SCIENCE CHINA
Information Sciences

• RESEARCH PAPER •

January 2019, Vol. 62 012207:1–012207:13
https://doi.org/10.1007/s11432-017-9411-0

Symmetry-based decomposition of finite games

Changxi LI[1], Fenghua HE[1*], Ting LIU[2] & Daizhan CHENG[2]

[1]*School of Astronautics, Harbin Institute of Technology, Harbin* 150001*, China;*
[2]*Key Laboratory of Systems and Control, Academy of Mathematics and Systems Sciences, Chinese Academy of Sciences, Beijing* 100190*, China*

Received 28 October 2017/Revised 14 January 2018/Accepted 5 March 2018/Published online 19 December 2018

Abstract The symmetry-based decompositions of finite games are investigated. First, the vector space of finite games is decomposed into a symmetric subspace and an orthogonal complement of the symmetric subspace. The bases of the symmetric subspace and those of its orthogonal complement are presented. Second, the potential-based orthogonal decompositions of two-player symmetric/antisymmetric games are presented. The bases and dimensions of all dual decomposed subspaces are revealed. Finally, some properties of these decomposed subspaces are obtained.

Keywords potential game, symmetric game, decomposition, Nash equilibrium, semi-tensor product of matrices

Citation Li C X, He F H, Liu T, et al. Symmetry-based decomposition of finite games. Sci China Inf Sci, 2019, 62(1): 012207, https://doi.org/10.1007/s11432-017-9411-0

1 Introduction

Game theory has drawn considerable attention in recent years owing to its widespread applications from social systems to engineering science [1–3]. Although most studies arise from social and economic fields, there is a trend of applying game theory in cybernetics, which is called game-theoretic control. A few representative applications are (i) optimization and control of power networks [4], (ii) consensus of multi-agent systems [5], and (iii) allocation problem of finite resource [6, 7]. Partly owing to the widespread applications of game theory, particularly, potential game theory, researchers have begun studying the topological structure of finite games.

To investigate the topological structure of finite games, different decompositions of finite games have been reported [8–11]. Potential-based decomposition of finite games was first proposed by Candogan et al. [8] using the Helmholtz decomposition theorem, where the space of finite games is decomposed into a canonical sum of the pure potential subspace $\mathcal{P}$, non-strategic subspace $\mathcal{N}$, and pure harmonic subspace $\mathcal{H}$. Kalai et al. [9] decomposed bimatrix game into cooperative and competitive components. Hwang et al. [10] considered the decomposition of two-player games. The bases of the subspaces of potential games, zero-sum games, and their orthogonal complements are provided in [10]. However, it is noteworthy that the work on orthogonal decomposition is still limited. The main difficulty is that the vector space structure for multi-player game cannot be imbedded into Euclidean space.

The above issues have been solved recently with the concept of semi-tensor product (STP) of matrices [12, 13]. Particularly, the verification of finite potential games is completely solved [13]. Then,

* Corresponding author (email: hefenghua@hit.edu.cn)

Ref. [12] marginally modified the potential-based decomposition of [8] by proposing the following vector space structure on nl-dimensional Euclidean space $\mathbb{R}^{nl}$:

$$\mathcal{G}_{[n;l_1,\ldots,l_n]} = \underbrace{\mathcal{P} \oplus \overbrace{\mathcal{N}}^{\text{Harmonic games}}}_{\text{Potential games}} \overbrace{\oplus \mathcal{H}}, \tag{1}$$

where $l = \prod_{i=1}^{n} l_i$, $\mathcal{G}_{[n;l_1,\ldots,l_n]}$ is the set of n-player finite games and the i-th player has l_i strategies. The bases of pure potential subspace $\mathcal{P}$, non-strategic subspace $\mathcal{N}$, and pure harmonic subspace $\mathcal{H}$ were presented in [12]. Note that although the decomposition (1), which was first proposed by Candogan, is called orthogonal, the inner product used in [8] was not the standard one in Euclidean space. Hence, when $G \in \mathcal{G}_{[n;l_1,\ldots,l_n]}$ is imbedded onto $\mathbb{R}^{nl}$, their decomposition is not orthogonal.

Recently, Ref. [14] investigated the structures of different symmetric games using a linear representation of finite symmetric group under the vector space structure of $\mathcal{G}_{[n;\kappa]}$, where $\mathcal{G}_{[n;\kappa]}$ is the set of n-player finite games with κ strategies for each player. The symmetry-based decomposition of finite games is proposed, where finite games $\mathcal{G}_{[n;\kappa]}$ are decomposed into an orthogonal sum of the subspace of symmetric games $\mathcal{S}_{[n;\kappa]}$ and its orthogonal complement $\mathcal{K}_{[n;\kappa]}$. That is,

$$\mathcal{G}_{[n;\kappa]} = \mathcal{S}_{[n;\kappa]} \oplus \mathcal{K}_{[n;\kappa]}. \tag{2}$$

However, the bases of symmetric games and its orthogonal complement are not explicit as of yet. Hence, the structures of $\mathcal{S}_{[n;\kappa]}$ and $\mathcal{K}_{[n;\kappa]}$ are still not very clear. The two-player symmetric games are particularly important in networked evolutionary games [15]; therefore, they have been emphasized in our later investigation. To consider the properties of different types of two-player games, further decomposition of two-player games is studied by combining the above two decomposition methods.

This study aims at clarifying the structures of symmetric games and their orthogonal complements, combining potential-based decomposition and symmetry-based decomposition for two-player games, and then revealing the structures of all dual decomposed subspaces. The contributions of this paper are threefold: (i) The bases of the subspace of symmetric games $\mathcal{S}_{[n;\kappa]}$ and its orthogonal complement $\mathcal{K}_{[n;\kappa]}$ are revealed. (ii) Potential-based orthogonal decompositions of two-player symmetric/antisymmetric games are presented, which are shown in (3) and (4); here, $\mathcal{K}_{[2;\kappa]}$ is the subspace of the antisymmetric games. (iii) Certain properties of the decomposed subspaces in (3) and (4) are obtained.

$$\mathcal{S}_{[2;\kappa]} = \underbrace{\mathcal{SP}_{[2;\kappa]} \oplus \mathcal{SN}_{[2;\kappa]}}_{\text{Potential component}} \oplus \mathcal{SH}_{[2;\kappa]}, \quad \text{Harmonic component: } \mathcal{SN}_{[2;\kappa]} \oplus \mathcal{SH}_{[2;\kappa]} \tag{3}$$

and

$$\mathcal{K}_{[2;\kappa]} = \underbrace{\mathcal{KP}_{[2;\kappa]} \oplus \mathcal{KN}_{[2;\kappa]}}_{\text{Potential component}} \oplus \mathcal{KH}_{[2;\kappa]}. \quad \text{Harmonic component: } \mathcal{KN}_{[2;\kappa]} \oplus \mathcal{KH}_{[2;\kappa]} \tag{4}$$

The rest of this paper is organized as follows: as the technique used in this paper is STP, Section 2 provides some preliminaries including STP and symmetric game theory. Section 3 presents the bases of the symmetric subspace $\mathcal{S}_{[n;\kappa]}$ and its orthogonal complement $\mathcal{K}_{[n;\kappa]}$. The orthogonal decompositions of two-player symmetric/antisymmetric games into pure potential component, non-strategic component, and pure harmonic component are studied in Section 4. The bases of all dual decomposed subspaces are obtained. Section 5 reveals certain properties of these orthogonal subspaces. Section 6 is a brief conclusion.

For convenience, we provide certain notations used throughout this paper.

(1) $\mathbb{R}^n$: n dimensional Euclidean space.

(2) $\mathcal{M}_{m\times n}$: the set of $m \times n$ real matrices.

(3) $\mathcal{D}_k := \{1, 2, \dots, k\}$, $k \geqslant 2$.
(4) δ_n^i: the i-th column of the identity matrix I_n.
(5) $\Delta_n := \{\delta_n^i | i = 1, \dots, n\}$.
(6) $\mathbf{1}_\ell = [\underbrace{1, 1, \dots, 1}_{\ell}]^{\mathrm{T}}$.
(7) P_n: the set of permutations of the elements of $N = \{1, 2, \dots, n\}$.
(8) $\mathrm{Col}(M)$: the set of columns of matrix M. $\mathrm{Col}_i(M)$: the i-th column of matrix M.

2 Preliminaries

This section provides a few preliminaries, including the semi-tensor product of matrices, symmetric game, and vector space structure of finite games.

2.1 Symmetric game

A finite non-cooperative game is denoted by $G = \{N, \{S_i\}_{i\in N}, \{c_i\}_{i\in N}\}$, where $N = \{1, 2, \dots, n\}$ is the set of players; $S_i = \{1, 2, \dots, l_i\}$ is the set of the strategies of player i for each $i \in N$; and $c_i : S \to \mathbb{R}$ is the payoff function of player i, with $S := \prod_{i=1}^n S_i$ being the strategy profile of the game. Let $S_{-i} := \prod_{j\neq i} S_j$ be the set of partial strategy profiles other than player i. The concept of symmetric game was first proposed by Nash in his famous paper [16]. The definition of symmetric game is as follows.

Definition 1 (Symmetric game [17])**.** A game $G \in \mathcal{G}_{[n;\kappa]}$ is called ordinary symmetric if for any permutation $\sigma \in P_n$

$$c_i(x_1, \dots, x_n) = c_{\sigma(i)}(x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \dots, x_{\sigma^{-1}(n)}), \quad i = 1, \dots, n,$$

where σ^{-1} is the inverse permutation of σ.

Considering a strategy profile $s \in S$, the strategy multiplicity vector of $s = (s_1, \dots, s_n)$ is defined as [17]

$$\sharp(s) = [\sharp(s, 1), \sharp(s, 2), \dots, \sharp(s, \kappa)] \in \mathbb{R}^\kappa,$$

where $\sharp(s, i) := |\{s_j \,|\, s_j = i\}|$, $i = 1, \dots, \kappa$, and κ is the number of strategies for each player in a symmetric game. For any subset of a profile, the strategy multiplicity vector can be defined in a similar way. Particularly, $\sharp(s_{-i}) \in \mathbb{R}^\kappa$ is well defined. Using the notation of strategy multiplicity, it is easy to verify the following equivalent condition of symmetric game.

Proposition 1 ([14])**.** A finite game $G \in \mathcal{G}_{[n;\kappa]}$ is symmetric if and only if

$$c_i(x_i, x_{-i}) = c_j(y_j, y_{-j}), \quad 1 \leqslant i, j \leqslant n,$$

where $x_i = y_j$ and $\sharp(x_{-i}) = \sharp(y_{-j})$.

Remark 1. Consider a two-player game $G \in \mathcal{G}_{[2;\kappa]}$. Suppose $A \in \mathcal{M}_{\kappa\times\kappa}$ is the payoff matrix of the first player, and $B \in \mathcal{M}_{\kappa\times\kappa}$ is the payoff matrix of the second player. It is easy to verify that G is symmetric if and only if $A = B^{\mathrm{T}}$. Similarly, G is antisymmetric if and only if $A = -B^{\mathrm{T}}$.

2.2 Semi-tensor product of matrices

The STP of matrices is a generalization of the conventional matrix product. This subsection provides a brief review on STP. Please refer to [18] for more details.

Definition 2 (STP of matrices)**.** Let $A \in \mathcal{M}_{m\times n}$, $B \in \mathcal{M}_{p\times q}$, and t be the least common multiple of n and p. The STP of A and B is defined as

$$A \ltimes B := \left(A \otimes I_{t/n}\right)\left(B \otimes I_{t/p}\right) \in \mathcal{M}_{mt/n\times qt/p}.$$

Hereafter, all the matrix products are assumed to be STP unless otherwise stated.

Definition 3 (Swap matrix [18]). A swap matrix $W_{[m,n]} \in \mathcal{M}_{mn\times mn}$ is defined as

$$\mathrm{Col}_{(i-1)n+j}(W_{[m,n]}) = \delta_{mn}^{i+(j-1)m}, \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots, n.$$

Proposition 2 ([18]). Let $X \in \mathbb{R}^m$ and $Y \in \mathbb{R}^n$ be two column vectors. Then,

$$W_{[m,n]}XY = YX.$$

Definition 4 (Mix-valued pseudo-logical function). A function $c : \prod_{i=1}^n \mathcal{D}_{l_i} \to \mathbb{R}$ is called a mix-valued pseudo-logical function.

Let $i \sim \delta_k^i$, which is called the vector expression of $i \in \mathcal{D}_k$. In the following, all the strategies are expressed in vector form.

Proposition 3. Let $c : \prod_{i=1}^n \mathcal{D}_{l_i} \to \mathbb{R}$ be a mix-valued pseudo-logical function. Using vector expression of x_i, there exists a unique row vector $V^c \in \mathbb{R}^l$ such that

$$c(x_1, \dots, x_n) = V^c \ltimes_{i=1}^n x_i,$$

V^c is called the structure vector of c, and $l = \prod_{i=1}^n l_i$.

Finally, we briefly introduce the vector space structure of finite games.

Using STP and the vector expression of strategies $x_i \in S_i$, each payoff function c_i becomes a mapping $c_i : \Delta_l \to \mathbb{R}$, where $l = \prod_{i=1}^n l_i$. Hence for each c_i, we can determine a unique row vector $V_i \in \mathbb{R}^l$ such that

$$c_i(x_1, \dots, x_n) = V_i \ltimes_{j=1}^n x_j, \quad i = 1, \dots, n.$$

The payoff vector $V_G := [V_1, V_2, \dots, V_n] \in \mathbb{R}^{nl}$, where V_i is the structure vector of c_i. Then, it is evident that $\mathcal{G}_{[n;l_1,\dots,l_n]}$ has a natural vector space structure as $\mathbb{R}^{nl}$.

3 Vector space structure of symmetry-based decomposition

The structures of different symmetric games using the linear representation of a finite symmetric group under the vector space structure of $\mathcal{G}_{[n;\kappa]}$ are investigated in [14]. The symmetry-based decomposition of finite games is proposed. However, the bases of symmetric games and their orthogonal complement are not explicit as of yet. We investigate the bases of symmetric games and their orthogonal complement in this section.

3.1 Vector space structure of symmetric games

The necessary and sufficient condition for a game $G \in \mathcal{G}_{[n;\kappa]}$ to be symmetric was provided in [14].

Theorem 1 ([14]). $G \in \mathcal{G}_{[n;\kappa]}$ is a symmetric game if and only if

(i) $V_1[I_\kappa \otimes (W_{[\kappa^{j-2},\kappa]}W_{[\kappa,\kappa^{j-1}]}) - I_{\kappa^{j+1}}] = 0,\ j = 2, 3, \dots, n-1;$

(ii) $V_i = V_1 W_{[\kappa^{i-1},\kappa]},\ i = 2, 3, \dots, n;$

where V_i is the payoff vector of player i and $W_{[m,n]}$ is the swap matrix.

To obtain the bases of symmetric games, we only need to consider the first player. According to Proposition 1, c_1 is a candidate payoff function of the first player for a symmetric game if and only if

$$c_1(x_1, x_{-1}) = c_1(x_1, y_{-1}), \tag{5}$$

where $\sharp(x_{-1}) = \sharp(y_{-1})$.

Let $\mathcal{S}_{-1}$ denote the set of strategy profiles with strict partial order except the first player, which is expressed as follows:

$$\mathcal{S}_{-1} := \{\mathfrak{s}_{-1} \,|\, \mathfrak{s}_{-1} = (i_2, \dots, i_n) \in S_{-1},\ i_2 \leqslant \cdots \leqslant i_n\}. \tag{6}$$

Set $\mathfrak{s}^*_{-1} := \{x_{-1} \,|\, \sharp(x_{-1}) = \sharp(\mathfrak{s}_{-1}),\ \mathfrak{s}_{-1} \in \mathcal{S}_{-1}\}$, and define a vector $h^{(i_1,\mathfrak{s}_{-1})}$ as

$$h^{(i_1,\mathfrak{s}_{-1})} =: \sum_{(i_2,\ldots,i_n)\in\mathfrak{s}^*_{-1}} \delta_\kappa^{i_1}\delta_\kappa^{i_2}\delta_\kappa^{i_3}\cdots\delta_\kappa^{i_n}.$$

According to (5), all vector $h^{(i_1,\mathfrak{s}_{-1})}$ form the basis of V_1. Define a matrix $H_n \in \mathcal{M}_{\kappa^n\times d^*}$, whose columns are

$$\left\{h^{(i_1,i_2,\ldots,i_n)} | (i_2,\ldots,i_n) \in \mathcal{S}_{-1},\ i_1 = 1,2,\ldots,\kappa\right\}.$$

The columns of $H_n \in \mathcal{M}_{\kappa^n\times d^*}$ should be organized in the lexicographical order, where $d^* = \kappa|\mathcal{S}_{-1}|$. It is apparent that all the columns of matrix H_n are linearly independent. Then, we have the following result.

Proposition 4. Consider a symmetric game $G \in \mathcal{G}_{[n;\kappa]}$. $V_1 \in \mathbb{R}^{\kappa^n}$ is the structure vector of the first player if and only if

$$V_1^{\mathrm{T}} \in \mathrm{Span}(H_n).$$

Define

$$W_n := \begin{bmatrix} I_{\kappa^n} \\ W_{[\kappa,\kappa]} \otimes I_{\kappa^{n-2}} \\ W_{[\kappa,\kappa^2]} \otimes I_{\kappa^{n-3}} \\ \vdots \\ W_{[\kappa,\kappa^{n-2}]} \otimes I_\kappa \\ W_{[\kappa,\kappa^{n-1}]} \end{bmatrix}.$$

Using Theorem 1 and Proposition 4, the structure of symmetric games $\mathcal{S}_{[n;\kappa]}$ is apparent and is depicted as follows.

Proposition 5. The subspace of symmetric games $\mathcal{S}_{[n;\kappa]}$ is

$$\mathcal{S}_{[n;\kappa]} = \mathrm{Span}(W_n H_n),$$

which has $\mathrm{Col}(W_n H_n)$ as its basis. Therefore, the dimension of $\mathcal{S}_{[n;\kappa]}$ is $\dim(\mathcal{S}_{[n;\kappa]}) = \kappa|\mathcal{S}_{-1}| = \kappa\binom{n+k-1}{n}$.

3.2 Orthogonal complement of symmetric games

The vector space structure of $\mathcal{K}_{[n;k]}$ is investigated in this subsection, where $\mathcal{K}_{[n;k]}$ is the orthogonal complement of $\mathcal{S}_{[n;k]}$.

Definition 5 ([19])**.** Consider a permutation $\sigma \in P_n$. A σ permutation matrix is defined as $W_{[\sigma;l_1,\ldots,l_n]} \in \mathcal{M}_{l\times l}$, whose columns are

$$\left\{\ltimes_{i=1}^n \delta_{l_{\sigma(i)}}^{i_{\sigma(i)}} \,|\, i_j = 1,2,\ldots,k_j\right\},$$

and should be organized in the order of $(\sigma^{-1}(1),\ldots,\sigma^{-1}(n))$, where $l = \prod_{i=1}^n l_i$. Particularly, $W_{[\sigma;\kappa]} := W_{[\sigma;\kappa,\ldots,\kappa]}$. The σ permutation matrix can be used to reorder n vectors according to permutation σ.

Proposition 6 ([19])**.** Let $X_i \in \mathbb{R}^{l_i}$, $i = 1,\ldots,n$. Then

$$W_{[\sigma;l_1,\ldots,l_n]} \ltimes_{i=1}^n X_i = \ltimes_{i=1}^n X_{\sigma(i)}.$$

Define σ_r and $\sigma^r \in P_n$ as follows:

$$\sigma_r := (r, r-1, \ldots, 1),\ \sigma^r := (1,2,\ldots,r),\quad r = 2,\ldots,n.$$

It is evident that σ_r and σ^r are the inverse permutations of each other.

Proposition 7.

$$W_{[\sigma_l;\kappa]} = W_{[\kappa^{l-1},\kappa]} \otimes I_{\kappa^{n-l}}, \quad l = 1, 2, \ldots, n. \tag{7}$$

Proof.

$$\begin{aligned}(W_{[\kappa^{l-1},\kappa]} \otimes I_{\kappa^{n-l}})\delta_\kappa^{i_1}\delta_\kappa^{i_2}\cdots\delta_\kappa^{i_n} &= \delta_\kappa^{i_l}\delta_\kappa^{i_1}\cdots\delta_\kappa^{i_{l-1}}\delta_\kappa^{i_{l+1}}\cdots\delta_\kappa^{i_n} \\ &= \delta_\kappa^{i_{\sigma_l(1)}}\delta_\kappa^{i_{\sigma_l(2)}}\cdots\delta_\kappa^{i_{\sigma_l(n)}} \\ &= W_{[\sigma_l;\kappa]}\delta_\kappa^{i_1}\delta_\kappa^{i_2}\cdots\delta_\kappa^{i_n}, \quad l = 1, 2, \ldots, n.\end{aligned}$$

Define a matrix $J_1 = [J_1^{(1,1,\ldots,1)}, \ldots, J_1^{(i_1,i_2,\ldots,i_n)}, \ldots, J_1^{(n,n,\ldots,n)}] \in \mathcal{M}_{n\kappa^n \times d^*}$, where $(i_2, \ldots, i_n) \in \mathcal{S}_{-1}$, and

$$J_1^{(i_1,i_2,\ldots,i_n)} = \begin{bmatrix} \delta_\kappa^{i_1}\delta_\kappa^{i_2}\delta_\kappa^{i_3}\cdots\delta_\kappa^{i_n} \\ -\delta_{n-1}^1\delta_\kappa^{i_{\sigma_2(1)}}\cdots\delta_\kappa^{i_{\sigma_2(n)}} \end{bmatrix}.$$

Furthermore, define a matrix $J_2 \in \mathcal{M}_{n\kappa^n \times (\kappa^n - d^*)}$, whose column is $J_2^{(i_1,i_2,\ldots,i_n)}$, $(i_2, \ldots, i_n) \in S_{-1} \backslash \mathcal{S}_{-1}$. According to (6), for any $(i_2, \ldots, i_n) \in S_{-1}$, there exists only an $\mathfrak{s}_{-1} = (j_2, \ldots, j_n) \in \mathcal{S}_{-1}$ such that $(i_2, \ldots, i_n) \in \mathfrak{s}_{-1}^*$. $J_2^{(i_1,i_2,\ldots,i_n)}$ is defined as follows:

$$J_2^{(i_1,i_2,\ldots,i_n)} = [\delta_n^1(\delta_\kappa^{i_1}\delta_\kappa^{j_2}\cdots\delta_\kappa^{j_n} - \delta_\kappa^{i_1}\delta_\kappa^{i_2}\cdots\delta_\kappa^{i_n})],$$

where $\mathfrak{s}_{-1} = (j_2, \ldots, j_n) \in \mathcal{S}_{-1}$, $(i_2, \ldots, i_n) \in \mathfrak{s}_{-1}^* \backslash \mathfrak{s}_{-1}$, and $j_1 = 1, 2, \ldots, \kappa$.

Proposition 8. Define $J := [J_1, J_2]$. Then, J has full column rank and

$$\mathcal{K}_{[n;\kappa]} = \mathrm{Span}(J).$$

Proof. First, it is obvious that all the columns of J are linearly independent. Then, J is full column rank.

Next, we prove that any column of J is a solution of the equation $H_n^{\mathrm{T}} W_n^{\mathrm{T}} x = 0$. This can be established by straightforward computations. According to (7), we have

$$\begin{aligned}H_n^{\mathrm{T}} W_n^{\mathrm{T}} &= H_n^{\mathrm{T}}\left[I_{k^n}, W_{[k,k]} \otimes I_{k^{n-2}}, W_{[k^2,k]} \otimes I_{k^{n-3}}, \ldots, W_{[k^{n-1},k]}\right] \\ &= \left[H_n^{\mathrm{T}} W_{[\sigma^1;k]}, H_n^{\mathrm{T}} W_{[\sigma^2;k]}, \ldots, H_n^{\mathrm{T}} W_{[\sigma^n;k]}\right].\end{aligned}$$

Consider any column $J_1^{(j_1,j_2,\ldots,j_n)}$ in J_1; then,

$$\begin{aligned}&\mathrm{Row}_{(i_1,i_2,\ldots,i_n)}(H_n^{\mathrm{T}} W_n) \cdot J_1^{(j_1,j_2,\ldots,j_n)} \\ &= \sum_{t=1}^n h^{(i_1,i_2,\ldots,i_n)} W_{[\sigma^t;k]} \cdot J_1^{(j_1,j_2,\ldots,j_n)} \\ &= \sum_{t=1}^n h^{(i_1,i_2,\ldots,i_n)} \cdot J_1^{(j_{\sigma^t(1)}, j_{\sigma^t(2)}, \ldots, j_{\sigma^t(n)})} \\ &= 0.\end{aligned} \tag{8}$$

The last equality in (8) arises from the definitions of $h^{(i_1,i_2,\ldots,i_n)}$ and $J_1^{(j_1,j_2,\ldots,j_n)}$. Similarly, for any column $J_2^{(j_1,\ldots,j_n)}$, we have

$$\mathrm{Row}_{(i_1,i_2,\ldots,i_n)}(H_n^{\mathrm{T}} W_n^{\mathrm{T}}) \cdot J_2^{(j_1,\ldots,j_n)} = 0.$$

4 Potential-based decomposition of two-player symmetric/antisymmetric games

This section aims at combining potential-based decomposition and symmetry-based decomposition for two-player games and then revealing the structures of all dual decomposed subspaces.

4.1 Symmetry-based and potential-based decomposition of two-player games

The structures of different subspace for two-player games under symmetry-based and potential-based decomposition are provided in this subsection. Using Remark 1, it is convenient to prove that $\mathcal{K}_{[2;\kappa]}$ is exactly the subspace of antisymmetric games. According to the results of Propositions 5 and 8, we have Propositions 9 and 10.

Proposition 9 (Symmetry-based decomposition of two-player games)**.** Two-player games $\mathcal{G}_{[2;\kappa]}$ under symmetry-based decomposition can be decomposed as $\mathcal{G}_{[2;\kappa]} = \mathcal{S}_{[2;\kappa]} \oplus \mathcal{K}_{[2;\kappa]}$. Moreover, (i) the subspace of two-player symmetric game $\mathcal{S}_{[2;\kappa]}$ is

$$\mathcal{S}_{[2;\kappa]} = \mathrm{Span}(S),$$

where

$$S = \begin{bmatrix} I_{\kappa^2} \\ W_{[\kappa,\kappa]} \end{bmatrix};$$

(ii) the subspace of two-player antisymmetric game $\mathcal{K}_{[2;\kappa]}$ is

$$\mathcal{K}_{[2;\kappa]} = \mathrm{Span}(K),$$

where

$$K = \begin{bmatrix} I_{\kappa^2} \\ -W_{[\kappa,\kappa]} \end{bmatrix}.$$

Potential-based decomposition is investigated in [12], and the following proposition is a special case of the general result in [12].

Proposition 10 (Potential-based decomposition of two-player games)**.** Two-player games $\mathcal{G}_{[2;\kappa]}$ under potential-based decomposition can be decomposed as $\mathcal{G}_{[2;\kappa]} = \mathcal{P}_{[2;\kappa]} \oplus \mathcal{N}_{[2;\kappa]} \oplus \mathcal{H}_{[2;\kappa]}$. Moreover, (i) the pure potential subspace $\mathcal{P}_{[2;\kappa]}$ is

$$\mathcal{P}_{[2;\kappa]} = \mathrm{Span}(E_P),$$

where

$$E_P = \begin{bmatrix} I_{\kappa^2} - \frac{1}{\kappa}E_1E_1^{\mathrm{T}} \\ I_{\kappa^2} - \frac{1}{\kappa}E_2E_2^{\mathrm{T}} \end{bmatrix} \in \mathcal{M}_{2\kappa^2\times\kappa^2}, \quad E_1 = \mathbf{1}_\kappa \otimes I_\kappa, \quad E_2 = I_\kappa \otimes \mathbf{1}_\kappa;$$

(ii) the non-strategic subspace $\mathcal{N}_{[2;\kappa]}$ is

$$\mathcal{N}_{[2;\kappa]} = \mathrm{Span}(E_N),$$

where

$$E_N := \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \in \mathcal{M}_{2\kappa^2\times 2\kappa};$$

(iii) the pure harmonic subspace $\mathcal{H}_{[2;\kappa]}$ is

$$\mathcal{H}_{[2;\kappa]} = \mathrm{Span}(E_H),$$

where

$$\mathrm{Col}(E_H) = \{J^{(i_1,i_2)}, i_1 \neq 1, i_2 \neq 1\}, \quad J^{(i_1,i_2)} = \begin{bmatrix} (\delta_\kappa^1 - \delta_\kappa^{i_1})(\delta_\kappa^1 - \delta_\kappa^{i_2}) \\ -(\delta_\kappa^1 - \delta_\kappa^{i_1})(\delta_\kappa^1 - \delta_\kappa^{i_2}) \end{bmatrix}.$$

Moreover, all the columns of E_H are organized in the lexicographical order.

4.2 Potential-based decomposition of two-player symmetric games

Potential-based decomposition of two-player symmetric games implies that two-player symmetric games are decomposed into three parts: non-strategic part $\mathcal{SN}_{[2;\kappa]}$, pure harmonic part $\mathcal{SH}_{[2;\kappa]}$, and pure potential part $\mathcal{SP}_{[2;\kappa]}$, which are presented in (3). The bases of the different parts in the decomposition are revealed in this subsection.

Theorem 2. The non-strategic subspace of two-player symmetric games $\mathcal{SN}_{[2;\kappa]}$ is

$$\mathcal{SN}_{[2;\kappa]} = \mathrm{Span}(B_{SN}),$$

where

$$B_{SN} = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}.$$

Proof. Define a variable ξ as follows:

$$\xi := [b_1, \ldots, b_\kappa, c_1, \ldots, c_\kappa]^{\mathrm{T}} \in \mathbb{R}^{2\kappa}.$$

The structure vector of a two-player non-strategic game can be described as $E_N\xi$. A two-player non-strategic game is symmetric if and only if $K^{\mathrm{T}}E_N\xi = 0$, which implies that $b_i = c_i$, $i = 1, 2, \ldots, \kappa$.

Equivalently, the two-player non-strategic symmetric subspace is

$$\mathrm{Span}\left(E_N \begin{bmatrix} I_\kappa \\ I_\kappa \end{bmatrix}\right) = \mathrm{Span}\,(B_{SN}).$$

Theorem 3. The pure harmonic subspace of two-player symmetric games $\mathcal{SH}_{[2;\kappa]}$ is

$$\mathcal{SH}_{[2;\kappa]} = \mathrm{Span}(B_{SH}),$$

where

$$\mathrm{Col}(B_{SH}) = \{J^{(i,j)} - J^{(j,i)} \mid i > j, i \neq 1, j \neq 1\}.$$

Moreover, all the columns of B_{SH} are organized in the lexicographical order.

Proof. The proof is similar to that of Theorem 2. Hence, we omit it here.

Theorem 4. The pure potential subspace of two-player symmetric games $\mathcal{SP}_{[2;\kappa]}$ is

$$\mathcal{SP}_{[2;\kappa]} = \mathrm{Span}(B_{SP}),$$

where

$$\mathrm{Col}(B_{SP}) = \left\{ \begin{bmatrix} \kappa(\delta_\kappa^i\delta_\kappa^j + \delta_\kappa^j\delta_\kappa^i) - \mathbf{1}_\kappa(\delta_\kappa^i + \delta_\kappa^j) \\ \kappa(\delta_\kappa^i\delta_\kappa^j + \delta_\kappa^j\delta_\kappa^i) - (\delta_\kappa^i + \delta_\kappa^j)\mathbf{1}_\kappa \end{bmatrix} \middle| i \geqslant j \right\}.$$

Moreover, all the columns of B_{SP} are organized in the lexicographical order.

Proof. The proof is similar to that of Theorem 2. Hence, we omit it here.

Corollary 1. The dimensions of two-player symmetric non-strategic subspace $\mathcal{SN}_{[2;\kappa]}$, pure potential subspace $\mathcal{SP}_{[2;\kappa]}$, and pure harmonic subspace $\mathcal{SH}_{[2;\kappa]}$ are

$$\begin{aligned} \dim(\mathcal{SN}_{[2;\kappa]}) &= \kappa, \\ \dim(\mathcal{SP}_{[2;\kappa]}) &= \frac{\kappa(\kappa+1)}{2} - 1, \\ \dim(\mathcal{SH}_{[2;\kappa]}) &= \frac{(\kappa-1)(\kappa-2)}{2}, \end{aligned}$$

respectively.

4.3 Potential-based decomposition of two-player antisymmetric games

The potential-based decomposition of two-player antisymmetric games means that two-player antisymmetric games are decomposed into three parts: non-strategic part $\mathcal{KN}_{[2;\kappa]}$, pure harmonic part $\mathcal{KH}_{[2;\kappa]}$, and pure potential part $\mathcal{KP}_{[2;\kappa]}$, which are shown in (4). The bases of different parts in the decomposition are revealed in this subsection.

Theorem 5. The non-strategic subspace of two-player antisymmetric games $\mathcal{KN}_{[2;\kappa]}$ is

$$\mathcal{KN}_{[2;\kappa]} = \mathrm{Span}(B_{KN}),$$

where

$$B_{KN} = \begin{bmatrix} E_1 \\ -E_2 \end{bmatrix}.$$

Moreover, $\mathcal{KN}_{[2;\kappa]}$ has $\mathrm{Col}(B_{KN})$ as its basis.

Proof. Define

$$\xi := [b_1, \ldots, b_\kappa, c_1, \ldots, c_\kappa]^{\mathrm{T}} \in \mathbb{R}^{2\kappa}.$$

The structure vector of a two-player non-strategic game can be described as $E_N\xi$. A two-player non-strategic game is antisymmetric if and only if $S^{\mathrm{T}}E_N\xi = 0$, which implies that $b_i = -c_i$, $i = 1, 2, \ldots, \kappa$.

Equivalently, the non-strategic subspace of two-player antisymmetric games is

$$\mathrm{Span}\left(E_N \begin{bmatrix} I_\kappa \\ -I_\kappa \end{bmatrix}\right) = \mathrm{Span}\left(B_{KN}\right).$$

Theorem 6. The pure harmonic subspace of two-player antisymmetric games $\mathcal{KH}_{[2;\kappa]}$ is

$$\mathcal{KH}_{[2;\kappa]} = \mathrm{Span}(B_{KH}),$$

where

$$\mathrm{Col}(B_{KH}) = \left\{ J^{(i,j)} + J^{(j,i)} | i \geqslant j, i \neq 1, j \neq 1 \right\}. \tag{9}$$

All the columns of B_{KH} are organized in the lexicographical order. $\mathcal{KH}_{[2;\kappa]}$ has $\mathrm{Col}(B_{KH})$ as its basis.

Proof. The proof is similar to that of Theorem 5. Hence, we omit it here.

Theorem 7. The pure potential subspace of two-player antisymmetric games $\mathcal{KP}_{[2;\kappa]}$ is

$$\mathcal{KP}_{[2;\kappa]} = \mathrm{Span}(B_{KP}),$$

where

$$\mathrm{Col}(B_{KP}) = \left\{ \begin{bmatrix} \kappa(\delta_\kappa^i \delta_\kappa^j - \delta_\kappa^j \delta_\kappa^i) - \mathbf{1}_\kappa(\delta_\kappa^i - \delta_\kappa^j) \\ \kappa(\delta_\kappa^i \delta_\kappa^j - \delta_\kappa^j \delta_\kappa^i) - (\delta_\kappa^i - \delta_\kappa^j)\mathbf{1}_\kappa \end{bmatrix} \middle| i > j \right\}.$$

All the columns of B_{KP} are organized in the lexicographical order.

Proof. The proof is similar to that of Theorem 5. Hence, we omit it here.

Corollary 2. The dimensions of two-player antisymmetric non-strategic subspace $\mathcal{KN}_{[2;\kappa]}$, pure potential subspace $\mathcal{KP}_{[2;\kappa]}$, and pure harmonic subspace $\mathcal{KH}_{[2;\kappa]}$ are

$$\begin{aligned} \dim(\mathcal{KN}_{[2;\kappa]}) &= \kappa, \\ \dim(\mathcal{KP}_{[2;\kappa]}) &= \frac{\kappa(\kappa-1)}{2}, \\ \dim(\mathcal{KH}_{[2;\kappa]}) &= \frac{\kappa(\kappa-1)}{2}, \end{aligned}$$

respectively.

5 Properties of potential-based decomposed subspaces

Nash equilibrium and Pareto optimal are two important concepts in non-cooperative games. We investigate the properties of the decomposed subspace of two player games under potential-based decomposition.

Definition 6 (Nash equilibrium [20]). A profile $s^* \in S$ is called a Nash equilibrium of a finite non-cooperative game $G = \{N, \{S_i\}_{i\in N}, \{c_i\}_{i\in N}\}$ if

$$c_i(s^*) \geqslant c_i(s_i, s^*_{-i}), \quad \forall s_i \in S_i, \quad i = 1, \ldots, n.$$

Definition 7 (Pareto optimal [8]). A strategy profile $s^* \in S$ is called Pareto optimal if and only if there does not exist another strategy profile $s \in S$ such that all the players increase their payoffs weakly and at least one increases its payoff strictly, i.e.,

$$\begin{aligned} &c_i(s) \geqslant c_i(s^*), \quad \text{for all } i,\\ &c_j(s) > c_j(s^*), \quad \text{for some } j. \end{aligned}$$

Propositions 11 and 12 described the pure Nash equilibria of two-player zero-sum game and pure harmonic game.

Proposition 11 ([20]). Consider a two-player zero-sum game G. If the strategy profiles (i_1, j_1) and (i_2, j_2) are pure Nash equilibria of G, then (i_1, j_2) and (i_2, j_1) are pure Nash equilibria of G.

Proposition 12 ([21]). Consider a two-player pure harmonic game $G \in \mathcal{G}_{[2;\kappa]}$. $A = (a_{ij})_{\kappa\times\kappa}$ is the payoff matrix of the first player. Then the strategy profile (i, j) is a pure Nash equilibrium of G if and only if

$$a_{is} = 0, \ s = 1, 2, \ldots, \kappa; \quad a_{tj} = 0, \ t = 1, 2, \ldots, \kappa.$$

5.1 Properties of two-player symmetric games

The following result reveals the property of two-player symmetric games.

Theorem 8. Consider a two-player symmetric game $G \in \mathcal{S}_{[2;\kappa]}$. If (i, j) is a pure Nash equilibrium of G, then (j, i) is also a pure Nash equilibrium of G. Moreover, if $G \in \mathcal{SH}_{[2;\kappa]}$ and (i, j) is a pure Nash equilibrium of G, then (i, i), (j, j) and (j, i) are also pure Nash equilibria of G.

Proof. Consider a two-player game $G \in \mathcal{S}_{[2;\kappa]}$. Suppose the payoff matrix of the first player is $A = (a_{ij})_{\kappa\times\kappa}$, and $B = (b_{ij})_{\kappa\times\kappa}$ is the one for the second player.

If (i^*, j^*) is a pure Nash equilibrium of G, we have

$$\begin{aligned} &a_{i^*j^*} \geqslant a_{ij^*}, \ i = 1, 2, \ldots, \kappa,\\ &b_{i^*j^*} \geqslant b_{i^*j}, \ j = 1, 2, \ldots, \kappa. \end{aligned}$$

As G is symmetric, we have

$$\begin{aligned} &b_{j^*i^*} \geqslant b_{j^*i}, \ i = 1, 2, \ldots, \kappa,\\ &a_{j^*i^*} \geqslant a_{ji^*}, \ j = 1, 2, \ldots, \kappa, \end{aligned}$$

which implies that (j^*, i^*) is a pure Nash equilibrium of G.

Using Proposition 12, we have (i^*, i^*) and (j^*, j^*) are pure Nash equilibria of $G \in \mathcal{SH}_{[2;\kappa]}$.

We provide an example to illustrate the properties of games in $\mathcal{SH}_{[2;\kappa]}$.

Example 1. (i) Assume $\kappa = 2$; then, $\dim(\mathcal{SH}_{[2;2]}) = 0$. It implies that all the two-player two-strategic symmetric games are potential game.

(ii) Assume $\kappa = 3$; then, $\dim(\mathcal{SH}_{[2;3]}) = 1$, and

$$B_{SH} = J^{(2,3)} - J^{(3,2)}.$$

Consider the game of rock-paper-scissors, whose payoff bi-matrix is shown in Table 1.

Table 1 Payoff bi-matrix of rock-paper-scissors

	Rock	Paper	Scissor
Rock	0, 0	−1, 1	1, −1
Paper	1, −1	0, 0	−1, 1
Scissor	−1, 1	1, −1	0, 0

Table 2 Payoff bi-matrix of matching pennies

	Heads	Tails
Heads	1, −1	−1, 1
Tails	−1, 1	1, −1

It is clear that

$$V_1 = [0, -1, 1, 1, 0, -1 - 1, 1, 0], \quad V_2 = [0, 1, -1, -1, 0, 1, 1, -1, 0].$$

Then, the payoff vector of G is $V_G = [V_1, V_2]$. It is convenient to verify that

$$V_G^{\mathrm{T}} = -(J^{(2,3)} - J^{(3,2)}).$$

We conclude that rock-paper-scissors game is a symmetric pure harmonic game. Moreover, all two-player three-strategic symmetric pure harmonic games are equivalent to rock-paper-scissors game with the multiplication of a common constant coefficient for all payoffs.

5.2 Properties of antisymmetric pure harmonic games

Theorem 9. Consider an antisymmetric pure harmonic game $G \in \mathcal{KH}_{[2;\kappa]}$. If (i, j) is a pure Nash equilibrium of G, then $(i, i), (j, j)$ and (j, i) are pure Nash equilibria of G. Moreover, all pure Nash equilibria are Pareto optimal.

Proof. Consider a two-player game $G \in \mathcal{KH}_{[2;\kappa]}$. Suppose the payoff matrix of the first player is $A = (a_{ij})_{\kappa\times\kappa}$. Observing the form of the basis (9), we have $a_{ij} = a_{ji}$, which implies that A is symmetric.

If (i, j) is a pure Nash equilibrium of G, then we have

$$a_{is} = 0, \ s = 1, 2, \ldots, \kappa; \quad a_{tj} = 0, \ t = 1, 2, \ldots, \kappa.$$

Note that A is symmetric, which implies

$$a_{si} = 0, \ s = 1, 2, \ldots, \kappa; \quad a_{jt} = 0, \ t = 1, 2, \ldots, \kappa.$$

Using Propositions 11 and 12, we obtain that (i, i), (j, j) and (j, i) are pure Nash equilibria of G.

Moreover, according to the definition of Pareto optimal, all pure Nash equilibria of G are Pareto optimal.

Example 2. (i) Assume $\kappa = 2$; then, $\dim(\mathcal{KH}_{[2;2]}) = 1$, and

$$B_{KH} = J^{(2,2)}.$$

Consider matching pennies game, whose payoff bi-matrix is shown in Table 2. It is convenient to verify that $V_G^{\mathrm{T}} = J^{(2,2)}$. We conclude that matching pennies game is an antisymmetric pure harmonic game, and it does not have a Nash equilibrium. Moreover, all two-player two-strategic antisymmetric pure harmonic games are equivalent to the matching pennies game with the multiplication of a common constant coefficient for all payoffs.

(ii) Assume $\kappa = 3$; then, $\dim(\mathcal{KH}_{[2;3]}) = 3$, and

$$B_{KH} = [J^{(2,2)}, J^{(2,3)} + J^{(3,2)}, J^{(3,3)}].$$

Consider an antisymmetric pure harmonic game $G \in \mathcal{KH}_{[2;3]}$. Then the payoff vector of G is of the form $V_G = B_{KH} \cdot [a, b, c]^{\mathrm{T}}$, where a, b, c are scalars. According to the form of V_G, the payoff matrix of the first player, denoted by A, is as follows:

$$A = \begin{bmatrix} a+2b+c, & -a-b, & -b-c \\ a-b, & a, & b \\ -b-c, & b, & c \end{bmatrix}.$$

Observing the form of A, one can verify that any antisymmetric pure harmonic game $G \in \mathcal{KH}_{[2;3]}$ either has only one Nash equilibrium (i, i), $i = 2, 3$, or does not have Nash equilibrium. Moreover, $(2, 2)$ is the Nash equilibrium of G if and only if $a = b = 0$, and $(3, 3)$ is the Nash equilibrium if and only if $b = c = 0$.

Before concluding this paper, we would like to discuss the possible applications. First, the representation of symmetric games can be simplified using the decomposition [14]. Second, as pointed out in [17], determining a Nash equilibrium of symmetric games with two players is still challenging because it is in general PPAD-complete. However, when we focus on special class two-player symmetric games, such as games in $\mathcal{S}_{[2;\kappa]}$ and $\mathcal{KH}_{[2;3]}$, we can reduce the computational problem according to Theorems 8 and 9. Third, two-player symmetric games play a key role in networked evolutionary games [15]. Different classes of two-player symmetric games may lead to different evolution results.

6 Conclusion

This study investigate the orthogonal decompositions of finite games. First, the bases of symmetric games and their orthogonal complement are obtained. Then, the potential-based decompositions of two-player symmetric/antisymmetric games are studied. The bases of these dual decomposed subspaces are obtained. Lastly, certain properties of pure harmonic subspaces of two-player symmetric/antisymmetric games are studied. We also find that all two-player three-strategic symmetric pure harmonic games are equivalent to rock-paper-scissors, and all two-player two-strategic antisymmetric pure harmonic games are equivalent to matching pennies game. Moreover, any pure Nash equilibrium of a game that belongs to the pure harmonic subspace of two-player antisymmetric games is Pareto efficient. We will study the potential-based decompositions of symmetric games in further work.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 61473099, 61273013, 61333001).

References

1 Nowak M A, May R M. Evolutionary games and spatial chaos. Nature, 1992, 359: 826–829
2 Tembine H, Altman E, El-Azouzi R, et al. Evolutionary games in wireless networks. IEEE Trans Syst Man Cybern B, 2010, 40: 634–646
3 Santos F C, Santos M D, Pacheco J M. Social diversity promotes the emergence of cooperation in public goods games. Nature, 2008, 454: 213–216
4 Heikkinen T. A potential game approach to distributed power control and scheduling. Comput Netw, 2006, 50: 2295–2311
5 Marden J R, Arslan G, Shamma J S. Cooperative control and potential games. IEEE Trans Syst Man Cybern B, 2009, 39: 1393–1407
6 Niyato D, Hossain E. A noncooperative game-theoretic framework for radio resource management in 4G heterogeneous wireless access networks. IEEE Trans Mobile Comput, 2008, 7: 332–345
7 Arslan G, Marden J R, Shamma J S. Autonomous vehicle-target assignment: a game-theoretical formulation. J Dyn Syst Meas Control, 2007, 129: 584–596
8 Candogan O, Menache I, Ozdaglar A, et al. Flows and decompositions of games: harmonic and potential games. Math Oper Res, 2011, 36: 474–503
9 Kalai A T, Kalai E. Cooperation and competition in strategic games with private information. In: Proceedings of the 11th ACM Conference on Electronic Commerce, Cambridge, 2010. 345–346
10 Hwang S H, Rey-Bellet L. Decompositions of two player games: potential, zero-sum, and stable games. 2011. ArXiv:1106.3552

11 Wang Y H, Liu T, Cheng D Z. From weighted potential game to weighted harmonic game. IET Control Theory Appl, 2017, 11: 2161–2169
12 Cheng D Z, Liu T, Zhang K Z, et al. On decomposed subspaces of finite games. IEEE Trans Autom Control, 2016, 61: 3651–3656
13 Cheng D Z. On finite potential games. Automatica, 2014, 50: 1793–1801
14 Cheng D Z, Liu T. Linear representation of symmetric games. IET Control Theory Appl, 2017, 11: 3278–3287
15 Cheng D Z, He F H, Qi H S, et al. Modeling, analysis and control of networked evolutionary games. IEEE Trans Autom Control, 2015, 60: 2402–2415
16 Nash J. Non-cooperative games. Ann Math, 1951, 54: 286–295
17 Cao Z G, Yang X G. Symmetric games revisited. 2016. https://ssrn.com/abstract=2637225
18 Cheng D Z, Qi H S, Zhao Y. An Introduction to Semi-Tensor Product of Matrices and Its Applications. Singapore: World Scientific, 2012
19 Wang Y H, Liu T, Cheng D Z. Some notes on semi-tensor product of matrices and swap matrix. J Syst Sci Math Sci, 2016, 36: 1367–1375
20 Webb J N. Game Theory. Berlin: Springer, 2007
21 Li C X, Liu T, He F H, et al. On finite harmonic games. In: Proceedings of the 55th Conference on Decision and Control, Las Vegas, 2016. 7024–7029