state is shared between the two intermediate nodes, which significantly reduces the consumption of the entanglement resource. Second, only $O(k)$ of classical information needs to be transmitted through the network, thereby considerably reducing the communication cost. In addition, the transmission of classical information is independent of the CNC as no classical transmission bottleneck exists in the decoding process, which is better suited to a large-scale quantum k-pair network. Therefore, our protocol offers a great advantage in terms of communication efficiency. Furthermore, we can attain the 1-max flow quantum communication of value k over the network $\mathcal{N}$, and the achievable rate region is $\{(r_1,\ldots,r_k)|\sum_{i=1}^k r_i \leqslant k\}$, extending the corresponding result over the butterfly network.

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