SCIENCE CHINA
Information Sciences

• RESEARCH PAPER •

January 2019, Vol. 62 012204:1–012204:15
https://doi.org/10.1007/s11432-017-9302-9

Basic theory and stability analysis for neutral stochastic functional differential equations with pure jumps

Mengling LI[1], Feiqi DENG[1*] & Xuerong MAO[2]

[1]*Systems Engineering Institute, South China University of Technology, Guangzhou* 510640*, China;*
[2]*Department of Statistics and Modelling Science, University of Strathclyde, Glasgow* G1 1XH*, UK*

Received 7 August 2017/Revised 7 October 2017/Accepted 6 November 2017/Published online 16 October 2018

Abstract This paper investigates the existence and uniqueness of solutions to neutral stochastic functional differential equations with pure jumps (NSFDEwPJs). The boundedness and almost sure exponential stability are also considered. In general, the classical existence and uniqueness theorem of solutions can be obtained under a local Lipschitz condition and linear growth condition. However, there are many equations that do not obey the linear growth condition. Therefore, our first aim is to establish new theorems where the linear growth condition is no longer required whereas the upper bound for the diffusion operator will play a leading role. Moreover, the pth moment boundedness and almost sure exponential stability are also obtained under some loose conditions. Finally, we present two examples to illustrate the effectiveness of our results.

Keywords neutral term, stochastic functional differential equations, pure jumps, existence and uniqueness theorem, stability analysis

Citation Li M L, Deng F Q, Mao X R. Basic theory and stability analysis for neutral stochastic functional differential equations with pure jumps. Sci China Inf Sci, 2019, 62(1): 012204, https://doi.org/10.1007/s11432-017-9302-9

1 Introduction

As is well known, many real systems are affected by stochastic factors. Thus, it is necessary to consider stochastic systems. Of course, there is extensive literature in this area, such as [1–3]. Delayed systems are suited to describing such systems that not only depend on the present states, but also on the past states. In this paper, we study a class of time delay systems depending on past and present values that involve derivatives with delays as well as the function itself. Such systems historically have been referred to as neutral systems. Moreover, with the development of the stochastic analysis and the requirements of applications, Poisson jumps have been considered by many researchers [4,5]. Moreover, there are practical applications for Poisson jumps, such as financial markets [6]. In this paper, we consider the neutral stochastic functional differential equations containing Poisson jumps, which are complex equations. Thus, various stochastic analysis tools are employed to analyze our problems.

Our paper is motivated by [7], where the existence and uniqueness of the solutions to neutral stochastic functional differential equations with pure jumps (NSFDEwPJs) whose coefficients satisfy the local Lipschitz condition and the linear growth condition was studied. However, the coefficients of many important equations, such as stochastic delay Lotka–Volterra equations [8], do not satisfy the linear growth

* Corresponding author (email: aufqdeng@scut.edu.cn)

condition. Thus, if we could find a wider condition to replace the linear growth condition, many problems would be solved. Fortunately, we finally find a more general test for NSFDEwPJs that covers a wide class of highly nonlinear NSFDEwPJs. Here, we refer to [9–12], which also allow that there are high-order terms consisting in their parameters. However, the above studies only focus on the stochastic models containing the continuous Brown motion. To the best of our knowledge, the stochastic models containing Poisson jump have not been investigated previously.

Based on the above discussion, our first result mainly utilizes the upper bounds of the diffusion operator to replace the linear growth condition. A unique global solution exists even if the coefficients of the equation are high order. In addition, in Section 4, we utilize the modified conditions to deal with the asymptotic moment estimation, which is also one of our main results. Moreover, if there exists a trivial solution for an equation, the trivial solution will be almost surely exponentially stable. Finally, we present two examples to illustrate the effectiveness of our theory.

2 Preliminaries

Throughout this paper, unless otherwise specified, we use the following notation. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathcal{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all $\mathcal{P}$-null sets). If A is a vector or matrix, A^{T} denotes its transpose and its trace norm is denoted by $|A| = \sqrt{\mathrm{trace}(A^{\mathrm{T}}A)}$. Let $\tau > 0$. $\mathcal{D}([-\tau, 0]; \mathbb{R}^n)$ denotes the family of all right-continuous functions with left limits ϕ from $[-\tau, 0]$ to $\mathbb{R}^n$ equipped with the norm $\|\phi\| = \sup_{-\tau\leqslant s\leqslant 0} |\phi(s)|$. Here $\mathcal{D}^b_{\mathcal{F}_0}$ denotes the family of all almost surely bounded, $\mathcal{F}_0$-measurable, $\mathcal{D}([-\tau, 0]; \mathbb{R}^n)$-valued random variables $\xi = \{\xi(\theta) : -\tau \leqslant \theta \leqslant 0\}$. We use $\mathrm{E}(x)$ to denote the mathematical expectation of a random variable x. Let $p \geqslant 2$. $\mathcal{L}^p_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ denotes the family of all $\mathcal{F}_0$-measurable, $\mathcal{D}([-\tau, 0]; \mathbb{R}^n)$-valued random variables $\phi = \{\phi(\theta) : -\tau \leqslant \theta \leqslant 0\}$ such that $\mathrm{E}\sup_{-\tau\leqslant\theta\leqslant 0} |\phi(\theta)|^p < \infty$. $C([-\tau, \infty) \times \mathbb{R}^n; \mathbb{R}^+)$ denotes the family of continuously nonnegative functions from $[-\tau, \infty) \times \mathbb{R}^n$ to $\mathbb{R}^+$. For $a, b \in \mathbb{R}$, $a \vee b$ (respectively, $a \wedge b$) means the maximum (respectively, minimum) of a and b.

Let $(U, \mathcal{B}(U))$ be a measurable space and $\bar{p}(t)(t \geqslant 0)$ be the jump at time t. Then, for each Borel set $A \in \mathcal{B}(U - \{0\})$, the Poisson counting measure $N_{\bar{p}}$ is defined by

$$N_{\bar{p}}(t, A) := \sum_{0<s\leqslant t} I_A(\bar{p}(s)) = \#\{0 < s \leqslant t, \bar{p}(s) \in A\},$$

where $I(\cdot)$ denotes the indicative function and $\#$ records the number of jumps from 0 to t. According to [13], if we fix t and A, $N_{\bar{p}}(t, A)$ is a random variable. However, if we fix $\omega \in \Omega$ and $t \geqslant 0$, $N_{\bar{p}}(t, \cdot)(\omega)$ is a measure. Therefore, if we fix A with a Lévy measure $\pi(A)$, $\{N_{\bar{p}}(t, A)\}_{t\geqslant 0}$ is a Poisson process with intensity $\pi(A)$. Moreover, we have

$$P(N_{\bar{p}}(t, A) = n) = \frac{\exp(-\pi(A)t)(\pi(A)t)^n}{n!},$$

and the measure $\tilde{N}_{\bar{p}}$ satisfying $\tilde{N}_{\bar{p}}(t, A) = N_{\bar{p}}(t, A) - \pi(A)t$ is a martingale measure.

Let $x(t)$ be an $\mathbb{R}^n$-valued stochastic process on $t \in [-\tau, \infty)$ and let $x_t = \{x(t + \theta) : -\tau \leqslant \theta \leqslant 0\}$ for $t \geqslant 0$ be regarded as a $\mathcal{D}([-\tau, 0]; \mathbb{R}^n)$-valued stochastic process. Consider the following NSFDEwPJs:

$$\mathrm{d}[x(t) - D(x_t)] = f(t, x_t)\mathrm{d}t + \int_U h(u, x_t)N_{\bar{p}}(\mathrm{d}t, \mathrm{d}u), \tag{1}$$

where $f : \mathbb{R}^+ \times \mathcal{D}([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^n$ and $h : U \times \mathcal{D}([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}^n$ are both Borel-measurable functions. Assume that the initial data is given by

$$x_0 = \xi = \{\xi(t) : -\tau \leqslant t \leqslant 0\} \in \mathcal{L}^p_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n).$$

In addition, $\tilde{N}_{\bar{p}}(\mathrm{d}t, \mathrm{d}u) = N_{\bar{p}}(\mathrm{d}t, \mathrm{d}u) - \pi(\mathrm{d}u)\mathrm{d}t$ is the compensated Poisson random measure.

The classical existence and uniqueness theorem requires the coefficients f, h to satisfy the local Lipschitz condition and the linear growth condition and the neutral term D satisfies the contractility condition (see [7]). In this paper, we retain the contractility condition and the local Lipschitz condition, but replace the linear growth condition by a more general condition. To state our condition, we need to introduce the well-known Lyapunov function and Itô formula.

Let $C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$ denote the family of all nonnegative functions $V(t,x)$ on $\mathbb{R}^+ \times \mathbb{R}^n$ that are continuously twice differentiable in x and once in t; moreover, define

$$\begin{aligned} V_x(t,x) &= \left(\frac{\partial V(t,x)}{\partial x_1}, \ldots, \frac{\partial V(t,x)}{\partial x_n}\right), \\ V_{xx}(t,x) &= \left(\frac{\partial^2 V(t,x)}{\partial x_k \partial x_l}\right)_{n\times n}, \\ V_t(t,x) &= \frac{\partial V(t,x)}{\partial t}. \end{aligned}$$

Then, we can define an operator $\mathcal{L}V : \mathbb{R}^+ \times \mathcal{D}([-\tau,0];\mathbb{R}^n) \to \mathbb{R}$ for the function $V(t,x) \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$ by

$$\begin{aligned} \mathcal{L}V(t,\phi) =& V_t(t,\phi(0)-D(\phi)) + V_x(t,\phi(0)-D(\phi))f(t,\phi) \\ &+ \int_U [V(t,\phi(0)-D(\phi)+h(u,\phi)) - V(t,\phi(0)-D(\phi)) \\ &- V_x(t,\phi(0)-D(\phi))h(u,\phi)]\pi(\mathrm{d}u), \end{aligned}$$

where $t \in \mathbb{R}^+$, $\phi \in \mathcal{D}([-\tau,0],\mathbb{R}^n)$.

Based on this, we can cite the Itô formula

$$\begin{aligned} \mathrm{d}V(t,x(t)-D(x_t)) =& \mathcal{L}V(t,x_t)\mathrm{d}t + \int_U [V(t,x(t)-D(x_t)+h(u,x_t)) \\ &- V(t,x(t)-D(x_t))]\tilde{N}_{\bar{p}}(\mathrm{d}t,\mathrm{d}u). \end{aligned} \tag{2}$$

We need to make the following assumptions.

Assumption 1 (Local Lipschitz condition). For arbitrary $\varphi, \psi \in \mathcal{D}([-\tau,0];\mathbb{R}^n)$ and $\|\varphi\| \vee \|\psi\| \leqslant n$, there is a positive constant k_n such that

$$|f(\varphi,t)-f(\psi,t)|^2 \vee \int_U |h(\varphi,u)-h(\psi,u)|^2\pi(\mathrm{d}u) \leqslant k_n\|\varphi-\psi\|^2,$$

where $n \in N^+$, $t \in \mathbb{R}^+$, $u \in U$. Moreover,

$$L = \sup\{|f(t,0)| \vee |h(u,0)| : t \geqslant 0, u \in U\} < \infty.$$

Assumption 2. There are functions $V \in C^{1,2}([-\tau,\infty) \times \mathbb{R}^n; \mathbb{R}^+)$, $H \in C([-\tau,\infty) \times \mathbb{R}^n; \mathbb{R}^+)$, and a nondecreasing function $K(t) \in C(\mathbb{R}^+;\mathbb{R}^+)$, three positive constants $p(\geqslant 2), c_1, c_2$, and two probability measures $\mu(\cdot)$ and $\nu(\cdot)$ that are real-valued functions defined on $[-\tau,0]$ with bounded variation such that

$$c_1|x|^p \leqslant V(t,x) \leqslant c_2|x|^p,$$

for any $x \in \mathbb{R}^n$ and

$$\begin{aligned} \mathcal{L}V(t,\phi) \leqslant& K(t)[1+V(t,\phi(0)) + \int_{-\tau}^0 V(t+\theta,\phi(\theta))\mathrm{d}\mu(\theta)] \\ &- H(t,\phi(0)) + \int_{-\tau}^0 H(t+\theta,\phi(\theta))\mathrm{d}\nu(\theta), \end{aligned} \tag{3}$$

where $\phi \in \mathcal{D}([-\tau,0];\mathbb{R}^n)$.

Assumption 3 (Contractility condition)**.** $D(0)=0$ and there is a constant $k_0\in[0,1)$ such that

$$\mathrm{E}|D(\phi)|^p\leqslant k_0^p\sup_{-\tau\leqslant\theta\leqslant 0}\mathrm{E}|\phi(\theta)|^p,$$

for all $\phi\in\mathcal{L}^p_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$.

In Section 3, the proof of our new existence and uniqueness theorem requires that there exists a maximal local solution for (1) at first. Thus, we should introduce the definition of a maximal local solution and provide a lemma about the existence of the maximal local solution.

Definition 1 (Maximal local solution)**.** Let σ_∞ be a stopping time, and there exists a finite time T such that $0\leqslant\sigma_\infty\leqslant T$ almost surely. An $\mathcal{F}_t$-adapted, $\mathbb{R}^n$-valued, cadlag process $\{x(t):-\tau\leqslant t\leqslant\sigma_\infty\}$ (a cadlag process refers to a stochastic process that is right continuous and has left limit) is called a local solution to (1) with initial data $x(t)=\xi(t)$ on $t\in[-\tau,0]$ if for any stopping time $\sigma_k\leqslant\sigma_\infty$ and any $t\in[0,T]$,

$$x(t\wedge\sigma_k)-D(x_{t\wedge\sigma_k})=\xi(0)-D(\xi)+\int_0^{t\wedge\sigma_k}f(x_s,s)\mathrm{d}s+\int_0^{t\wedge\sigma_k}\int_U h(x_s,u)N_{\bar{p}}(\mathrm{d}s,\mathrm{d}u)$$

holds with probability 1. Moreover, if

$$\lim_{t\to\sigma_\infty}\sup|x(t)|=\infty,\quad\text{whenever }\sigma_\infty<T,$$

$\{x(t)\}$ and σ_∞ are called a maximal local solution of (1) and the explosion time, respectively.

Lemma 1. If Assumptions 1 and 3 hold, for any given initial data $\xi(t)$ on $[-\tau,0]$, there exists a unique maximal local solution to (1).

Proof. From Assumption 3, we have $|D(\phi)|\leqslant k_0\|\phi\|$ almost surely. There is a sufficiently large number $k^0\in\mathbb{R}^+$ such that $\|\xi\|\leqslant k^0$. In addition, for any positive integer $k\geqslant k^0$, define

$$z^{[k]}=\frac{|z|\wedge k}{|z|}z,\quad 0^{[k]}=0$$

for any $z\in\mathbb{R}^n$. Then we can define the following truncation functions:

$$f_k(t,y)=f(t,y^{[k]}),\quad h_k(u,y)=h(u,y^{[k]})$$

for $y\in\mathbb{R}^n$. Next, we consider the following equation:

$$\mathrm{d}[x^k(t)-D(x_t^k)]=f_k(t,x_t^k)\mathrm{d}t+\int_U h_k(u,x_t^k)N_{\bar{p}}(\mathrm{d}t,\mathrm{d}u)\tag{4}$$

on $t\in[0,T]$ with initial data $x^k(t)=\xi(t)$ on $t\in[-\tau,0]$. Then, we have

$$|f_k(t,\phi)|^2\leqslant 2|f_k(t,\phi)-f(0,t)|^2+2L\leqslant L_1(1+\|\phi\|^2),$$

where $L_1=\max\{2k_k,2L\}$. Similarly,

$$\int_U|h_k(u,\phi)|^2\leqslant L_1(1+\|\phi\|^2).$$

By Assumption 1, Eq. (4) satisfies the global Lipschitz condition and the linear growth condition. Therefore, according to the literature [7], there is a unique global solution $x^k(t)$ to (4). Then, we define the stopping time

$$\sigma_k=\inf\{t\in[0,T]:|x^k(t)|\geqslant k\}$$

for $k\geqslant k_0$ and where we set $\inf\emptyset=\infty$ throughout our paper (as usual $\emptyset$ denotes the empty set). Moreover, we can see that

$$x^k(t)=x^{k+1}(t),\quad -\tau\leqslant t\leqslant\sigma_k,\tag{5}$$

which means that $\{\sigma_k\}$ is a nondecreasing sequence and then let $\lim_{k\to\infty}\sigma_k=\sigma_\infty$ almost surely. Consequently, we can define $\{x(t):-\tau\leqslant t<\sigma_\infty\}$ with the initial data $x(t)=\xi(t)$ on $t\in[-\tau,0]$ and define

$$x(t)=x^k(t),\quad t\in[\sigma_{k-1},\sigma_k),\ k\geqslant 1,\tag{6}$$

where $\sigma_0=0$. From (4)–(6), we can obtain

$$x(t\wedge\sigma_k)-D(x_{t\wedge\sigma_k})=\xi(0)-D(\xi)+\int_0^{t\wedge\sigma_k}f(x_s,s)\mathrm{d}s+\int_0^{t\wedge\sigma_k}\int_U h(x_s,u)N_{\bar{p}}(\mathrm{d}s,\mathrm{d}u)$$

for any $t\in[0,T]$. Moreover, if $\sigma_\infty<T$,

$$\lim_{t\to\sigma_\infty}\sup|x(t)|\geqslant\lim_{k\to\infty}\sup|x(\sigma_k)|=\lim_{k\to\infty}\sup|x_k(\sigma_k)|=\infty.$$

Hence, $\{x(t):-\tau\leqslant t<\sigma_\infty\}$ is a maximal local solution to (1) according to Definition 1.

3 The existence and uniqueness theorem

In this section, we prove that there is a unique global solution to (1) if the coefficients of (1) satisfy the contractility condition, local Lipschitz condition, and Assumption 3.

Theorem 1. Let Assumptions 1–3 hold, then for a given initial value $x_0=\xi\in\mathcal{D}([-\tau,0];\mathbb{R}^n)$, there exists a unique global solution $x(t;t_0,\xi)$ to (1). Moreover, the solution has the property that

$$\mathrm{E}V(t,x(t))\leqslant(0.5+c_3)\mathrm{e}^{\frac{2c_2}{c_1(1-k_0)^p}\int_0^t K(\rho+\tau)\mathrm{d}\rho},$$

where

$$c_3=\frac{c_2k_0}{1-k_0}\mathrm{E}\|\xi\|^p+\frac{c_2M}{c_1(1-k_0)^p},$$

$$M=\mathrm{E}\left(V(0,x(0)-D(x_0))+K(\tau)\int_{-\tau}^0 V(\rho,x(\rho))\mathrm{d}\rho+\int_{-\tau}^0 H(\rho,x(\rho))\mathrm{d}\rho\right).$$

Proof. By Lemma 1, there is a unique maximal local solution $x(t)$ on $[-\tau,\sigma_\infty)$, where σ_∞ is called the explosion time. Let $k^0\in\mathbb{R}^+$ be sufficiently large for $\|\xi\|\leqslant k^0$. For any integer $k\geqslant k^0$, define

$$\tau_k=\inf\{t\in[0,\sigma_\infty):|x(t)|\geqslant k\},$$

where $\inf\emptyset=\infty$. Obviously, the sequence $\{\tau_k\}$ is increasing. Thus, we have a limit $\tau_\infty=\lim_{k\to\infty}\tau_k$, whence $\tau_\infty\leqslant\sigma_\infty$. If we can show that $\tau_\infty=\infty$, we will have $\sigma_\infty=\infty$. Therefore, we only need to focus on proving $\tau_\infty=\infty$.

For any $s>0$, by the Itô formula and Assumption 2, we have

$$\begin{aligned}V(s,x(s)-D(x_s))&=V(0,x(0)-D(x_0))+\int_0^s\mathcal{L}V(\rho,x_\rho)\mathrm{d}\rho\\&\quad+\int_0^s\int_U[V(\rho,x(\rho)-D(x_\rho)+h(x_\rho,u))-V(\rho,x(\rho)-D(x_\rho))]\tilde{N}_{\bar{p}}(\mathrm{d}\rho,\mathrm{d}u)\\&\leqslant V(0,x(0)-D(x_0))+\int_0^s K(\rho)[1+V(\rho,x(\rho))]\mathrm{d}\rho\\&\quad+\int_0^s\int_{-\tau}^0 K(\rho)V(\rho+\theta,x(\rho+\theta))\mathrm{d}\mu(\theta)\mathrm{d}\rho\\&\quad-\int_0^s H(\rho,x(\rho))\mathrm{d}\rho+\int_0^s\int_{-\tau}^0 H(\rho+\theta,x(\rho+\theta))\mathrm{d}\nu(\theta)\mathrm{d}\rho\\&\quad+\int_0^s\int_U[V(\rho,x(\rho)-D(x_\rho)+h(x_\rho,u))-V(\rho,x(\rho)-D(x_\rho))]\tilde{N}_p(\mathrm{d}\rho,\mathrm{d}u).\end{aligned}\tag{7}$$

At the same time, by the Fubini theorem and the fact that $K(t)$ is a nondecreasing function, we thus obtain

$$\begin{aligned}\int_0^s K(\rho)\int_{-\tau}^0 V(\rho+\theta,x(\rho+\theta))\mathrm{d}\mu(\theta)\mathrm{d}\rho &= \int_{-\tau}^0 \mathrm{d}\mu(\theta)\int_0^s K(\rho)V(\rho+\theta,x(\rho+\theta))\mathrm{d}\rho\\ &\leqslant \int_{-\tau}^s K(\rho+\tau)V(\rho,x(\rho))\mathrm{d}\rho\\ &\leqslant K(\tau)\int_{-\tau}^0 V(\rho,x(\rho))\mathrm{d}\rho\\ &\quad+\int_0^s K(\rho+\tau)V(\rho,x(\rho))\mathrm{d}\rho. \end{aligned}\tag{8}$$

Similarly, we have

$$\int_0^s\int_{-\tau}^0 H(\rho+\theta,x(\rho+\theta))\mathrm{d}\nu(\theta)\mathrm{d}\rho \leqslant \int_{-\tau}^s H(\rho,x(\rho))\mathrm{d}\rho.\tag{9}$$

Substituting (8) and (9) into (7) yields

$$\mathrm{E}V(s\wedge\tau_k,x(s\wedge\tau_k)-D(x_{s\wedge\tau_k}))\leqslant M+\mathrm{E}\int_0^{s\wedge\tau_k}K(\rho+\tau)[1+2V(\rho,x(\rho))]\mathrm{d}\rho,\tag{10}$$

where $M=\mathrm{E}(V(0,x(0)-D(x_0))+K(\tau)\int_{-\tau}^0 V(\rho,x(\rho))\mathrm{d}\rho+\int_{-\tau}^0 H(\rho,x(\rho))\mathrm{d}\rho)$. Because $c_1|x|^p\leqslant V(x,t)\leqslant c_2|x|^p$, by inequality (10), we have

$$c_1\mathrm{E}|x(s\wedge\tau_k)-D(x_{s\wedge\tau_k})|^p\leqslant M+\mathrm{E}\int_0^{s\wedge\tau_k}K(\rho+\tau)[1+2V(\rho,x(\rho))]\mathrm{d}\rho.\tag{11}$$

Applying elementary inequality $|a+b|^p\leqslant(1-\varepsilon)^{1-p}|a|^p+\varepsilon^{1-p}|b|^p$ for any $a,b\in\mathbb{R},\varepsilon\in(0,1)$ and setting $\varepsilon=k_0$, we have

$$|x(s)|^p\leqslant(1-k_0)^{1-p}|x(s)-D(x_s)|^p+k_0^{1-p}|D(x_s)|^p,$$

which implies

$$\begin{aligned}\mathrm{E}|x(s\wedge\tau_k)|^p &\leqslant (1-k_0)^{1-p}\mathrm{E}|x(s\wedge\tau_k)-D(x_{s\wedge\tau_k})|^p+k_0^{1-p}\mathrm{E}|D(x_{s\wedge\tau_k})|^p\\ &\leqslant (1-k_0)^{1-p}\mathrm{E}|x(s\wedge\tau_k)-D(x_{s\wedge\tau_k})|^p+k_0\sup_{-\tau\leqslant\theta\leqslant0}\mathrm{E}|x(s\wedge\tau_k+\theta)|^p\\ &\leqslant (1-k_0)^{1-p}\mathrm{E}|x(s\wedge\tau_k)-D(x_{s\wedge\tau_k})|^p+k_0\mathrm{E}\|\xi\|^p+k_0\sup_{0\leqslant u\leqslant s\wedge\tau_k}\mathrm{E}|x(u)|^p\\ &\leqslant (1-k_0)^{1-p}\mathrm{E}|x(s\wedge\tau_k)-D(x_{s\wedge\tau_k})|^p+k_0\mathrm{E}\|\xi\|^p+k_0\sup_{0\leqslant u\leqslant s}\mathrm{E}|x(u\wedge\tau_k)|^p. \end{aligned}\tag{12}$$

Then for any $t\geqslant s$, we have

$$\begin{aligned}\sup_{0\leqslant s\leqslant t}\mathrm{E}|x(s\wedge\tau_k)|^p\leqslant(1-k_0)^{1-p}&\sup_{0\leqslant s\leqslant t}\mathrm{E}|x(s\wedge\tau_k)-D(x_{s\wedge\tau_k})|^p\\ &+k_0\mathrm{E}\|\xi\|^p+k_0\sup_{0\leqslant s\leqslant t}\mathrm{E}|x(s\wedge\tau_k)|^p,\end{aligned}$$

which implies

$$\sup_{0\leqslant s\leqslant t}\mathrm{E}|x(s\wedge\tau_k)|^p\leqslant\frac{k_0}{1-k_0}\mathrm{E}\|\xi\|^p+\frac{1}{(1-k_0)^p}\sup_{0\leqslant s\leqslant t}\mathrm{E}|x(s\wedge\tau_k)-D(x_{s\wedge\tau_k})|^p.\tag{13}$$

By (11) and (13), we have

$$\sup_{0\leqslant s\leqslant t}\mathrm{E}|x(s\wedge\tau_k)|^p\leqslant\frac{k_0}{1-k_0}\mathrm{E}\|\xi\|^p+\frac{M}{c_1(1-k_0)^p}$$

$$+\frac{1}{c_1(1-k_0)^p}\sup_{0\leqslant s\leqslant t}\mathrm{E}\int_0^{s\wedge\tau_k}K(\rho+\tau)[1+2V(\rho,x(\rho))]\mathrm{d}\rho. \tag{14}$$

Applying $c_1|x|^p \leqslant V(x,t) \leqslant c_2|x|^p$ once again and by inequality (14), we can obtain

$$\begin{aligned}\sup_{0\leqslant s\leqslant t}\mathrm{E}V(s\wedge\tau_k,x(s\wedge\tau_k)) &\leqslant c_3+\frac{c_2}{c_1(1-k_0)^p}\sup_{0\leqslant s\leqslant t}\mathrm{E}\int_0^{s\wedge\tau_k}K(\rho+\tau)[1+2V(\rho,x(\rho))]\mathrm{d}\rho\\ &\leqslant c_3+\frac{c_2}{c_1(1-k_0)^p}\sup_{0\leqslant s\leqslant t}\mathrm{E}\int_0^{s}K(\rho+\tau)[1+2V(\rho\wedge\tau_k,x(\rho\wedge\tau_k))]\mathrm{d}\rho,\end{aligned} \tag{15}$$

where $c_3=\frac{c_2k_0}{1-k_0}\mathrm{E}\|\xi\|^p+\frac{c_2M}{c_1(1-k_0)^p}$.

Inequality (15) implies that

$$\sup_{0\leqslant s\leqslant t}\mathrm{E}V(x(s\wedge\tau_k),s\wedge\tau_k)\leqslant c_3+\frac{2c_2}{c_1(1-k_0)^p}\int_0^t K(\rho+\tau)\left[0.5+\sup_{0\leqslant\beta\leqslant\rho}\mathrm{E}V(\beta\wedge\tau_k,x(\beta\wedge\tau_k))\right]\mathrm{d}\rho.$$

By the Gronwall inequality [14], we therefore obtain

$$0.5+\mathrm{E}V(x(t\wedge\tau_k),t\wedge\tau_k)\leqslant(0.5+c_3)\mathrm{e}^{\frac{2c_2}{c_1(1-k_0)^p}\int_0^t K(\rho+\tau)\mathrm{d}\rho}.$$

Consequently,

$$\begin{aligned}c_1k^pP(\tau_k\leqslant t)&\leqslant c_1\mathrm{E}(|x(\tau_k)|^pI_{\{\tau_k\leqslant t\}})\leqslant c_1\mathrm{E}|x(t\wedge\tau_k)|^p\\ &\leqslant \mathrm{E}V(x(t\wedge\tau_k),t\wedge\tau_k)\leqslant(0.5+c_3)\mathrm{e}^{\frac{2c_2}{c_1(1-k_0)^p}\int_0^t K(\rho+\tau)\mathrm{d}\rho}.\end{aligned} \tag{16}$$

Letting $k\to\infty$ in the above inequality, we can obtain $\lim_{k\to\infty}P(\tau_k\leqslant t)=0$. Because t is arbitrary, we have $P(\tau_k<\infty)=0$. Hence, $\tau_\infty=\infty$ almost surely and by inequality (16), we have

$$\mathrm{E}V(t,x(t))\leqslant(0.5+c_3)\mathrm{e}^{\frac{2c_2}{c_1(1-k_0)^p}\int_0^t K(\rho+\tau)\mathrm{d}\rho}.$$

The proof is complete.

4 Boundedness and almost sure exponential stability for solution

In this section, with the notation introduced in the previous section, we discuss the boundedness of the pth moment of the solution. Moreover, if $x(t)=0$ is the trivial solution, we can also obtain that the trivial solution is almost surely exponentially stable under certain conditions.

Definition 2 (Almost sure exponential stability). If $f(t,0)=h(u,0)=D(0)\equiv 0$ for $t\geqslant 0$, $u\in U$. Then the trivial solution of (1) is said to be almost surely exponentially stable if the following formula

$$\limsup_{t\to\infty}\frac{1}{t}\log|x(t;0,\xi)|\leqslant 0 \ \text{ almost surely}$$

can be satisfied for all $\xi\in\mathbb{R}^n$.

Theorem 2. Let Assumptions 1–3 hold except (3) which is replaced by

$$\begin{aligned}\mathcal{L}V(t,\phi)\leqslant&-\alpha_1V(t,\phi(0))+\alpha_2\int_{-\tau}^0V(t+\theta,\phi(\theta))\mathrm{d}\mu(\theta)\\ &-H(t,\phi(0))+\alpha_3\int_{-\tau}^0H(t+\theta,\phi(\theta))\mathrm{d}\nu(\theta),\end{aligned} \tag{17}$$

where $\alpha_1>\alpha_2\geqslant 0$ and $\alpha_3\in(0,1)$. Then for any given initial data ξ, there is a unique global solution $x(t)$ to (1) that has the property that

$$\mathrm{E}|x(t)|^p\leqslant\frac{\frac{C(1-k_0)^{1-p}}{c_1}+k_0\mathrm{E}\|\xi\|^p}{1-k_0} \tag{18}$$

for any $t \geqslant 0$, where $C = V(0, x(0) - D(x_0)) + \alpha_2 \mathrm{e}^{\varepsilon\tau} \mathrm{E} \int_{-\tau}^{0} V(s, x(s))\mathrm{d}s + \alpha_3 \mathrm{e}^{\varepsilon\tau} \int_{-\tau}^{0} H(s, x(s))\mathrm{d}s$ while $\varepsilon = \varepsilon_1 \wedge \varepsilon_2$. In addition, ε_1 and ε_2 satisfy

$$\alpha_1 - \alpha_2 \mathrm{e}^{\varepsilon_1 \tau} = 0, \quad 1 - \alpha_3 \mathrm{e}^{\varepsilon_2 \tau} = 0,$$

respectively. Moreover,

$$\int_0^{\infty} H(t, x(t))\mathrm{d}t \leqslant \frac{M}{1 - \alpha_3}, \tag{19}$$

where $M = \mathrm{E}(V(0, x(0) - D(x_0)) + \alpha_2 \int_{-\tau}^{0} V(s, x(s))\mathrm{d}s + \alpha_3 \int_{-\tau}^{0} H(s, x(s))\mathrm{d}s)$.

Proof. First, we can observe that Eq. (17) is stronger than (3). Thus, we can obtain that there is a unique global solution $x(t)$ to (1) by Theorem 1. By the Itô formula and inequality (17), we can compute

$$\begin{aligned}
&\mathrm{E}(\mathrm{e}^{\varepsilon(t\wedge\tau_k)} V(t\wedge\tau_k, x(t\wedge\tau_k) - D(x_{t\wedge\tau_k})) - V(0, x(0) - D(x_0))\\
&= \mathrm{E}\int_0^{t\wedge\tau_k} \varepsilon \mathrm{e}^{\varepsilon s} V(s, x(s) - D(x_s))\mathrm{d}s + \mathrm{E}\int_0^{t\wedge\tau_k} \mathrm{e}^{\varepsilon s} \mathcal{L}V(s, x_s)\mathrm{d}s\\
&\leqslant \mathrm{E}\int_0^{t\wedge\tau_k} \varepsilon \mathrm{e}^{\varepsilon s} V(s, x(s) - D(x_s))\mathrm{d}s - \alpha_1 \mathrm{E}\int_0^{t\wedge\tau_k} \mathrm{e}^{\varepsilon s} V(s, x(s))\mathrm{d}s\\
&\quad + \alpha_2 \mathrm{E}\int_0^{t\wedge\tau_k} \int_{-\tau}^{0} \mathrm{e}^{\varepsilon s} V(s+\theta, x(s+\theta))\mathrm{d}\mu(\theta)\mathrm{d}s - \mathrm{E}\int_0^{t\wedge\tau_k} \mathrm{e}^{\varepsilon s} H(s, x(s))\mathrm{d}s\\
&\quad + \alpha_3 \mathrm{E}\int_0^{t\wedge\tau_k} \int_{-\tau}^{0} \mathrm{e}^{\varepsilon s} H(s+\theta, x(s+\theta))\mathrm{d}\nu(\theta)\mathrm{d}s\\
&\leqslant \mathrm{E}\int_0^{t\wedge\tau_k} \varepsilon \mathrm{e}^{\varepsilon s} V(s, x(s) - D(x_s))\mathrm{d}s - \alpha_1 \mathrm{E}\int_0^{t\wedge\tau_k} \mathrm{e}^{\varepsilon s} V(s, x(s))\mathrm{d}s\\
&\quad + \alpha_2 \mathrm{E}\int_{-\tau}^{t\wedge\tau_k} \mathrm{e}^{\varepsilon(s+\tau)} V(s, x(s))\mathrm{d}s - \mathrm{E}\int_0^{t\wedge\tau_k} \mathrm{e}^{\varepsilon s} H(s, x(s))\mathrm{d}s\\
&\quad + \alpha_3 \mathrm{E}\int_{-\tau}^{t\wedge\tau_k} \mathrm{e}^{\varepsilon(s+\tau)} H(s, x(s))\mathrm{d}s
\end{aligned} \tag{20}$$

for any $t \geqslant 0$ and τ_k is as defined in Theorem 1. We take $\varepsilon = \varepsilon_1 \wedge \varepsilon_2$, and $\varepsilon_1, \varepsilon_2$ satisfy

$$\alpha_1 - \alpha_2 \mathrm{e}^{\varepsilon_1 \tau} = 0, \quad 1 - \alpha_3 \mathrm{e}^{\varepsilon_2 \tau} = 0,$$

respectively. Then the inequality (20) leads to

$$\begin{aligned}
&E(\mathrm{e}^{\varepsilon(t\wedge\tau_k)}) V(t\wedge\tau_k, x(t\wedge\tau_k) - D(x_{t\wedge\tau_k}))\\
&\quad \leqslant V(0, x(0) - D(x_0)) + \alpha_2 \mathrm{e}^{\varepsilon\tau} \mathrm{E}\int_{-\tau}^{0} V(s, x(s))\mathrm{d}s + \alpha_3 \mathrm{e}^{\varepsilon\tau} \mathrm{E}\int_{-\tau}^{0} H(s, x(s))\\
&\quad + \varepsilon \mathrm{E}\int_0^{t\wedge\tau_k} \mathrm{e}^{\varepsilon s} V(s, x(s) - D(x_s))\mathrm{d}s = C + \varepsilon \mathrm{E}\int_0^{t\wedge\tau_k} \mathrm{e}^{\varepsilon s} V(s, x(s) - D(x_s))\mathrm{d}s,
\end{aligned} \tag{21}$$

where $C = V(0, x(0) - D(x_0)) + \alpha_2 \mathrm{e}^{\varepsilon\tau} \mathrm{E} \int_{-\tau}^{0} V(s, x(s))\mathrm{d}s + \alpha_3 \mathrm{e}^{\varepsilon\tau} \mathrm{E} \int_{-\tau}^{0} H(s, x(s))\mathrm{d}s$.

By inequality (21) and the Fubini theorem [15], we can obtain

$$\begin{aligned}
&\mathrm{E}(\mathrm{e}^{\varepsilon(t\wedge\tau_k)}) V(t\wedge\tau_k, x(t\wedge\tau_k) - D(x_{t\wedge\tau_k}))\\
&\quad \leqslant C + \varepsilon \mathrm{E}\int_0^{t} \mathrm{e}^{\varepsilon(s\wedge\tau_k)} V(s\wedge\tau_k, x(s\wedge\tau_k) - D(x_{s\wedge\tau_k}))\mathrm{d}s\\
&\quad = C + \varepsilon \int_0^{t} \mathrm{E}(\mathrm{e}^{\varepsilon(s\wedge\tau_k)} V(s\wedge\tau_k, x(s\wedge\tau_k) - D(x_{s\wedge\tau_k})))\mathrm{d}s.
\end{aligned}$$

Hence, using the Gronwall inequality [14], we derive that

$$\mathrm{E}(\mathrm{e}^{\varepsilon(t\wedge\tau_k)} V(t\wedge\tau_k, x(t\wedge\tau_k) - D(x_{t\wedge\tau_k}))) \leqslant C\mathrm{e}^{\varepsilon t}.$$

Letting $k \to \infty$, we obtain that

$$\mathrm{E}(V(t, x(t) - D(x_t))) \leqslant C. \tag{22}$$

By Assumption 2 and inequality (22), we know that

$$c_1 \mathrm{E}|x(t) - D(x_t)|^p \leqslant \mathrm{E}(V(t, x(t) - D(x_t))) \leqslant C. \tag{23}$$

Recall the fundamental inequality: $|a+b|^p \leqslant (1-k)^{1-p}|a|^p + k^{1-p}|b|^p$ for any $a, b \in \mathbb{R}$, $p \geqslant 1$, $k \in (0,1)$. Then,

$$|x(t) - D(x_t)|^p \geqslant \frac{|x(t)|^p - k_0^{1-p}|D(x_t)|^p}{(1-k_0)^{1-p}}. \tag{24}$$

Substituting (24) into (23) yields

$$\mathrm{E}|x(t)|^p \leqslant \frac{C(1-k_0)^{1-p}}{c_1} + k_0^{1-p}\mathrm{E}|D(x_t)|^p. \tag{25}$$

By Assumption 3 and inequality (25), we therefore obtain

$$\begin{aligned}
\sup_{0 \leqslant s \leqslant t} \mathrm{E}|x(s)|^p &\leqslant \frac{C(1-k_0)^{1-p}}{c_1} + k_0^{1-p} \sup_{0 \leqslant s \leqslant t} \mathrm{E}|D(x_s)|^p \\
&\leqslant \frac{C(1-k_0)^{1-p}}{c_1} + k_0 \sup_{0 \leqslant s \leqslant t} \sup_{-\tau \leqslant \theta \leqslant 0} \mathrm{E}|x(s+\theta)|^p \\
&\leqslant \frac{C(1-k_0)^{1-p}}{c_1} + k_0 \sup_{-\tau \leqslant s \leqslant t} \mathrm{E}|x(s)|^p \\
&\leqslant \frac{C(1-k_0)^{1-p}}{c_1} + k_0 \left(\mathrm{E}\|\xi\|^p + \sup_{0 \leqslant s \leqslant t} \mathrm{E}|x(s)|^p\right)
\end{aligned}$$

for any $t \geqslant 0$, which yields

$$\mathrm{E}|x(t)|^p \leqslant \sup_{0 \leqslant s \leqslant t} \mathrm{E}|x(s)|^p \leqslant \frac{\frac{C(1-k_0)^{1-p}}{c_1} + k_0 \mathrm{E}\|\xi\|^p}{1-k_0}$$

and the assertion (18) follows.

To prove the other assertion (16), we apply Itô formula to $V(t,x)$ directly:

$$\begin{aligned}
\mathrm{E}V(t \wedge \tau_k, x(t \wedge \tau_k) - D(x_{t \wedge \tau_k})) &= \mathrm{E}V(0, x(0) - D(x_0)) + \mathrm{E}\int_0^{t \wedge \tau_k} \mathcal{L}V(s, x_s)\mathrm{d}s \\
&\leqslant \mathrm{E}V(0, x(0) - D(x_0)) - \alpha_1 \mathrm{E}\int_0^{t \wedge \tau_k} V(s, x(s))\mathrm{d}s \\
&\quad + \alpha_2 \mathrm{E}\int_0^{t \wedge \tau_k} \int_{-\tau}^0 V(s+\theta, x(s+\theta))\mathrm{d}\mu(\theta)\mathrm{d}s \\
&\quad + \alpha_3 \mathrm{E}\int_0^{t \wedge \tau_k} \int_{-\tau}^0 H(s+\theta, x(s+\theta))\mathrm{d}\nu(\theta)\mathrm{d}s \\
&\quad - \mathrm{E}\int_0^{t \wedge \tau_k} H(s, x(s))\mathrm{d}s.
\end{aligned} \tag{26}$$

By Fubini theorem [15], we can compute

$$\begin{aligned}
\int_0^{t \wedge \tau_k} \int_{-\tau}^0 V(s+\theta, x(s+\theta))\mathrm{d}\mu(\theta)\mathrm{d}s &= \int_{-\tau}^0 \int_0^{t \wedge \tau_k} V(s+\theta, x(s+\theta))\mathrm{d}s\mathrm{d}\mu(\theta) \\
&\leqslant \int_{-\tau}^0 \left(\int_0^{t \wedge \tau_k} V(s, x(s))\mathrm{d}s\right)\mathrm{d}\mu(\theta) \\
&\leqslant \int_{-\tau}^{t \wedge \tau_k} V(s, x(s))\mathrm{d}s.
\end{aligned} \tag{27}$$

Similarly,

$$\int_0^{t\wedge\tau_k}\int_{-\tau}^0 H(s+\theta,x(s+\theta))\mathrm{d}\mu(\theta)\mathrm{d}s \leqslant \int_{-\tau}^{t\wedge\tau_k} H(s,x(s))\mathrm{d}s. \tag{28}$$

Substituting (27) and (28) into (26), we obtain

$$\begin{aligned}\mathrm{E}V(t\wedge\tau_k,x(t\wedge\tau_k)-D(x_{t\wedge\tau_k})) &\leqslant M+(\alpha_2-\alpha_1)\mathrm{E}\int_0^{t\wedge\tau_k} V(s,x(s))\mathrm{d}s\\ &\quad+(\alpha_3-1)\mathrm{E}\int_0^{t\wedge\tau_k} H(s,x(s))\mathrm{d}s\\ &\leqslant M+(\alpha_3-1)\mathrm{E}\int_0^{t\wedge\tau_k} H(s,x(s))\mathrm{d}s, \end{aligned}\tag{29}$$

where $M=\mathrm{E}(V(0,x(0)-D(x_0))+\alpha_2\int_{-\tau}^0 V(s,x(s))\mathrm{d}s+\alpha_3\int_{-\tau}^0 H(s,x(s))\mathrm{d}s)$. Because $\alpha_3\in(0,1)$ and $V(t,x)\geqslant c_1|x|^p\geqslant 0$, inequality (29) yields the assertion (19).

Remark 1. Compared with the other known results, such as [9,11,12], the right-hand side of inequality (16) is missing a positive constant. The reason is that our system is a neutral system, so we cannot obtain the same result if there is a positive constant on the right-hand side of inequality (16) with our current technique. That is, we can roughly explain that the positive constant can "compensate" for the neutral system.

We can obtain the upper bound of the pth moment of the solution from Theorem 2. Next, we prove that if there is a trivial solution, the boundedness of the pth moment of the solution implies that the trivial solution is almost surely exponentially stable.

Theorem 3. Let all the assumptions of Theorem 2 hold and $f(t,0)=h(u,0)=D(0)\equiv 0$ for $t\geqslant 0$, $u\in U$. Then the trivial solution to equation (1) is almost surely exponentially stable. That is, the unique global solution $x(t)$ has the property that

$$\limsup_{t\to\infty}\frac{1}{t}\log|x(t;0,\xi)|\leqslant 0 \quad \text{almost surely}$$

for all $\xi\in\mathbb{R}^n$.

Proof. If $f(t,0)=h(u,0)=D(0)\equiv 0$ for $t\geqslant 0$, $u\in U$, the equation admits a trivial solution $x(t;0)\equiv 0$ corresponding to the initial data $x_0=0$.

For simplify, we set $M\triangleq\frac{\frac{C(1-k_0)^{1-p}}{c_1}+k_0\mathrm{E}\|\xi\|^p}{1-k_0}$. Next, for each $n=1,2,\ldots$ and any $\epsilon>0$, it follows from the Markov inequality [16] and Theorem 2 that

$$P\{\omega:|x(t,\omega)|^p>\mathrm{e}^{\epsilon n}\}\leqslant\frac{\mathrm{E}|x(t)|^p}{\mathrm{e}^{\epsilon n}}\leqslant M\mathrm{e}^{-\epsilon n}$$

for any $t\in[n-1,n]$.

Because $\sum_{n=0}^{\infty}M\mathrm{e}^{-\epsilon n}<\infty$, by the Borel–Cantelli lemma [14], there is an integer n_0 such that

$$|x(t)|^p\leqslant\mathrm{e}^{\epsilon n}\quad\text{almost surely}$$

for all $n\geqslant n_0$ and $t\in[n-1,n]$. Then, we can obtain

$$\frac{1}{t}\log|x(t)|=\frac{1}{pt}\log|x(t)|^p\leqslant\frac{\epsilon n}{p(n-1)}. \tag{30}$$

Letting $n\to\infty$ on both sides of inequality (26),

$$\limsup_{t\to\infty}\frac{1}{t}\log|x(t)|\leqslant\frac{\epsilon}{p},$$

which is the required assertion because $\epsilon>0$ is arbitrary.

Theorem 4. Let Assumptions 1–3 hold except for (3), which is replaced by

$$\begin{aligned}\mathcal{L}V(t,\phi) \leqslant& \alpha - \alpha_1 V(t,\phi(0)) + \alpha_2 \int_{-\tau}^{0} V(t+\theta,\phi(\theta))\mathrm{d}\mu(\theta) \\ &- H(t,\phi(0)) + \alpha_3 \int_{-\tau}^{0} H(t+\theta,\phi(\theta))\mathrm{d}\nu(\theta) - \alpha_4 V(t,\phi(0)-D(\phi)),\end{aligned} \tag{31}$$

where α is a constant, and $\alpha_1 > \alpha_2 \geqslant 0$, $\alpha_3 \in (0,1)$, $\alpha_4 > 0$. Then for any given initial data ξ, there is a unique global solution $x(t)$ to (1) that has the property that

$$\limsup_{t\to\infty} \mathrm{E}|x(t)|^p \leqslant \frac{\alpha}{\varepsilon c_1(1-k_0)^p} + \frac{k_0 \mathrm{E}\|\xi\|^p}{1-k_0} \tag{32}$$

for any $t \geqslant 0$.

Here ξ is the initial data and $\varepsilon = \varepsilon_1 \wedge \varepsilon_2 \wedge \alpha_4$. In addition, $\varepsilon_1, \varepsilon_2$ satisfy

$$\alpha_1 - \alpha_2 \mathrm{e}^{\varepsilon_1 \tau} = 0, \quad 1 - \alpha_3 \mathrm{e}^{\varepsilon_2 \tau} = 0, \tag{33}$$

respectively.

If $f(t,0) = h(u,0) = D(0) \equiv 0$ for $t \geqslant 0$, $u \in U$, the trivial solution to (1) is almost surely exponentially stable. That is, the unique global solution $x(t)$ has the property that

$$\limsup_{t\to\infty} \frac{1}{t} \log|x(t;0,\xi)| \leqslant 0 \ \text{ almost surely} \tag{34}$$

for all $\xi \in \mathbb{R}^n$.

Proof. Similar to the proof of Theorem 2, we obtain

$$\begin{aligned}&\mathrm{E}(\mathrm{e}^{\varepsilon(t\wedge\tau_k)} V(t\wedge\tau_k, x(t\wedge\tau_k) - D(x_{t\wedge\tau_k}))) - \mathrm{E}V(0, x(0) - D(x_0)) \\ &= \mathrm{E}\int_0^{t\wedge\tau_k} \varepsilon \mathrm{e}^{\varepsilon s} V(s, x(s) - D(x_s))\mathrm{d}s + \mathrm{E}\int_0^{t\wedge\tau_k} \mathrm{e}^{\varepsilon s} \mathcal{L}V(s, x_s)\mathrm{d}s \\ &\leqslant \mathrm{E}\int_0^{t\wedge\tau_k} \mathrm{e}^{\varepsilon s} V(s, x(s))\mathrm{d}s + \mathrm{E}\int_0^{t\wedge\tau_k} \alpha \mathrm{e}^{\varepsilon s}\mathrm{d}s \\ &\quad - \alpha_1 \mathrm{E}\int_0^{t\wedge\tau_k} \mathrm{e}^{\varepsilon s} V(s, x(s))\mathrm{d}s + \alpha_2 \mathrm{E}\int_0^{t\wedge\tau_k}\int_{-\tau}^{0} \mathrm{e}^{\varepsilon s} V(s+\theta, x(s+\theta))\mathrm{d}\mu(\theta)\mathrm{d}s \\ &\quad - \mathrm{E}\int_0^{t\wedge\tau_k} \mathrm{e}^{\varepsilon s} H(s, x(s))\mathrm{d}s + \alpha_3 \mathrm{E}\int_0^{t\wedge\tau_k}\int_{-\tau}^{0} \mathrm{e}^{\varepsilon s} H(s+\theta, x(s+\theta))\mathrm{d}\nu(\theta)\mathrm{d}s \\ &\quad - \alpha_4 \mathrm{E}\int_0^{t\wedge\tau_k} \mathrm{e}^{\varepsilon s} V(s, x(s) - D(x_s))\mathrm{d}s \\ &\leqslant \alpha_2 \mathrm{e}^{\varepsilon\tau} \mathrm{E}\int_{-\tau}^{0} V(s, x(s))\mathrm{d}s + \alpha_3 \mathrm{e}^{\varepsilon\tau} \mathrm{E}\int_{-\tau}^{0} H(s, x(s))\mathrm{d}s \\ &\quad + \alpha \frac{\mathrm{e}^{\varepsilon t}}{\varepsilon} - (\alpha_4 - \varepsilon)\mathrm{E}\int_0^{t\wedge\tau_k} \mathrm{e}^{\varepsilon s} V(s, x(s) - D(x_s))\mathrm{d}s \\ &\quad - (\alpha_1 - \alpha_2 \mathrm{e}^{\varepsilon\tau})\mathrm{E}\int_0^{t\wedge\tau_k} \mathrm{e}^{\varepsilon s} V(s, x(s))\mathrm{d}s - (1 - \alpha_3 \mathrm{e}^{\varepsilon\tau})\mathrm{E}\int_0^{t\wedge\tau_k} \mathrm{e}^{\varepsilon s} H(s, x(s))\mathrm{d}s \\ &\leqslant \alpha_2 \mathrm{e}^{\varepsilon\tau} \mathrm{E}\int_{-\tau}^{0} V(s, x(s))\mathrm{d}s + \alpha_3 \mathrm{e}^{\varepsilon\tau} \mathrm{E}\int_{-\tau}^{0} H(s, x(s))\mathrm{d}s + \alpha \frac{\mathrm{e}^{\varepsilon t}}{\varepsilon}.\end{aligned}$$

Taking $k \to \infty$ on both sides, we obtain

$$\mathrm{E}(\mathrm{e}^{\varepsilon t} V(t, x(t) - D(x_t)) \leqslant R + \frac{\alpha \mathrm{e}^{\varepsilon t}}{\varepsilon}, \tag{35}$$

where $R = \mathrm{E}V(0, x(0) - D(x_0)) + \mathrm{e}^{\varepsilon\tau}\mathrm{E}\int_{-\tau}^{0} V(s, x(s))\mathrm{d}s + \alpha_3\mathrm{e}^{\varepsilon\tau}\mathrm{E}\int_{-\tau}^{0} H(s, x(s))\mathrm{d}s$ is a constant. By inequality (35), we can compute

$$\mathrm{E}V(t, x(t) - D(x_t)) \leqslant R\mathrm{e}^{-\varepsilon t} + \frac{\alpha}{\varepsilon}. \tag{36}$$

Using the following two inequalities, Assumption 3 and inequality (35),

$$c_1|x(t) - D(x_t)|^p \leqslant V(t, x(t) - D(x_t)),$$
$$|x(t)|^p \leqslant (1 - k_0)^{1-p}|x(t) - D(x_t)|^p + k_0^{1-p}|D(x_t)|^p,$$

we can derive

$$\begin{aligned} \mathrm{E}|x(t)|^p &\leqslant \frac{(1-k_0)^{1-p}}{c_1}\mathrm{E}V(t, x(t) - D(x_t)) + k_0^{1-p}\mathrm{E}|D(x_t)|^p \\ &\leqslant \frac{(1-k_0)^{1-p}}{c_1}\left(R\mathrm{e}^{-\varepsilon t} + \frac{\alpha}{\varepsilon}\right) + k_0 \sup_{-\tau\leqslant\theta\leqslant 0} \mathrm{E}|x(t+\theta)|^p \end{aligned}$$

for any $t \geqslant 0$. Then

$$\begin{aligned} \sup_{0\leqslant s\leqslant t} \mathrm{E}|x(s)|^p &\leqslant \frac{(1-k_0)^{1-p}}{c_1}\left(R\mathrm{e}^{-\varepsilon t} + \frac{\alpha}{\varepsilon}\right) + k_0 \sup_{0\leqslant s\leqslant t}\ \sup_{-\tau\leqslant\theta\leqslant 0} \mathrm{E}|x(s+\theta)|^p \\ &\leqslant \frac{(1-k_0)^{1-p}}{c_1}\left(R\mathrm{e}^{-\varepsilon t} + \frac{\alpha}{\varepsilon}\right) + k_0 \sup_{-\tau\leqslant s\leqslant t} \mathrm{E}|x(s)|^p \\ &\leqslant \frac{(1-k_0)^{1-p}}{c_1}\left(R\mathrm{e}^{-\varepsilon t} + \frac{\alpha}{\varepsilon}\right) + k_0\mathrm{E}\|\xi\|^p + k_0 \sup_{0\leqslant s\leqslant t} \mathrm{E}|x(s)|^p, \end{aligned} \tag{37}$$

and inequality (37) implies

$$\mathrm{E}|x(t)|^p \leqslant \sup_{0\leqslant s\leqslant t} \mathrm{E}|x(s)|^p \leqslant \frac{R\mathrm{e}^{-\varepsilon t} + \frac{\alpha}{\varepsilon}}{c_1(1-k_0)^p} + \frac{k_0\mathrm{E}\|\xi\|^p}{1-k_0}.$$

Consequently, we can obtain

$$\limsup_{t\to\infty} \mathrm{E}|x(t)|^p \leqslant \frac{\alpha}{\varepsilon c_1(1-k_0)^p} + \frac{k_0\mathrm{E}\|\xi\|^p}{1-k_0},$$

which is the assertion (31). The proof of the assertion (34) is similar to the proof of Theorem 3.

Remark 2. Note that the right-hand side of inequality (31) contains a constant α. That is because the right-hand side of the inequality also includes another negative term $-\alpha_4 V(t, \phi(0) - D(\phi))$, which can "compensate" for the constant. Moreover, if $\alpha \leqslant 0$, we can obtain that the pth moment of the solution is stable.

5 Examples

Example 1. Consider the following equation

$$\mathrm{d}\left[x(t) - \frac{1}{2}\int_{-1}^{0} x(t+\theta)\mathrm{d}\theta\right] = \left(-\frac{1}{2}x^3(t) + t\int_{-1}^{0} x(t+\theta)\mathrm{d}\theta\right)\mathrm{d}t + \sqrt{t}\int_{-1}^{0} x(t+\theta)\mathrm{d}\theta\mathrm{d}N(t). \tag{38}$$

In this example, we set $\tau = 1, U = \{1\}$. Define

$$D(\varphi) = \frac{1}{2}\int_{-1}^{0} \phi(\theta)\mathrm{d}\theta,$$
$$f(t, \varphi) = -\frac{1}{2}x^3(t) + t\int_{-1}^{0} \varphi(\theta)\mathrm{d}\theta,$$
$$h(1, \varphi) = \sqrt{t}\int_{-1}^{0} \varphi(\theta)\mathrm{d}\theta.$$

Choose $V(t,x)=x^2$, by Itô formula and Hölder inequality [14], we can compute

$$\begin{aligned}
&\mathcal{L}V(t,\varphi)\\
&=2\left(\varphi(0)-\frac{1}{2}\int_{-1}^{0}\phi(\theta)\mathrm{d}\theta\right)\left(-\frac{1}{2}\varphi^3(0)+t\int_{-1}^{0}\varphi(\theta)\mathrm{d}\theta\right)+\left(\varphi(0)-\frac{1}{2}\int_{-1}^{0}\phi(\theta)\mathrm{d}\theta+\sqrt{t}\int_{-1}^{0}\varphi(\theta)\mathrm{d}\theta\right)^2\\
&\quad-\left(\varphi(0)-\frac{1}{2}\int_{-1}^{0}\phi(\theta)\mathrm{d}\theta\right)^2-2\sqrt{t}\left(\varphi(0)-\frac{1}{2}\int_{-1}^{0}\phi(\theta)\mathrm{d}\theta\right)\int_{-1}^{0}\varphi(\theta)\mathrm{d}\theta\\
&=\frac{1}{2}t\left(\int_{-1}^{0}\phi(\theta)\mathrm{d}\theta\right)^2+\frac{1}{4}\phi^3(0)\int_{-1}^{0}\phi(\theta)\mathrm{d}\theta+2t\phi(0)\int_{-1}^{0}\phi(\theta)\mathrm{d}\theta-\phi^4(0)\\
&\leqslant\frac{3}{2}t\left[\varphi^2(0)+\int_{-1}^{0}\varphi^2(\theta)\mathrm{d}\theta\right]-\frac{1}{16}\phi^4(0)+\frac{1}{16}\int_{-1}^{0}\varphi^4(\theta)\mathrm{d}\theta.
\end{aligned}$$

Taking $K(t)=\frac{3}{2}t, H(t,x)=\frac{1}{2}x$, we can conclude that Eq. (38) has a unique global solution for any given initial data by Theorem 1.

Remark 3. Example 1 has checked the conclusion of Theorem 1. The following example mainly shows the effectiveness of Theorem 2.

Example 2. Let us consider the equation

$$\mathrm{d}\left[x(t)-\frac{2}{3}\int_{-1}^{0}x(t+\theta)\mathrm{d}\theta\right]=(-x(t)-x^3(t))\mathrm{d}t+\frac{1}{2}\int_{-1}^{0}x(t+\theta)\mathrm{d}\theta\mathrm{d}N(t). \tag{39}$$

In this example, we set $U=\{1\}$ and define

$$\begin{aligned}
D(\varphi)&=\frac{2}{3}\int_{-1}^{0}\varphi(\theta)\mathrm{d}\theta,\\
F(t,\varphi)&=-\varphi(0)-\varphi^3(0),\\
h(1,\varphi)&=\frac{1}{2}\int_{-1}^{0}\varphi(\theta)\mathrm{d}\theta
\end{aligned}$$

for $t\geqslant 0$, $\varphi\in D([-1,0];\mathbb{R})$. Then the coefficients satisfy Assumption 1. If we choose $p=2$, the Hölder inequality [14] yields

$$\mathrm{E}|D(\varphi)|^2=\frac{4}{9}\mathrm{E}\left|\int_{-1}^{0}\varphi(\theta)\mathrm{d}\theta\right|^2\leqslant\frac{4}{9}\sup_{-1\leqslant\theta\leqslant 0}\mathrm{E}|\varphi(\theta)|^2,$$

which implies Assumption 3.

Define $V(x)=|x|^2$ and employing Itô formula, Young inequality [17] and Hölder inequality [14], we can compute

$$\begin{aligned}
\mathcal{L}V(t,\varphi)&=2\left(\varphi(0)-\frac{2}{3}\int_{-1}^{0}\varphi(\theta)\mathrm{d}\theta\right)(-\varphi(0)-\varphi^3(0))\\
&\quad+\left(\varphi(0)-\frac{2}{3}\int_{-1}^{0}\varphi(\theta)\mathrm{d}\theta+\frac{1}{2}\int_{-1}^{0}\varphi(\theta)\mathrm{d}\theta\right)^2-\left(\varphi(0)-\frac{2}{3}\int_{-1}^{0}\varphi(\theta)\mathrm{d}\theta\right)^2\\
&\quad-\left(\varphi(0)-\frac{2}{3}\int_{-1}^{0}\varphi(\theta)\mathrm{d}\theta\right)\int_{-1}^{0}\varphi(\theta)\mathrm{d}\theta\\
&\leqslant-\frac{4}{3}|\varphi(0)|^2+\frac{11}{12}\left|\int_{-1}^{0}\varphi(\theta)\mathrm{d}\theta\right|^2-|\varphi(0)|^4+\frac{1}{3}\left|\int_{-1}^{0}\varphi(\theta)\mathrm{d}\theta\right|^4\\
&\leqslant-\frac{4}{3}\varphi^2(0)+\frac{11}{12}\int_{-1}^{0}\varphi^2(\theta)\mathrm{d}\theta-\varphi^4(0)+\frac{1}{3}\int_{-1}^{0}\varphi^4(\theta)\mathrm{d}\theta.
\end{aligned}$$

Applying Theorem 1, if we choose $H(t,x)=|x|^4$, we can obtain that there is a unique global solution for any given initial data. The conditions in Theorem 2 are satisfied when we choose $\alpha_1=\frac{4}{3},\alpha_2=\frac{11}{12},\alpha_3=\frac{1}{3}$. Therefore, if we take initial data $\xi(\theta)=\theta+1$, $-1\leqslant\theta\leqslant 0$, we can compute $C=0.986$ and $\mathrm{E}|x(t)|^2\leqslant 4.194$.

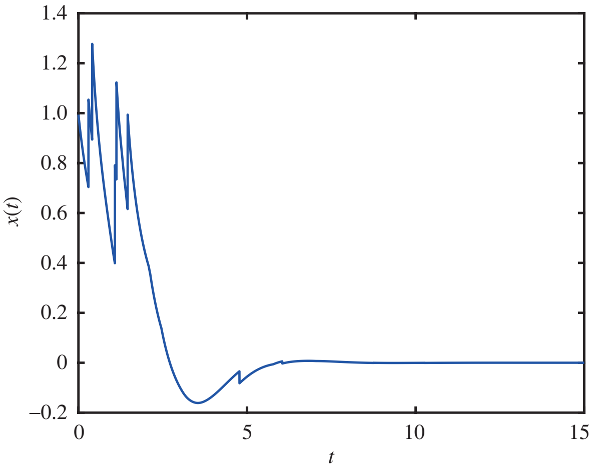

Figure 1 (Color online) Trajectory of state.

In particular, the trivial solution $x(t) \equiv 0$ of (39) is almost surely exponentially stable in theory. Moreover, it can be seen from Figure 1 that the trivial solution is stable.

6 Conclusion

In this paper, the existence and uniqueness of the solution to the NSFDEwPJs under the local Lipschitz condition and the Khasminskii-type condition has been solved. The linear growth condition was not required, so we could deal with the problems that the coefficients of the equation are high order. Moreover, we have obtained the boundedness of the pth moment of the solution. The almost surely exponential stability has also been proved in this paper.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 61573156, 61273126, 61503142), the Ph.D. Start-up Fund of Natural Science Foundation of Guangdong Province (Grant No. 2014A030310388), and Fundamental Research Funds for the Central Universities (Grant No. x2zdD2153620).

References

1 Wu L, Zheng W X, Gao H. Dissipativity-based sliding mode control of switched stochastic systems. IEEE Trans Automat Contr, 2013, 58: 785–791

2 Zhang B, Deng F Q, Zhao X Y, et al. Hybrid control of stochastic chaotic system based on memristive Lorenz system with discrete and distributed time-varying delays. IET Contr Theor Appl, 2016, 10: 1513–1523

3 Wu D, Luo X, Zhu S. Stochastic system with coupling between non-Gaussian and Gaussian noise terms. Phys A-Stat Mech Appl, 2007, 373: 203–214

4 Shao J, Yuan C. Transportation-cost inequalities for diffusions with jumps and its application to regime-switching processes. J Math Anal Appl, 2015, 425: 632–654

5 Elliott R J, Osakwe C J U. Option pricing for pure jump processes with markov switching compensators. Finance Stochast, 2006, 10: 250–275

6 Lee S S, Mykland P A. Jumps in financial markets: a new nonparametric test and jump dynamics. Rev Financ Stud, 2008, 21: 2535–2563

7 Mao W, Zhu Q, Mao X. Existence, uniqueness and almost surely asymptotic estimations of the solutions to neutral stochastic functional differential equations driven by pure jumps. Appl Math Comput, 2015, 254: 252–265

8 Agarwal R P. Editorial announcement. J Inequal Appl, 2011, 2011: 1

9 Song M H, Hu L J, Mao X R, et al. Khasminskii-type theorems for stochastic functional differential equations. Discrete Cont Dyn Syst - Ser B, 2013, 18: 1697–1714

10 Luo Q, Mao X, Shen Y. Generalised theory on asymptotic stability and boundedness of stochastic functional differential equations. Automatica, 2011, 47: 2075–2081

11 Wu F, Hu S. Khasminskii-type theorems for stochastic functional differential equations with infinite delay. Stat Probab Lett, 2011, 81: 1690–1694

12 Mao X, Rassias M J. Khasminskii-type theorems for stochastic differential delay equations. Stochastic Anal Appl, 2005, 23: 1045–1069

13 Applebaum D. Lévy Processes and Stochastic Calculus. Cambridge: Cambridge University Press, 2009

14 Mao X R. Stochastic Differential Equations and Applications. 2nd ed. Cambridge: Woodhead Publishing, 2008

15 Meyer P A. Probability and Potentials. Waltham: Blaisdell, 1966

16 Pawłucki W, Pleśniak W. Markov's inequality and C^∞ functions on sets with polynomial cusps. Math Ann, 1986, 275: 467–480

17 Beckner W. Inequalities in fourier analysis. Ann Math, 1975, 102: 159–182