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On strong structural controllability and observability of linear time-varying systems: a constructive method

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Abstract In this paper, we consider the controllability and observability of generalized linear time-varying (LTV) systems whose coefficients are not exactly known. All that is known about these systems is the placement of non-zero entries in their coefficient matrices (A, B). We provide the characterizations in order to judge whether the placements can guarantee the controllability/observability of such LTV systems, regardless of the exact value of each non-zero coefficient. We also present a direct and efficient algorithm with an associated time cost of $O(n+m+\nu)$ to verify the conditions of our characterizations, where n and m denote the number of columns of A and B, respectively, and ν is number of non-zero entries in (A, B).

Keywords linear time-varying (LTV) systems, strong structural properties, controllability, observability, duality

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1 Introduction

A typical LTV system is formulated as

$$\dot{x} = A(t)x + B(t)u, \quad y = C(t)x + D(t)u, \quad x(0) = \xi, \quad t \in [0, T]. \tag{1}$$

The research on controllability and observability of LTV systems, which is originated by Kalman et al. [1], is among the classical topics of modern control theory. After Kalman's pioneering work on the controllability and observability of controlled systems, there followed other powerful results that provided much simpler criteria of controllability and observability, especially for the linear time-invariant (LTI) case, such as Rosenbrock's criterion in [2] and the famous Popov-Belevitch-Hautus (PBH) criterion in [3]. However, these criteria require the exact value of every coefficient of a linear system to check its controllability and observability. This is challenging in practice, since the acquisition of these coefficients may involve considerable work pertaining to measurement and identification. In fact, the controllability and observability of a broad class of linear systems could be determined merely by their zero patterns. This has motivated research on structural and strong structural controllability and observability.

A zero pattern formulated by quadruple structural matrices, such as $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$, stands for the placement of zero and non-zero entries in the coefficient matrices of a linear system. Its impact on

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controllability was first considered by Lin [4]. In [4], Lin introduced the structural controllability (SC) for LTI systems and characterized the structurally controllable zero patterns for LTI systems with a single input. Shields and Pearson extended Lin's work to the multi-input cases and provided Form I and II to analyze structural controllability of linear systems [5]. The corresponding graph theoretic interpretation was derived in [6], and developed for structural controllability of multi-agent systems under some communication topologies [7, 8]. Ref. [9] proposed a graph method to analyze the controllability of directed networks, and the graph method was extended to analyze distributed systems with local structure changes in [10]. The main drawback of the definition of structural controllability is that a structurally controllable pattern may have uncontrollable numerical systems as its evaluations, since the definition only requires the pattern to have at least one controllable evaluation.

Mayeda and Yamada were the first to fill in this gap by defining strong structurally controllable (SSC) systems instead. A zero pattern of linear systems is said to be SSC if every numerical system of this pattern is controllable. In [11], Mayeda and Yamada found the SSC patterns for LTI systems in a relatively direct manner using graph theoretic, compared to PBH criterion. Though Mayeda and Yamada's work seems classical, the much more general problem of characterizing SSC patterns for LTV systems had long been open until the main breakthrough was made in a series of papers by Reissig, Hartung and Svaricek (see [12–15]). In [13], Form III is defined to analyze the SSC property of LTI systems. In [15], Reissig, Hartung and Svaricek characterized SSC patterns for LTV systems in the most general sense, stating that a numerical system with these structures may have measurable maps of t as its coefficient matrices and input vector. However, their characterization did not tend to provide a direct algorithm to check the SSC property of structural LTV systems.

In this paper, we present a new characterization of SSC systems that is equivalent to that of Reissig, Hartung and Svaricek. Our method for this characterization is quite different from that in [15], and is completely constructive. The constructive method gives rise to an efficient algorithm for checking the strong structural controllability of the system. The proposed algorithm is based on the bipartite graphical representation of structural LTV systems rather than the directed graphical representation as in [15]. The algorithm is shown to be quite efficient with an associated time cost of $O(n+m+\nu)$, where n and m denote the number of columns of A and B, respectively, and ν is the number of non-zero entries in (A, B).

The characterization of structural LTV systems that are strong structurally observable (SSO) can also be considered as a dual case of the characterization of SSC systems. We refer to a modified version of the principle of duality, claiming that a pattern of structural LTV systems is SSO if and only if its dual pattern is SSC. Thus, we can deal with SSO LTV systems in the same manner as with the SSC ones.

The rest of this paper is organized as follows. In Section 2, we list some essential results on the controllability and observability of generalized LTV systems and introduce the bipartite graphical representation of structural systems. Section 3 is devoted to characterization of SSC LTV systems. In Section 4, we propose an efficient algorithm for SSC checking and provide an example of a non-SSC LTV system. In Section 5, we discuss SSO LTV systems. Finally, in Section 6, we conclude this paper and propose an extended problem that continues to remain open.

2 Preliminaries

2.1 Generalized LTV systems

A generalized LTV system, which we denote by $\Sigma_{(A(\cdot),B(\cdot),C(\cdot),D(\cdot))}$, has been formulated formally in (1). To make it rigours, we assume that:

(1) The entries of $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$, $C(t) \in \mathbb{R}^{r\times n}$ and $D(t) \in \mathbb{R}^{r\times m}$ are measurable functions defined on $[0, T]$;

(2) The components of $u(t) \in \mathbb{R}^m$ are also measurable functions on $[0, T]$;

(3) The vector function $x(t)$ has a generalized 1-order derivative $\dot{x} = A(t)x + B(t)u$ for almost every $t \in [0, T]$;

(4) The components of $y(t)$ are measurable functions.

As is done for normal LTV systems, one can also define the family of transition matrices $\Phi(\cdot,\cdot)$ of $\Sigma_{(A(\cdot),B(\cdot),C(\cdot),D(\cdot))}$, to be as follows:

$$\dot{\Phi}(t,t_0)=A(t)\Phi(t,t_0),\ \Phi(t_0,t_0)=I_n,$$

where $\Phi(\cdot,\cdot)\in\mathbb{R}^{n\times n}$, t_0 is initial time, t is observing time and I_n stands for the $n\times n$ identity matrix [16, 17]. Transition matrices $\Phi(\cdot,\cdot)$ have characteristics as follows:

(1) $\Phi(t,t)=I_n,\ t\in[0,T]$;

(2) $\Phi(t_3,t_2)\Phi(t_2,t_1)=\Phi(t_3,t_1),\ \Phi(t_1,t_2)\Phi(t_2,t_1)=I_n,\ t_1,t_2,t_3\in[0,T]$;

(3) $\partial\Phi(t,\tau)/\partial t=A(t)\Phi(t,\tau),\ \partial\Phi(t,\tau)/\partial\tau=-\Phi(t,\tau)A(\tau),\ t,\tau\in[0,T]$.

A dynamical system is controllable if, with a suitable choice of input u, it can be driven from any initial state $x(t_0)$ to any desired final state $x(t_f)$ in a finite time; whereas a dynamical system is observable at time t_f if, given arbitrary input u, the initial state $x(t_0)$ can be uniquely determined from measurements of the output $y(\tau)$ over the finite interval $t_0\leqslant\tau\leqslant t_f$ (see [1, 17]).

It should be noted that since we have assumed each coefficient matrix to be merely measurable rather than square Lebesgue integrable, the classical criteria of observability and controllability which involve the Gramians, become invalid in this much more general case. Now, we refer to another well-known result stated in [18].

Proposition 1. An LTV system $\Sigma_{(A(\cdot),B(\cdot),C(\cdot),D(\cdot))}$ is controllable for $[0,t]$, if and only if $p^{\mathrm{T}}\Phi(t,\tau)B(\tau)\equiv 0$ for almost every (a.e.) $\tau\in[0,t]$ implies $p=0$, where $p\in\mathbb{R}^{n\times 1}$; while the system is observable for $[0,t]$, if and only if $C(\tau)\Phi(\tau,t)\xi\equiv 0$ for a.e. $\tau\in[0,t]$ implies $\xi=0$, where $\xi\in\mathbb{R}^{n\times 1}$.

The dual system of $\Sigma_{(A(\cdot),B(\cdot),C(\cdot),D(\cdot))}$, which we denote by $\Sigma_{(-A^{\mathrm{T}}(\cdot),C^{\mathrm{T}}(\cdot),B^{\mathrm{T}}(\cdot),D^{\mathrm{T}}(\cdot))}$, is given as

$$\begin{aligned}&\dot{x}=-A^{\mathrm{T}}(t)x+C^{\mathrm{T}}(t)u,\\&y=B^{\mathrm{T}}(t)x+D^{\mathrm{T}}(t)u,\\&x(T)=\eta.\end{aligned}$$

It is clear that the relationship of duality is mutual, that is to say, $\Sigma_{(-A^{\mathrm{T}}(\cdot),C^{\mathrm{T}}(\cdot),B^{\mathrm{T}}(\cdot),D^{\mathrm{T}}(\cdot))}$ and $\Sigma_{(A(\cdot),B(\cdot),C(\cdot),D(\cdot))}$ are dual systems of each other (see [17, 18]). Each transition matrix for $\Psi(\cdot,\cdot)$ of $\Sigma_{(-A^{\mathrm{T}}(\cdot),C^{\mathrm{T}}(\cdot),B^{\mathrm{T}}(\cdot),D^{\mathrm{T}}(\cdot))}$ corresponds to a transition matrix for $\Phi(\cdot,\cdot)$ of $\Sigma_{(A(\cdot),B(\cdot),C(\cdot),D(\cdot))}$:

$$\Psi(t,\tau)=\Phi^{\mathrm{T}}(\tau,t).$$

The principle of duality states that the controllability of an LTV system is equivalent to the observability of its dual system. We refer to this principle in order to show in Section 5 that one can consider the characterization of the SSO LTV systems to be the dual case of the characterization of the SSC LTV systems.

2.2 Strong structural controllability

A structural LTV system which we denote by $\Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})}$ represents a certain class of numerical systems with a common as well as fixed placement of zero and non-zero entries in their coefficient matrices $A(t),B(t),C(t)$ and $D(t)$. The placement is also named as a zero pattern in [14] as well as other related papers, and is given by quadruple structural matrices $(\bar{A},\bar{B},\bar{C},\bar{D})$. For example,

$$\bar{A}=\begin{bmatrix}* & 0 & * & *\\ 0 & * & 0 & 0\\ 0 & 0 & * & *\\ * & 0 & 0 & *\end{bmatrix},\quad \bar{B}=\begin{bmatrix}* & 0\\ * & *\\ 0 & *\\ * & 0\end{bmatrix},\quad \bar{C}=\begin{bmatrix}* & 0 & * & *\\ 0 & * & 0 & 0\end{bmatrix},\quad \bar{D}=\begin{bmatrix}* & 0\\ 0 & *\end{bmatrix},$$

where a "0" stands for a zero constant while a "$*$" stands for a non-zero indeterminate parameter. A numerical system $\Sigma_{(A(\cdot),B(\cdot),C(\cdot),D(\cdot))}$ is called an evaluation of $\Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})}$, if the placement of zero and non-zero entries in $A(\cdot), B(\cdot), C(\cdot)$ and $D(\cdot)$ is consistent with that in $\bar{A}, \bar{B}, \bar{C}$ and $\bar{D}$ for almost every time. $\Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})}$ has its dual structural system $\Sigma_{(\bar{A^{\mathrm{T}}},\bar{C^{\mathrm{T}}},\bar{B^{\mathrm{T}}},\bar{D^{\mathrm{T}}})}$ made up of the dual systems of its evaluations.

One can define strong structural controllability and observability for structural LTV systems as follows.

Definition 1. A structural LTV system $\Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})}$ is strong structurally controllable (SSC) in $[0,T]$, if its evaluations are all controllable in $[0,T]$ and is strong structurally observable (SSO) in $[0,T]$ if its evaluations are all observable in $[0,T]$.

Actually, the strong structural controllability and observability can be defined in a more general sense where they are independent of the given time interval.

Proposition 2. If $\Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})}$ is SSC for some $[0,T],\ T>0$, then it is SSC for any $[0,t],\ t>0$.

Proof. We first assume the contrary that there is a $\Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})}$ that is SSC for $[0,T]$ but is not SSC for another $[0,t]$. Consider some evaluation $\Sigma_{(A(\cdot),B(\cdot),C(\cdot),D(\cdot))}$ in $[0,t]$ such that there exists a non-zero vector p satisfying

$$p^{\mathrm{T}}\Phi(t,\tau)B(\tau)\equiv 0,$$

for a.e. $\tau\in[0,t]$. Then, consider the system $\Sigma_{(\tilde{A}(\cdot),\tilde{B}(\cdot),\tilde{C}(\cdot),\tilde{D}(\cdot))}$ in $[0,T]$, where

$$\begin{aligned}&\tilde{A}(\tau)=\lambda A(\lambda\tau), \quad \tilde{B}(\tau)=\lambda B(\lambda\tau),\\ &\tilde{C}(\tau)=C(\lambda\tau), \quad \tilde{D}(\tau)=D(\lambda\tau), \qquad \lambda=t/T,\end{aligned}$$

then its transition matrix $\tilde{\Phi}(T,\tau)=\Phi(t,\lambda\tau)$. Thus,

$$p^{\mathrm{T}}\tilde{\Phi}(T,\tau)\tilde{B}(\tau)=\lambda p^{\mathrm{T}}\Phi(t,\lambda\tau)B(\lambda\tau). \tag{2}$$

With (2), we have $p^{\mathrm{T}}\tilde{\Phi}(T,\tau)\tilde{B}(\tau)\equiv 0$ for a.e. $\tau\in[0,T]$. Given that $p\neq 0$, we have $\Sigma_{(\tilde{A}(\cdot),\tilde{B}(\cdot),\tilde{C}(\cdot),\tilde{D}(\cdot))}$ is not controllable in $[0,T]$, which leads to a contradiction.

With arguments that are analogous to those that we have made above to prove Proposition 2, one can readily see that the following proposition.

Proposition 3. If $\Sigma_{(-\bar{A}^{\mathrm{T}},\bar{C^{\mathrm{T}}},\bar{B^{\mathrm{T}}},\bar{D^{\mathrm{T}}})}$ is SSO for some $[0,T],\ T>0$, then it is SSO for any $[0,t]$, where t can be any positive real number.

We now introduce a typical form of structural matrix pair $(\bar{A},\bar{B})$ that is closely related to the SSC property of structural LTV systems.

Definition 2. Structural system $(\bar{A},\bar{B})$ from $\Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})}$ is said to be of Form IV if there exists a permutation matrix $P\in\mathbb{R}^{n\times n}$ such that

$$P\begin{bmatrix}\bar{A} & \bar{B}\end{bmatrix}\begin{bmatrix}P^{\mathrm{T}} & \\ & I_m\end{bmatrix}=\begin{array}{c}\begin{bmatrix}\bar{Q} & \bar{R}\\ \bar{S} & \bar{T}\end{bmatrix}\begin{array}{l}\}n-s\\ \}s\end{array}\\ \underbrace{\quad}_{n-s}\ \underbrace{\quad}_{s+m}\qquad\quad\end{array}, \tag{3}$$

where $0\leqslant s\leqslant(n-1)$ and $\bar{R}$ has no column with exactly one "$*$".

The main result in Section 3 is that such a form characterizes each non-SSC structure.

2.3 Bipartite graphical representation of structural systems

As we will subsequently show, a structural system $\Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})}$ can be represented as a bipartite undirected graph $\mathcal{G}=(U,V,E)$. Take $G=[\bar{A}\ \bar{B}]$. Then by definition, its columns and rows can be considered as column vertices in $V=\{v_1,v_2,\ldots,v_{n+m}\}$ and row vertices in $U=\{u_1,u_2,\ldots,u_n\}$. Its non-zero entries, such as $G_{ij}=*$, can be considered as undirected edges in E connecting u_i and v_j. For $\mathcal{G}$, each vertex x in $U\cup V$ has its neighborhood $N(x)$ made up of the vertices that connect to it, i.e.,

$$N(x)=\{y\mid\exists\text{ an edge }e\in E\text{ between }x\text{ and }y\},$$

and a neighbor of x stands for an element in $N(x)$. The degree of x, which we denote by $|N(x)|$, stands for the total number of its neighbors. One should note that because $\mathcal{G}$ is bipartite, the vertices in U can only have vertices in V as their neighbors and vice versa. We refer to this representation of structural systems in order to present an efficient algorithm for verifying the conditions of our characterization in Section 4.

3 Characterization of SSC systems

This section is devoted to the characterization of strong structural controllability of LTV systems based on Form IV. Firstly define an index set

$$M = \left\{ j | p^{\mathrm{T}} \Phi(T,t) B(t) \equiv 0 \text{ for a.e. } t \in [0,T] \text{ implies } p_j = 0 \text{ for } \forall [0,T] \text{ and } \forall (A(t),B(t)) \in \sum_{(\bar{A},\bar{B},\bar{C},\bar{D})} \right\}, \tag{4}$$

which is to be shown as the key feature of $\Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})}$ pertaining to its strong structural controllability.

For an arbitrary non-SSC $\Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})}$, one can consider its index set M as defined in (4). Then, it follows that

$$M \subsetneqq \{1, 2, \ldots, n\},$$

because $\Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})}$ being non-SSC indicates that its evaluations are not always controllable in intervals such as $[0,T]$. Thus, $p^{\mathrm{T}}\Phi(t,\tau)B(\tau) \equiv 0$ for a.e. $\tau \in [0,t]$ does not always imply $p = 0$ for any evaluation of $\Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})}$ in $[0,T]$. Furthermore, we have the following lemma.

Lemma 1. If $j \in M$, then $p^{\mathrm{T}}\Phi(t,\tau)B(\tau) \equiv 0$ for a.e. $\tau \in [0,t]$ can always imply $(\Phi^{\mathrm{T}}(t,\tau)p)_j \equiv 0$ for any $[A(\tau)\ B(\tau)] \in [\bar{A}\ \bar{B}]$ defined on any $[0,t]$.

Proof. Given arbitrary $[A(\tau)\ B(\tau)] \in [\bar{A}\ \bar{B}]$ in $[0,t]$, consider a vector p such that $p^{\mathrm{T}}\Phi(t,\tau)B(\tau) \equiv 0$ for a.e. $\tau \in [0,t]$. Notice that $p^{\mathrm{T}}\Phi(t,\tau)B(\tau) \equiv 0$ for a.e. $\tau \in [0,t]$ implies

$$p^{\mathrm{T}}\Phi(t,\tau)\Phi(\tau,\varepsilon)B(\varepsilon) \equiv 0, \tag{5}$$

i.e.,

$$(\Phi(t,\tau)p)^{\mathrm{T}}\Phi(\tau,\varepsilon)B(\varepsilon) \equiv 0,$$

for a.e. (ε,τ) with $0 \leqslant \varepsilon \leqslant \tau \leqslant t$.

For each $\tau \in [0,t]$, consider $[A(\varepsilon)\ B(\varepsilon)] \in [\bar{A}\ \bar{B}]$ with $\varepsilon \in [0,\tau]$ and take $\Phi^{\mathrm{T}}(t,\tau)p$ as p_τ, and then

$$(\Phi^{\mathrm{T}}(t,\tau)p)_j = (p_\tau)_j = 0. \tag{6}$$

Thus, we have $(\Phi^{\mathrm{T}}(t,\tau)p)_j \equiv 0$ for each $\tau \in [0,t]$.

Lemma 2. Given

$$\begin{bmatrix} \bar{A} & \bar{B} \end{bmatrix} = \begin{array}{c} \begin{bmatrix} \bar{Q} & \bar{R} \\ \bar{S} & \bar{T} \end{bmatrix} \begin{array}{l} \}n-s \\ \}s \end{array} \\ \begin{array}{cc} \underbrace{\quad}_{n-s} & \underbrace{\quad}_{s+m} \end{array} \end{array},$$

and the corresponding index set M, we have

(1) If $\bar{R}$ has no column with exactly one "$*$", then $i \notin M$ for $i = 1, \ldots, n-s$;

(2) If $(n+1-i) \in M$ for $i = 1, \ldots, s$ and $\bar{R}$ has a column $R_{(\cdot)l}$ with exactly one "$*$" as $R_{(n-s)l}$, then $(n-s) \in M$.

Proof. (1) For convenience, we additionally take blocks of $\bar{R}$ and $\bar{T}$ as

$$\begin{bmatrix} \bar{R} \\ \bar{T} \end{bmatrix} = \begin{array}{c} \begin{bmatrix} \bar{R}_1 & \bar{R}_2 \\ \bar{T}_1 & \bar{T}_2 \end{bmatrix} \\ \begin{array}{cc} \underbrace{\quad}_{s} & \underbrace{\quad}_{m} \end{array} \end{array}.$$

Then, we get

$$\begin{bmatrix}\bar{Q} & \bar{R}_1\\ \bar{S} & \bar{T}_1\end{bmatrix} = \bar{A},\ \begin{bmatrix}\bar{R}_2\\ \bar{T}_2\end{bmatrix} = \bar{B}.$$

We now construct an evaluation of $\Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})}$ that is not controllable for some $[0,T]$. Consider the $(n-s)$-dimensional linear system,

$$\dot{r}(t) = -Q^{\mathrm{T}}(t)r,\quad r(T) = 1_{n-s},$$

where $Q(t) \in \bar{Q}$ for a.e. $t \in [0,T]$ and 1_{n-s} stands for the $(n-s)\times 1$ column vector with all elements to be 1. Choose the absolute value of each entry in $Q^{\mathrm{T}}(t)$ to be sufficiently small, so that

$$r_i(t) \neq 0 \text{ for } \forall\, t \in [0,T],\ i = 1,2,\ldots,n-s.$$

Because $\bar{R}$ has no column with exactly one "$*$", one can take $R(T) \in \bar{R}$ such that $R^{\mathrm{T}}(T)r(T) = 0_{s+m}$. If we choose $(R^{\mathrm{T}})_{ij}(t) = (R^{\mathrm{T}})_{ij}(T)r_j(T)/r_j(t)$, then $R^{\mathrm{T}}(t)r(t) = R^{\mathrm{T}}(T)r(T) = 0_{s+m}$. Considering that $R(t) = [\bar{R}_1(t)\ \bar{R}_2(t)]$, $R_1^{\mathrm{T}}(t)r(t) = 0_s$ and $R_2^{\mathrm{T}}(t)r(t) = 0_m$.

Taking $p(t) = \left[\begin{smallmatrix} r(t)\\ 0_s\end{smallmatrix}\right]$ and $q(t) = R_2^{\mathrm{T}}(t)r(t)$,

$$\dot{p}(t) = \begin{bmatrix}-Q^{\mathrm{T}}(t)r(t)\\ 0\end{bmatrix} = \begin{bmatrix}-Q^{\mathrm{T}}(t)r(t)\\ -R_1^{\mathrm{T}}(t)r(t)\end{bmatrix}, \tag{7}$$

and

$$q(t) = R_2^{\mathrm{T}}(t)r(t) = 0_m. \tag{8}$$

By combining (7) with (8), we have

$$\begin{bmatrix}\dot{p}(t)\\ 0_m\end{bmatrix} = \begin{bmatrix}-Q^{\mathrm{T}}(t)r(t)\\ -R_1^{\mathrm{T}}(t)r(t)\\ R_2^{\mathrm{T}}(t)r(t)\end{bmatrix}.$$

Then, we can take $S(t) \in \bar{S}$ and $T(t) \in \bar{T}$ arbitrarily for a.e. $t \in [0,T]$ and make sure that both $S(t) \in \bar{S}$ and $T(t) \in \bar{T}$ are measurable functions on the interval. Now, we have

$$\begin{bmatrix}\dot{p}(t)\\ 0_m\end{bmatrix} = \begin{bmatrix}-Q(t) & -S^{\mathrm{T}}(t)\\ -R_1^{\mathrm{T}}(t) & -T_1^{\mathrm{T}}(t)\\ R_2^{\mathrm{T}}(t) & T_2^{\mathrm{T}}(t)\end{bmatrix}\begin{bmatrix}r(t)\\ 0_s\end{bmatrix} = \begin{bmatrix}-A^{\mathrm{T}}(t)\\ B^{\mathrm{T}}(t)\end{bmatrix}p(t), \tag{9}$$

where $[A(t)\ B(t)] \in [\bar{A}\ \bar{B}]$ for a.e. $t \in [0,T]$. It follows from (9) that

$$\dot{p}(t) = -A^{\mathrm{T}}(t)p(t).$$

According to Subsection 2.1, we have $\dot{\Phi}(t,T) = A(t)\Phi(t,T)$, $\Phi(T,T) = I_n$ and $\Phi(t,T)\Phi(T,t) = I_n$. Then

$$\begin{aligned}
&\dot{p}(t) = -(\dot{\Phi}(t,T)/\Phi(t,T))^{\mathrm{T}}p(t),\\
&\int_t^{\mathrm{T}} \dot{p}(\tau)/p(\tau)\mathrm{d}\tau = -\left(\int_t^{\mathrm{T}} \dot{\Phi}(\tau,T)/\Phi(\tau,T)\mathrm{d}\tau\right)^{\mathrm{T}},\\
&\ln p(\tau)|_t^{\mathrm{T}} = -(\ln \Phi(\tau,T)|_t^{\mathrm{T}})^{\mathrm{T}},\\
&\ln(p(T)/p(t)) = (-\ln(\Phi(T,T)/\Phi(t,T)))^{\mathrm{T}},\\
&p(T)/p(t) = \Phi^{\mathrm{T}}(t,T).
\end{aligned}$$

So

$$p(t) = \Phi^{\mathrm{T}}(T,t)p(T),$$

and

$$p^{\mathrm{T}}(t)B(t) = p^{\mathrm{T}}(T)\Phi(T,t)B(t) = 0_m^{\mathrm{T}}.$$

Because $p(T) = \left[\begin{smallmatrix} r(T) \\ 0_s \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1_{n-s} \\ 0_s \end{smallmatrix}\right]$, there exist $[A(t)\ B(t)] \in [\bar{A}\ \bar{B}]$ for every $t \in [0,T]$ and $p = \left[\begin{smallmatrix} 1_{n-s} \\ 0_s \end{smallmatrix}\right]$ such that $p^{\mathrm{T}}\Phi(T,t)B(t) \equiv 0$. Thus (1) is proved.

Now, we prove (2). Take $[A(t)\ B(t)] \in [\bar{A}\ \bar{B}]$ arbitrarily for a.e. $t \in [0,T]$ and consider each vector p such that $p^{\mathrm{T}}\Phi(T,t)B(t) \equiv 0$ for a.e. $t \in [0,T]$. For convenience, take

$$p(t) = \Phi^{\mathrm{T}}(T,t)p, \quad q(t) = B^{\mathrm{T}}(t)p(t).$$

Then

$$\dot{p}(t) = -A^{\mathrm{T}}(t)p(t), \quad q(t) = B^{\mathrm{T}}(t)p(t), \quad p(T) = p. \tag{10}$$

Given that $(n+1-i) \in M$ for $i = 1,\ldots,s$, we have $p = \left[\begin{smallmatrix} r \\ 0_s \end{smallmatrix}\right]$ for p satisfying $q^{\mathrm{T}}(t) \triangleq p^{\mathrm{T}}\Phi(T,t)B(t) \equiv 0$ for a.e. $t \in [0,T]$. It also follows from Lemma 1 that $p(t) = \left[\begin{smallmatrix} r(t) \\ 0_s \end{smallmatrix}\right]$. By combining this with (10), we get

$$\begin{bmatrix} \dot{r}(t) \\ 0_{s+m} \end{bmatrix} = \begin{bmatrix} -Q(t) & -S^{\mathrm{T}}(t) \\ -R_1^{\mathrm{T}}(t) & -T_1^{\mathrm{T}}(t) \\ R_2^{\mathrm{T}}(t) & T_2^{\mathrm{T}}(t) \end{bmatrix} \begin{bmatrix} r(t) \\ 0_s \end{bmatrix}.$$

Thus, $0_{s+m} = R^{\mathrm{T}}(t)r(t)$ for a.e. $t \in [0,T]$. Given that $R_{(n-s)l}$ is the only component of $R_{(\cdot)l}$ that equals to "$*$", we have $(R^{\mathrm{T}})_{l(n-s)}(t)r_{(n-s)}(t) = 0$ for a.e. $t \in [0,T]$ and thus $r_{(n-s)}(t) = 0$ for a.e. $t \in [0,T]$. Furthermore, it follows from the absolute continuity of $r(t)$ that $r_{(n-s)}(t) \equiv 0,\ t \in [0,T]$ and thus $(n-s) \in M$.

Lemma 2 is the key lemma to prove the lemma that follows.

Lemma 3. If $\Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})}$ is non-SSC, then $(\bar{A},\bar{B})$ is of Form IV.

Proof. For $\Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})}$, consider a permutation Λ such that,

$$\Lambda(M_i) = n+1-i,\ i = 1,2,\ldots,s,$$

where $M = \{M_1,\ldots,M_s\}$. There is a permutation matrix P_Λ corresponding to Λ such that,

$$(P_\Lambda p)_{n+1-i} = p_{M_i},\ i = 1,2,\ldots,s,$$

which maps the M_i-th component of p to the $(n+1-i)$-th component of $P_\Lambda p$. Take the equivalent transform

$$P_\Lambda \begin{bmatrix} \bar{A} & \bar{B} \end{bmatrix} \begin{bmatrix} P_\Lambda^{\mathrm{T}} & \\ & I_m \end{bmatrix} = \begin{array}{c} \begin{bmatrix} \bar{Q}_\Lambda & \bar{R}_\Lambda \\ \bar{S}_\Lambda & \bar{T}_\Lambda \end{bmatrix} \begin{matrix} \}n-s \\ \}s \end{matrix} \\ \begin{matrix} \underbrace{\hphantom{\bar{Q}_\Lambda}}_{n-s} & \underbrace{\hphantom{\bar{T}_\Lambda}}_{s+m} \end{matrix} \end{array}.$$

Now we are going to show that $\bar{R}_\Lambda$ cannot have a column with exactly one "$*$".

Take

$$\begin{bmatrix} \bar{\alpha} & \bar{\beta} \end{bmatrix} = \begin{array}{c} \begin{bmatrix} \bar{Q}_\Lambda & \bar{R}_\Lambda \\ \bar{S}_\Lambda & \bar{T}_\Lambda \end{bmatrix} \begin{matrix} \}n-s \\ \}s \end{matrix} \\ \begin{matrix} \underbrace{\hphantom{\bar{Q}_\Lambda}}_{n-s} & \underbrace{\hphantom{\bar{T}_\Lambda}}_{s+m} \end{matrix} \end{array},$$

where $\bar{\alpha} \in \mathbb{R}^{n\times n}$ and $\bar{\beta} \in \mathbb{R}^{n\times m}$. Consider $[\bar{\alpha}\ \bar{\beta}]$ as $[\bar{A}\ \bar{B}]$ in Lemma 2. Assume the contrary that the l-th column of $\bar{R}_\Lambda$, which we denote by $(\bar{R}_\Lambda)_{(\cdot)l}$, contains exactly one "$*$". Without loss of generality, one can choose the free rows of the permutation matrix P_Λ such that $(\bar{R}_\Lambda)_{(n-s)l}$ is the "$*$" indeterminate. Now, consider the $\tilde{M}$ corresponding to $[\bar{\alpha}\ \bar{\beta}]$, we have $(n+1-i) \in \tilde{M},\ i = 1,\ldots,s$, since $\Lambda^{-1}(n+1-i) = M_i \in M$.

Given that $(\bar{R}_\Lambda)_{(n-s)l}$ is the only "$*$" in $(\bar{R}_\Lambda)_{(\cdot)l}$, it follows from Lemma 2 that $(n-s) \in \tilde{M}$. Thus $\Lambda^{-1}(n-s) \in M$, while we have $\Lambda^{-1}(n-s) \notin M$, since $\Lambda^{-1}(n-s) \neq M_i,\ i = 1, \ldots, s$. This leads to a contradiction.

From the above, we get the following theorem.

Theorem 1. A structural LTV system $\Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})}$ is SSC if and only if $(\bar{A}, \bar{B})$ is not of Form IV.

We compare our statement with Theorem III.5 in [15], which can be interpreted in the words of structural matrices as Theorem 2.

Theorem 2. A structural LTV system $\Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})}$ is SSC if and only if, for any permutation matrix P and any integer $0 \leqslant s \leqslant (n-1)$, one can consider the blocks as in (3) and there is always some column in $\bar{R}$ with exactly one "$*$".

It follows immediately from the notions we made for Form IV, that our statement is equivalent to that of Reissig, Hartung and Svaricek. Now we are going to prove Theorem 1 in a purely constructive way.

Proof. (The proof of Theorem 1) We first consider the "if" part, claiming that each $\Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})}$ with $(\bar{A}, \bar{B})$ not of Form IV should be SSC. Since we have proved Lemma 3 which has an equivalent statement, "if" part is proved and omitted.

Now, we have to prove the "only if" part. Assume the contrary that there exists a structural LTV system $\Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})}$ that is of both Form IV and SSC. Given that $\Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})}$ is of Form IV, there should be a permutation matrix P satisfying (3). When $P = I_n$, the following arguments are equivalent to those in proof of the statement (1) in Lemma 2. Similarly, consider blocks of $\bar{R}$ and $\bar{T}$ as

$$\begin{bmatrix} \bar{R} \\ \bar{T} \end{bmatrix} = \begin{bmatrix} \bar{R}_1 & \bar{R}_2 \\ \bar{T}_1 & \bar{T}_2 \end{bmatrix}.$$
$$\qquad\qquad \underbrace{\qquad}_{s} \ \underbrace{\qquad}_{m}$$

By combining this with (3), we get

$$\begin{bmatrix} \bar{Q} & \bar{R}_1 \\ \bar{S} & \bar{T}_1 \end{bmatrix} = P\bar{A}P^{\mathrm{T}}, \quad \begin{bmatrix} \bar{R}_2 \\ \bar{T}_2 \end{bmatrix} = P\bar{B}.$$

We now construct an evaluation of $\Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})}$ that is not controllable for some $[0, T]$. Consider the $(n-s)$-dimensional linear system

$$\dot{r}(t) = -Q^{\mathrm{T}}(t)r, \quad r(T) = 1_{n-s},$$

where $Q(t) \in \bar{Q}$ for a.e. $t \in [0, T]$. Choose the absolute value of each entry in $Q^{\mathrm{T}}(t)$ to be sufficiently small, and then

$$r_i(t) \neq 0 \text{ for } \forall\ t \in [0, T],\ i = 1, 2, \ldots, n-s.$$

Because $\bar{R}$ has no column with exactly one "$*$", one can take $R(T) \in \bar{R}$ such that $R^{\mathrm{T}}(T)r(T) = 0_{s+m}$. If we choose $(R^{\mathrm{T}})_{ij}(t) = (R^{\mathrm{T}})_{ij}(T)r_j(T)/r_j(t)$, then $R^{\mathrm{T}}(t)r(t) = R^{\mathrm{T}}(T)r(T) = 0_{s+m}$. Considering the blocks of $R(t)$, $R_1^{\mathrm{T}}(t)r(t) = 0_s$ and $R_2^{\mathrm{T}}(t)r(t) = 0_m$.

Consider $p(t) = P^{\mathrm{T}}\left[\begin{smallmatrix} r(t) \\ 0_s \end{smallmatrix}\right]$ and $q(t) = R_2^{\mathrm{T}}(t)r(t)$, so we have

$$P\dot{p}(t) = \begin{bmatrix} -Q^{\mathrm{T}}(t)r(t) \\ 0 \end{bmatrix} = \begin{bmatrix} -Q^{\mathrm{T}}(t)r(t) \\ -R_1^{\mathrm{T}}(t)r(t) \end{bmatrix}, \tag{11}$$

and

$$q(t) = R_2^{\mathrm{T}}(t)r(t) = 0_m. \tag{12}$$

By combining (11) with (12), we have

$$\begin{bmatrix} P\dot{p}(t) \\ 0_m \end{bmatrix} = \begin{bmatrix} -Q^{\mathrm{T}}(t)r(t) \\ -R_1^{\mathrm{T}}(t)r(t) \\ R_2^{\mathrm{T}}(t)r(t) \end{bmatrix}.$$

Then, choose $S(t) \in \bar{S}$ and $T(t) \in \bar{T}$ arbitrarily for a.e. $t \in [0, T]$ and make sure that both $S(t) \in \bar{S}$ and $T(t) \in \bar{T}$ are measurable functions on the interval. Now, we have

$$\begin{bmatrix} P\dot{p}(t) \\ 0_m \end{bmatrix} = \begin{bmatrix} -Q(t) & -S^{\mathrm{T}}(t) \\ -R_1^{\mathrm{T}}(t) & -T_1^{\mathrm{T}}(t) \\ R_2^{\mathrm{T}}(t) & T_2^{\mathrm{T}}(t) \end{bmatrix} \begin{bmatrix} r(t) \\ 0_s \end{bmatrix} = \begin{bmatrix} P & \\ & I_m \end{bmatrix} \begin{bmatrix} -A^{\mathrm{T}}(t) \\ B^{\mathrm{T}}(t) \end{bmatrix} P^{\mathrm{T}} P p(t) = \begin{bmatrix} -PA^{\mathrm{T}}(t)p(t) \\ B^{\mathrm{T}}(t)p(t) \end{bmatrix}, \tag{13}$$

where $[A(t)\ B(t)] \in [\bar{A}\ \bar{B}]$ for a.e. $t \in [0, T]$. It follows from (13) that $\dot{p}(t) = -A^{\mathrm{T}}(t)p(t)$. So $p(t) = \Phi^{\mathrm{T}}(T, t)p(T)$ and $p^{\mathrm{T}}(t)B(t) = p^{\mathrm{T}}(T)\Phi(T, t)B(t) = 0_m^{\mathrm{T}}$.

Because $p(T) = P^{\mathrm{T}}\left[\begin{smallmatrix} r(T) \\ 0_s \end{smallmatrix}\right] = P^{\mathrm{T}}\left[\begin{smallmatrix} 1_{n-s} \\ 0_s \end{smallmatrix}\right]$, there exists a non-zero $p(T)$ such that $p^{\mathrm{T}}(T)\Phi(T, t)B(t) = 0_m^{\mathrm{T}}$ for $t \in [0, T]$. Thus, $\dot{x} = A(t)x + B(t)u$ is not controllable for $[0, T]$, which implies that $\Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})}$ is not SSC, thus leading to a contradiction. Therefore, Theorem 1 is finally proved.

4 An efficient algorithm for SSC examination

As shown in the preceding sections, given $\Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})}$, the corresponding index set M acts as the key feature reflecting the system's strong structural controllability. Now we develop an efficient algorithm to determine M for $\Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})}$ which can be used to examine its SSC property. The algorithm is described using the bipartite graphical representation $\mathcal{G} = (U, V, E)$ as shown in Section 2, to make the algorithm intuitive.

4.1 The effectiveness of Algorithm 1

We are now going to prove the effectiveness of Algorithm 1. First, we need to define some necessary terms. Consider $\tilde{U}_k$, W_k and $\tilde{W}_k$ as the $\tilde{U}$, W and $\tilde{W}$ in Algorithm 1 when $t = k$. When the execution of Algorithm 1 terminates, we consider $t = K$ as the final value of t. There exist $(K-1)$ vertices that were deleted in sequence from U. If $K > 1$, the indices of each u_{j_t} for $t = 1, \ldots, K-1$ will make up an index set E_K. If $K = 1$, we just take $E_1 = \emptyset$.

The effectiveness of Algorithm 1 lies in the following equation:

$$M = E_K, \tag{14}$$

given $\Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})}$. Thus, the main goal of this subsection is to prove (14). We further remark that (14) also implies that one can choose each u_{j_t} arbitrarily when there are multiple alternatives, since it makes no difference to E_K and the returned result of Algorithm 1. That is to say, the final result of M remains unchanged, regardless of how we obtain each element of M in sequence according to the algorithm.

The case where $K = 1$ is obvious, since $E_1 = \emptyset$. On the other hand, if $K > 1$, we should consider a permutation matrix P_K such that

$$(\tilde{p})_{n+1-t} = (P_K p)_{n+1-t} = p_{j_t},\ t = 1, 2, \ldots, K-1, \tag{15}$$

and the equivalent transform

$$\tilde{G} = \left[P_K \bar{A} {P_K}^{\mathrm{T}}\ P_K \bar{B}\right] = [\bar{\alpha}\ \bar{\beta}]. \tag{16}$$

For each $t = 1, 2, \ldots, K-1$, consider blocks of $\tilde{G} = [\bar{\alpha}\ \bar{\beta}]$ to be

$$\tilde{G} = \begin{array}{c} \begin{bmatrix} \bar{\alpha}_{11}^t & \bar{\alpha}_{12}^t & \bar{\beta}_1^t \\ \bar{\alpha}_{21}^t & \bar{\alpha}_{22}^t & \bar{\beta}_2^t \end{bmatrix} \begin{array}{l} \}n-t+1 \\ \}t-1 \end{array} \\ \begin{array}{ccc} \underbrace{\quad}_{n-t+1} & \underbrace{\quad}_{t-1} & \underbrace{\quad}_{m} \end{array} \end{array}. \tag{17}$$

Then, we have the following lemma characterizing a column in each $R^t = [\bar{\alpha}_{12}^t\ \bar{\beta}_1^t]$, $t = 1, 2, \ldots, K-1$.

Algorithm 1 SSC checking for structural LTV system

Input
$U=\{u_1,\ldots,u_n\}, V=\{v_1,\ldots,v_{n+m}\}$,
$N(u_1),\ldots,N(u_n),N(v_1),\ldots,N(v_{n+m})$,
Initialization
$t=1$,
$W=\{v_{n+1},\ldots,v_{n+m}\}$,
$\tilde{W}=W$,
$\tilde{U}=U$,
for $x\in U\cup V$ **do**
    $\tilde{N}(x)=N(x)$,
end for
Main Algorithm
while $\tilde{U}\neq\emptyset$ **do**
    $x=$ Null,
    for each $w\in W$ **do**
        if $|\tilde{N}(w)|=1$ with $\tilde{N}(w)=u_{i_t}$ **then**
            $x=w$,
            $y=u_{i_t}$,
            Break,
        end if
    end for
    if $x=\emptyset$ **then**
        Return "False",
        Break,
    end if
    $t=t+1$,
    for each $z\in\tilde{N}(y)$ **do**
        $\tilde{N}(z)=\tilde{N}(z)-\{y\}$,
    end for
    $\tilde{U}=\tilde{U}-\{y\}$,
    $W=W\cup\{y\}-\{x\}$,
    $\tilde{W}=\tilde{W}\cup\{y\}$,
    Delete $\tilde{N}(y)$,
end while
if $\tilde{U}=\emptyset$ **then**
    Return "True",
end if

Lemma 4. There exists a column μ_t in each R^t, $t=1,2,\ldots,K-1$ such that

$$r_t=\begin{bmatrix}0\\ *\end{bmatrix}\begin{array}{l}\}n-t\\ \}1\end{array}.$$

Proof. As $G=[\bar{A}\ \bar{B}]$, R^t can also be considered as the adjacency matrix of the subgraph $\mathcal{M}_t=(U_t,\tilde{W}_t,E(U_t,\tilde{W}_t))$ of $\mathcal{G}$. In $\mathcal{M}_t$, each vertex in U_t corresponds to a row of R^t, each vertex in $\tilde{W}_t$ corresponds to a column of M_t and each undirected edge in $E(U_t,\tilde{W}_t)$ corresponds to a non-zero entry of R^t. Given that u_{j_t} is deleted from U_t, there should be a vertex $w_t\in W_t$ with u_{j_t} as its only neighbor in U_t. Furthermore, we have $w_t\in\tilde{W}_t$ since W_t is a subset of $\tilde{W}_t$. Thus, in R^t, there exists a column with a single "$*$" in its $(n+1-t)$-th row, corresponding to u_{j_t}.

If the algorithm returns "True" when it terminates, then $\tilde{U}_K=\tilde{U}_{n+1}=\emptyset$. Thus the row vertices deleted during the execution of the algorithm make up the entire U and their indices make up $E_K=E_{n+1}=\{1,\ldots,n\}$. Given that $M=E_{n+1}$, we have $M=\{1,\ldots,n\}$, which implies $\Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})}$ should be SSC.

If the algorithm terminates when $t=K\leqslant n$ and returns "False", then $U_K\neq\emptyset$ and $U_K^C=U-U_K\subsetneqq U$. Thus $M=E_K\neq\{1,\ldots,n\}$, which indicates that there should exist a non-zero vector p for $[A(t)\ B(t)]\in[\bar{A}\ \bar{B}]$ in some $[0,T]$ such that $p^{\mathrm{T}}\Phi(T,t)B(t)\equiv 0$ for a.e. $t\in[0,T]$, i.e., the evaluation $\Sigma_{(A(\cdot),B(\cdot),C(\cdot),D(\cdot))}$ is not controllable in $[0,T]$ and thus $\Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})}$ is non-SSC. Still we can take the equivalent transform as in (16) and we also have $\tilde{G} = [\bar{\alpha}\ \bar{\beta}]$. Now, by taking blocks of $\tilde{G}$ as in (17) when $t = K$, we get

$$\tilde{G} = \begin{array}{c} \begin{bmatrix} \bar{\alpha}_{11}^K & \bar{\alpha}_{12}^K & \bar{\beta}_1^K \\ \bar{\alpha}_{21}^K & \bar{\alpha}_{22}^K & \bar{\beta}_2^K \end{bmatrix} \begin{array}{l} \}n-K+1 \\ \}K-1 \end{array} \\ \underbrace{\quad}_{n-K+1} \underbrace{\quad}_{K-1} \underbrace{\quad}_{m} \end{array}.$$

Now, we show that there is no column in the submatrix $R^K = [\bar{\alpha}_{12}^K\ \bar{\beta}_2^K]$ with exactly one "$*$". Considering vertices in $(\tilde{W}_K - W_K)$, we have the following lemma.

Lemma 5. $(\tilde{W}_t - W_t)$ has an empty intersection with U_t for $t = 1, \dots, K$.

Proof. The statement is clear for $t = 1$ since $(\tilde{W}_1 - W_1) = \emptyset$. If $t > 1$, given $v \in (\tilde{W}_t - W_t)$, it follows from Algorithm 1 that v is once added to W_{t_1} for some t_1 and then removed from W_{t_2} for other $t_2 < t$. Then, there exists a $u_{j_{t_2}} \in U_{t_2}$ as the only element in $U_{t_2} \bigcap N(v)$. Thus, v is disjoint with any vertex in $U_{t_2+1} = U_{t_2} - \{u_{j_{t_2}}\}$. Furthermore, we have $U_t \subseteq U_{t_2+1}$ since $t_2 + 1 \leqslant t$, so v should be disjoint with any vertex in U_t.

Then, we can prove (14) as follows.

Proof. (The proof of (14)) We first show that $E_K \subseteq M$. Considering $\tilde{M}$ corresponding to the pair $(\bar{\alpha}, \bar{\beta})$, we have to show that $(n - t + 1) \in \tilde{M},\ t = 1, \dots, K - 1$. For $t = 1$, we have $\bar{R}^t = \bar{\beta}$, thus the column

$$r_1 = \begin{bmatrix} 0 \\ * \end{bmatrix} \begin{array}{l} \}n-1 \\ \}1 \end{array}$$

is exactly in $\bar{\beta}$. Once we take arbitrary $(\alpha(\tau), \beta(\tau)) \in (\bar{\alpha}, \bar{\beta})$ for $\tau \in [0, T]$, then there is a column

$$r_1(\tau) = \begin{bmatrix} 0 \\ \mu(\tau) \end{bmatrix} \begin{array}{l} \}n-1 \\ \}1 \end{array}$$

in $\beta(\tau)$. Consider $\Psi(\cdot, \cdot)$ as the family of transition matrices generated by $\alpha(t)$ and consider p satisfying $p^{\mathrm{T}}\Psi(T, \tau)\beta(\tau) \equiv 0$ for a.e. $\tau \in [0, T]$. For $p^{\mathrm{T}}\Psi(T, \tau)\beta(\tau)$, it has a component $Q(\tau)$ such that

$$Q(\tau) = p^{\mathrm{T}}\Psi(T, \tau) r_1(\tau) = \left(p^{\mathrm{T}}\Psi(T, \tau)\right)_n \mu(\tau),$$

where $(p^{\mathrm{T}}\Psi(T, \tau))_n$ stands for the n-th component of $p^{\mathrm{T}}\Psi(T, \tau)$. Then $(p^{\mathrm{T}}\Psi(T, \tau))_n \mu(\tau) \equiv 0$ for a.e. $\tau \in [0, T]$ and thus $(p^{\mathrm{T}}\Psi(T, \tau))_n \equiv 0$ in $[0, T]$. Once we take $\tau = T$, then $p^{\mathrm{T}}\Psi(T, T) = p^{\mathrm{T}}$ and $(p^{\mathrm{T}}\Psi(T, T))_n = (p^{\mathrm{T}})_n = 0$. So $(n - t + 1) \in \tilde{M}$ for $t = 1$. If $K > 2$, then we have to show $(n - t + 1) \in \tilde{M}$ for $t = 1, \dots, K - 1$. We prove this by induction. If $(n - t + 1) \in \tilde{M}$ for $t = 1, \dots, s$, where $s < K - 1$, take blocks of $\tilde{G} = [\bar{\alpha}\ \bar{\beta}]$ such that

$$[\bar{\alpha}\ \bar{\beta}] = \begin{array}{c} \begin{bmatrix} \bar{Q} & \bar{R} \\ \bar{S} & \bar{T} \end{bmatrix} \begin{array}{l} \}n-s \\ \}s \end{array} \\ \underbrace{\quad}_{n-s} \underbrace{\quad}_{s+m} \end{array},$$

then there is a column

$$r_t = \begin{bmatrix} 0 \\ * \end{bmatrix} \begin{array}{l} \}n-s-1 \\ \}1 \end{array}$$

in $\bar{R}$. With Lemma 2, we have $(n - s) \in \tilde{M}$, and thus $(n - t + 1) \in \tilde{M}$ for $t = 1, \dots, s + 1$. By repeating preceding arguments, we finally obtain $(n - t + 1) \in \tilde{M}$ for $t = 1, \dots, K - 1$. Note that $[\bar{\alpha}\ \bar{\beta}]$ is obtained with the equivalent transform (16) from $[\bar{A}\ \bar{B}]$, and then for $(\bar{A}, \bar{B})$, $j_t \in M,\ t = 1, \dots, K - 1$. Thus, we have proved $E_K \subseteq M$.

Now, we need to prove $E_K \supseteq M$. Considering $t = K$, it follows from Lemma 5 that $(\tilde{W}_K - W_K)$ has an empty intersection with U_K. We see also from Algorithm 1 that when "False" is returned, there

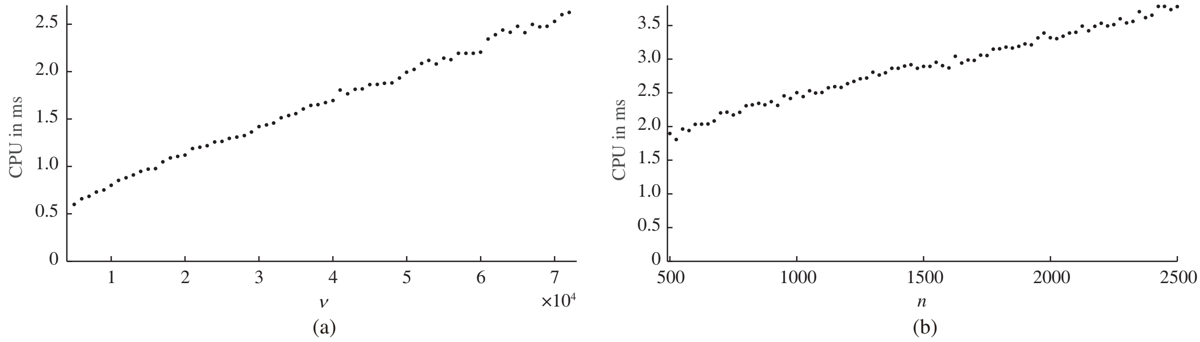

Figure 1 Run time of Algorithm 1 to verify strong structurally controllable property which depends on ν and n for randomly chosen structural matrices $(\bar{A}, \bar{B}) \in \{0, *\}^{n\times(n+m)}$ such that each LTV system $(\bar{A}, \bar{B})$ is strong structurally controllable. ν denotes the number of non-zero entries in $(\bar{A}, \bar{B})$. The underlying implementation of Algorithm 1 was executed in C programming language on a Intel Core i3-2120 (3.3 GHz). (a) $n = 1000, m = 250$; (b) $m = 500, \nu = 50000$.

should be no $v \in W_K$ with exactly one neighbor in U_K. So $\bar{R}^K$, which is the adjacency matrix of $\mathcal{M}_K = (U_K, \tilde{W}_K, E(U_K, \tilde{W}_K))$ has no column with exactly one "$*$".

Considering the index set $\tilde{M}$, it follows from Lemma 2 that $t \notin \tilde{M},\ t = 1, \dots, n-K+1$ since $\bar{R}^K$ has no column with exactly one "$*$". We see from (15) that if $j \notin E_K$, then $p_j = (P_K p)_t$ for some integer $t \in \{1, \dots, n-K+1\} = (\tilde{M})^C$. So $t \notin \tilde{M}$, which implies $j \notin M$. Now that we have shown that $j \notin E_K$ implies $j \notin M$, it is equivalent to say that $E_K \supseteq M$. Combine this with $E_K \subseteq M$, we finally obtain (14).

4.2 The time cost of Algorithm 1

We need to consider the time consumption during application of Algorithm 1 when $\Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})}$ with n state variables and m input variables is available in bipartite graphical representation described in Subsection 2.3. To make it rigorous, we take each deletion of an undirected edge of $\mathcal{G}$ as a unit operation of Algorithm 1. We denote the number of all undirected edges as ν, i.e., number of non-entries in $[\bar{A}\ \bar{B}]$.

With respect to Algorithm 1, the Initialization (lines 4–11) is executed in linear time $O(n+m)$. Since $\tilde{U}$ deletes one element each time (line 30), the 'while' loop (lines 13–34) runs no more than n times before $\tilde{U} = \emptyset$. The 'for' loop in line 15 terminates after at most m iterations. Commands contained in lines 27–29 pertain to deletion of non-zero entries and there are ν non-zero entries totally. So taking the 'for' loop in line 15 and the 'for' loop in line 27 into consideration, the 'while' loop (lines 13–34) runs with a time complexity of $O(n+m+\nu)$.

We conclude this in the theorem as follows.

Theorem 3. When applied to a structural system with n state variables, m input variables and ν non-zero entries, Algorithm 1 has a time cost of $O(n+m+\nu)$.

The linearity in ν and n of Algorithm 1 is illustrated in Figure 1. The matrices are randomly chosen on the promise that each LTV system $(\bar{A}, \bar{B})$ is strong structurally controllable.

4.3 A non-SSC example

We now present an example of a non-SSC LTV system with the structural pair $(\bar{A}, \bar{B})$ such that

$$\bar{G} = [\bar{A}\ \bar{B}] = \begin{bmatrix} * & 0 & * & * & 0 \\ 0 & * & * & * & 0 \\ * & 0 & * & 0 & * \end{bmatrix},$$

where $(\bar{A}, \bar{B})$ is taken from $\Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})}$. We initialize Algorithm 1 by taking

(1) $U_1 = \{u_1, u_2, u_3\},\ V_1 = \{v_1, v_2, v_3, v_4, v_5\},\ W_1 = \{v_4, v_5\},\ \tilde{W}_1 = \{v_4, v_5\}$;

(2) $N_1(u_1) = \{v_1, v_3, v_4\},\ N_1(u_2) = \{v_2, v_3, v_4\},\ N_1(u_3) = \{v_1, v_3, v_5\},\ N_1(v_1) = \{u_1, u_3\},\ N_1(v_2) = \{u_2\},\ N_1(v_3) = \{u_1, u_2, u_3\},\ N_1(v_4) = \{u_1, u_2\},\ N_1(v_5) = \{u_3\}$.

Then for $t = 1$, we have

(1) $u_{j_1} = u_3,\ U_2 = U_1 - \{u_3\},\ V_2 = V_1,\ W_2 = W_1 \cup \{v_3\} - \{v_5\},\ \tilde{W}_2 = \tilde{W}_1 \cup \{v_3\}$,

(2) $N_2(u_i) = N_1(u_i) - \{v_5\},\ i = 1, 2,\ N_2(v_j) = N_1(v_j) - \{u_3\},\ j = 1, 2, 3, 4, 5$,

i.e.,

(1) $U_2 = \{u_1, u_2\},\ V_2 = \{v_1, v_2, v_3, v_4, v_5\},\ W_2 = \{v_3, v_4\},\ \tilde{W}_2 = \{v_3, v_4, v_5\}$,

(2) $N_2(u_1) = \{v_1, v_3, v_4\},\ N_2(u_2) = \{v_2, v_3, v_4\},\ N_2(v_1) = \{u_1\},\ N_2(v_2) = \{u_2\},\ N_2(v_3) = \{u_1, u_2\},\ N_2(v_4) = \{u_1, u_2\},\ N_2(v_5) = \emptyset$.

Then for $t = 2$, there is no column vertex in W_2 that has exactly one neighbor since it has no one-degree elements. So Algorithm 1 should return "False" and the structural LTV system is non-SSC.

5 Results on SSO systems

In Section 2, we referred to the classical principle of duality for numerical systems, which states that an observable LTV system can be considered as the dual system of a controllable one. Actually, this principle also has its counterpart for structural LTV systems.

Corollary 1. A structural LTV system $\Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})}$ is SSO if and only if its dual structural system $\Sigma_{(\bar{A}^{\mathrm{T}},\bar{C}^{\mathrm{T}},\bar{B}^{\mathrm{T}},\bar{D}^{\mathrm{T}})}$ is SSC.

Proof. Choose $\Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})}$ and $\Sigma_{(\bar{A}^{\mathrm{T}},\bar{C}^{\mathrm{T}},\bar{B}^{\mathrm{T}},\bar{D}^{\mathrm{T}})}$ as sets of numerical systems. Then, we have the subjective between them as follows:

$$\mathcal{D} : \Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})} \leftrightarrow \Sigma_{(\bar{A}^{\mathrm{T}},\bar{C}^{\mathrm{T}},\bar{B}^{\mathrm{T}},\bar{D}^{\mathrm{T}})},$$
$$\mathcal{D}(\Sigma_{(A(\cdot),B(\cdot),C(\cdot),D(\cdot))}) = \Sigma_{(-A^{\mathrm{T}}(\cdot),C^{\mathrm{T}}(\cdot),B^{\mathrm{T}}(\cdot),D^{\mathrm{T}}(\cdot))}.$$

Choose arbitrary $\Sigma_{(A(\cdot),B(\cdot),C(\cdot),D(\cdot))} \in \Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})}$. It follows from the principle of duality that, in any $[0, T]$, the observability of $\Sigma_{(A(\cdot),B(\cdot),C(\cdot),D(\cdot))}$ is equivalent to the controllability of its map in $\Sigma_{(\bar{A}^{\mathrm{T}},\bar{C}^{\mathrm{T}},\bar{B}^{\mathrm{T}},\bar{D}^{\mathrm{T}})}$. Therefore, the SSO property of $\Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})}$ should be equivalent to the SSC property of $\Sigma_{(\bar{A}^{\mathrm{T}},\bar{C}^{\mathrm{T}},\bar{B}^{\mathrm{T}},\bar{D}^{\mathrm{T}})}$.

By combining this with Theorem 1, we have the following corollary.

Corollary 2. A structural LTV system $\Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})}$ is SSO if and only if the pair $(\bar{A}^{\mathrm{T}}, \bar{C}^{\mathrm{T}})$ is not of Form IV.

Because we have proved that the SSC property of $\Sigma_{(\bar{A}^{\mathrm{T}},\bar{C}^{\mathrm{T}},\bar{B}^{\mathrm{T}},\bar{D}^{\mathrm{T}})}$ is equivalent to the SSO property of $\Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})}$, the Algorithm 1 for SSC checking can also be used to check the SSO property of $\Sigma_{(\bar{A},\bar{B},\bar{C},\bar{D})}$ by checking the SSC property of $\Sigma_{(\bar{A}^{\mathrm{T}},\bar{C}^{\mathrm{T}},\bar{B}^{\mathrm{T}},\bar{D}^{\mathrm{T}})}$.

6 Conclusion

In this paper, we define a new class of structural matrix type – Form IV, and characterize that an LTV system is strong structurally controllable (SSC) if and only if its structural matrices $(\bar{A}, \bar{B})$ are not of Form IV. By introducing an index set, a direct constructive method is proposed to check the SSC property for a given LTV system with its bipartite graph representation. This method was summarized as an efficient algorithm to examine the SSC property. Besides, we also consider the characterization of SSO LTV systems as a dual case of that of SSC LTV systems.

In our opinion, there still exists a problem of fundamental importance that needs to be solved. This is the problem of characterizing the SSC LTV systems with switching structures. The related problem of characterizing structurally controllable LTI systems with switching structures has been proposed and solved in [19], while the problem of characterizing SSC LTV systems with switching structures appears to be still open. It is natural and reasonable to consider such a problem since we always expect LTV systems to be completely controllable barring multiple structural switches. We plan to work on the problem and look forward to a breakthrough in the future.

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