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Performance analysis of switched linear systems under arbitrary switching via generalized coordinate transformations

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Abstract For a continuous-time switched linear system, the spectral abscissa is defined as the worst-case divergence rate under arbitrary switching, which is critical for characterizing the asymptotic performance of the switched system. In this study, based on the generalized coordinate transformations approach, we develop a computational scheme that iteratively produces sequences of minimums of matrix set μ_1 measures, where the limits of the sequences are upper bound estimates of the spectral abscissa. A simulation example is presented to illustrate the effectiveness of the proposed scheme.

Keywords generalized coordinate transformation, matrix set measure, spectral abscissa, switched linear system

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1 Introduction

In this study, we focus on the continuous-time switched linear system given by

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad x(t) \in \mathbb{R}^n, \tag{1}$$

where $\sigma(t)$ is a switching signal that takes values from the index set $\{1,\dots,\kappa\}$ and $A_i \in \mathbb{R}^{n\times n}$ are real constant matrices for $i \in \{1,\dots,\kappa\}$.

Interest in the study of switched systems has grown due to their success in various applications [1–3] and their theoretical importance [4–6] (e.g., see [7–15], and the references therein).

A primary issue that affects switched systems is the guaranteed stability, i.e., asymptotic stability under arbitrary switching. Remarkable progress has been made in planar switched linear systems where the "most destabilizing" phase portrait could be constructed to provide verifiable stability criteria [16–18]. For general switched linear systems of higher order, the common Lyapunov function method played a major role in stability analysis [10, 11, 19]. As quadratic Lyapunov functions are not sufficient for characterizing stability [20], larger candidate functional sets, for instance, the set of piecewise linear/quadratic functions and/or the set of (sum-of-squares) polynomials, are identified for providing universal common Lyapunov

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functions [14,19], where effective search seems to be an intractable problem. To overcome the limitation of the common Lyapunov function method, the multiple Lyapunov function approach was proposed [21,22], which makes the search for a Lyapunov candidate easier and more flexible. However, a constructive and tractable algorithm is still unavailable in the general case.

Inspired by the achievements in spectral radius analysis for discrete-time switched linear systems [23, 24], some researchers have developed an algebraic approach for quantitatively characterizing the asymptotic performance to substitute for the qualitative stability scheme. On the other hand, for continuous-time switched linear systems, by introducing the notion of the spectral abscissa of matrix sets, it was proved that the spectral abscissa is exactly the worst-case divergence rate, i.e., the top Lyapunov exponent of the switched system [11]. It was established that, for any switched linear system (1), a positive real number c and a non-negative integer $k \leqslant n-1$ exist such that

$$\|x(t)\| \leqslant c(1+t^k)\mathrm{e}^{\varrho t}\|x(0)\|$$

for any system solution $x(\cdot)$, where ϱ is the spectral abscissa of the matrix set $\{A_1, \ldots, A_\kappa\}$ [25,26]. The estimate is tight in the sense that

$$\lim_{t\to\infty} \sup \frac{\|x(t)\|}{(1+t^k)\mathrm{e}^{\varrho t}\|x(0)\|} > 0, \quad \text{for almost all } x(0) \in \mathbb{R}^n,$$

where the superior is taken along all possible switching signals. Therefore, the spectral abscissa is important for characterizing both the transient and the asymptotic performances of continuous-time switched linear systems. It was proved that the spectral abscissa is equal to the least common (matrix) measure of matrix set over all possible matrix measures [27,28]. This equity provides a powerful tool for computing the spectral abscissa based on the matrix set measure scheme. In addition to this scheme, several computational algorithms have been proposed to approximate the least common measure of a matrix set [29, 30]. In particular, as shown in Lemma 2, by using proper coordinate transformations, the least common measure over all possible matrix measures can be approximated by the least common μ_1 measure of the transformed matrix set. In our previous study [30], we examined the case of square transformations by decomposing a non-singular matrix into a multiplication of elementary matrices. However, this idea cannot be extended to the case of non-square matrix transformations. For a class of third-order switched linear systems, we also proposed an approach for approximating the spectral abscissa based on 3×4 coordinate transformations [31]. In the present study, we extend this approach to the case where the generalized transformation matrix is of dimensions $n \times (n+1)$. A key idea here is that the transformation matrices are properly represented so we can make full use of the free parameters in the augmented dimension in order to decrease the μ_1 measure step by step. By performing the transformations iteratively, we obtain sequences of minimums of μ_1 measures that are convergent, where the limits are upper bound estimates of the spectral abscissa.

2 Preliminaries

For a positive integer k, let $\overline{k} = \{1, \ldots, k\}$. Let I_n be the $n \times n$ identity matrix. Λ denotes the set of non-trivial solutions of system (1).

Definition 1. For the switched linear system (1), the spectral abscissa is

$$\varrho(\mathcal{A}) = \limsup_{t\to\infty, x(\cdot)\in\Lambda} \frac{\ln\|x(t)\| - \ln\|x(0)\|}{t},$$

where $\mathcal{A} = \{A_1, \ldots, A_\kappa\}$.

Note that the spectral abscissa in Definition 1 is also referred to as the top Lyapunov exponent or the largest divergence rate in previous studies [26, 32–35].

Let $\lambda(\mathcal{A})$ be the largest real part of the eigenvalues of matrices A_i, $i = 1, \ldots, \kappa$. Clearly, $\lambda(\mathcal{A})$ is a lower bound of the spectral abscissa, i.e., $\lambda(\mathcal{A}) \leqslant \varrho(\mathcal{A})$ [36].

Definition 2 ([37]). For any vector norm $v(\cdot)$ in $\mathbb{R}^n$, the induced matrix measure of $A \in \mathbb{R}^{n\times n}$ is

$$\mu_v(A) = \limsup_{\tau\to 0^+, 0\neq z\in\mathbb{R}^n} \frac{v(z+\tau Az)-v(z)}{\tau v(z)}.$$

It is well known that for a matrix $A = (a_{ij})_{n\times n}$, the induced matrix measure of norm ℓ_1, which denotes the $\mu_1(A)$ measure for simplicity, is the maximum of the column sum

$$\mu_1(A) = \max_{j\in\overline{n}} \psi_j(A), \tag{2}$$

where the j-th column sum is defined as

$$\psi_j(A) = a_{jj} + \sum_{i=1, i\neq j}^{n} |a_{ij}|. \tag{3}$$

The definition of a matrix measure can be extended to a set of matrices $\mathcal{A} = \{A_1, \ldots, A_\kappa\}$, which is defined as Definition 3.

Definition 3 ([11]). For any vector norm $v(\cdot)$ in $\mathbb{R}^n$ and a given set of matrices $\mathcal{A} = \{A_1, A_2, \ldots, A_\kappa\}$, the induced matrix set measure is defined as

$$\mu_v(\mathcal{A}) = \max\{\mu_v(A_1), \ldots, \mu_v(A_\kappa)\}.$$

The least measure is defined as

$$\mu_*(\mathcal{A}) = \inf_{v\in\mathcal{V}} \mu_v(\mathcal{A}),$$

where $\mathcal{V}$ is the set of vector norms.

For any norm v, we have the inequality

$$v(x(t)) \leqslant \mathrm{e}^{\mu_v(\mathcal{A})t} v(x(0)), \quad \forall\, x(\cdot)\in\Lambda,\ t\geqslant 0.$$

It follows that $\mu_v(\mathcal{A})$ is an upper bound of the spectral abscissa, i.e., $\mu_v(\mathcal{A}) \geqslant \varrho(\mathcal{A})$. Furthermore, Lemma 1 establishes that the least matrix set measure is exactly the spectral abscissa.

Lemma 1 ([27, 28]). For any matrix set $\mathcal{A}$, we have

$$\mu_*(\mathcal{A}) = \varrho(\mathcal{A}).$$

Lemma 2, which is an extension of Theorem 4.1 given by [38], indicates that the least measure can be approximated at arbitrary precision by the μ_1 measure of matrices obtained by generalized coordinate transformations.

Lemma 2 ([30]). For any matrix set $\mathcal{A}$ and any $\epsilon > 0$, a natural number $r \geqslant n$, matrix $T_{n\times r}$ of rank n, and $r\times r$ matrices H_i exist such that

$$A_i T = T H_i, \quad i = 1, 2, \ldots, \kappa, \tag{4}$$

and

$$\mu_1(H_1, \ldots, H_\kappa) < \mu_*(\mathcal{A}) + \epsilon.$$

Remark 1. As discussed by [38], Lemma 2 requires the identification of a polytope with n linear constraints that approximates the level set $\{x : v(x) \leqslant \varepsilon\}$, where v is a norm that induces a measure near the least measure and ε is a sufficiently small positive real number. Obviously, when the polytope has more facets, i.e., r is large, the approximation could be more accurate. Therefore, the upper bound estimate of the spectral abscissa obtained by generalized transformations here is sharper than that obtained by square transformations.

Lemma 3 is a classic result of the matrix equation, which plays a very important role in this study.

Lemma 3 ([39, 40])**.** The matrix equation

$$A_{n\times n}Y_{n\times m} = Y_{n\times m}P_{m\times m} \tag{5}$$

with $A \in \mathbb{R}^{n\times n}$ and $Y \in \mathbb{R}^{n\times m}$ can be represented as

$$Gy = 0, \tag{6}$$

where $G = I_m \bigotimes A - P^{\mathrm{T}} \bigotimes I_n$, and $y = [y_{11}, \dots, y_{n1}, \dots, y_{1m}, \dots, y_{nm}]^{\mathrm{T}}$.

Remark 2. For a given matrix $A \in \mathbb{R}^{n\times n}$, let $m = n+1$, $P = (p_{ij})_{m\times m}$ and $Y = (y_{ij})_{n\times m}$. According to Lemma 3, for matrix equation (5), we can obtain a family of mn equations. For these equations, we treat y_{ij} as constants and then determine whether numbers p_{ij}, $i, j = 1, \dots, m$ exist that satisfy each of the equations simultaneously. We note that there are m^2 unknowns but mn equations, so we have m free independent parameters, which are denoted by p_j, $j = 1, \dots, m$, and these mn dependent unknowns p_{ij} can then be expressed in terms of the remaining independent parameters p_j, $j = 1, \dots, m$. In particular, the elements in the same column share the same independent parameter, i.e., p_{ij} are functions of y and p_j. $p_1, \dots, p_m$ are free independent parameters that need to be optimized, so we write $p = [p_1, \dots, p_m]^{\mathrm{T}}$ and $P = P(y, p)$.

Remark 3. Define $A(j)$, $j \in \overline{n}$ as the j-th column vector of $A \in \mathbb{R}^{n\times n}$. Observe that $Y_{n\times m} = (Z_1)_{n\times n}(Z_2)_{n\times m}$, $n \leqslant m$, where $Z_1 = (Y(1), \dots, Y(n))$, and $Z_2 = (I_n, Z_2(n+1), \dots, Z_2(m))$. If Z_1 is invertible, it holds that $AY = YP$ is equivalent to $Z_1^{-1}AZ_1Z_2 = Z_2P$. Therefore, the above coordinate transformation can be treated as a composition of the square transformation $AZ_1 = Z_1\hat{P}$, as shown in [30], and the generalized transformation $\hat{P}Z_2 = Z_2P$; otherwise, we suppose that $\mathrm{rank}(Z_1) = n-1$. We can first reduce the $n\times n$ system matrix to an $(n-1)\times(n-1)$ system matrix denoted by $\tilde{A}$ and then solve $\tilde{A}\tilde{Y} = \tilde{Y}\tilde{P}$ with $\tilde{Y} = (\tilde{Z}_1)_{(n-1)\times(n-1)}(\tilde{Z}_2)_{(n-1)\times(m-1)}$ and $\mathrm{rank}(\tilde{Z}_1) = n-1$, which can be treated as a combination of square and generalized transformations, as discussed above. Therefore, we focus on the case where the transformation matrix is of the form Z_2.

3 Calculation of the least μ_1 measure

Next, we search for a proper generalized transformation matrix $T \in \mathbb{R}^{n\times(n+1)}$ such that

$$A_iT = TP_i^*, \quad i = 1, 2, \dots, \kappa,$$

where $T = (I_n, z)$, $z = [z_1, \dots, z_n]^{\mathrm{T}} \in \mathbb{R}^n$, $z_i, i \in \overline{n}$ are unknown scalars that need to be determined, and we then examine the property of the least μ_1 measure of the generalized transformed matrices P_i^*. For clarity, we focus on the switched systems with two subsystems, i.e., $\mathcal{A} = \{A_1, A_2\}$ with $A_1 = (a_{ij})_{n\times n}$, $A_2 = (b_{ij})_{n\times n}$.

By Remark 2, we can verify that the matrices P_1^* and P_2^* are

$$P_1^* = P_1^*(z, p) \quad \text{and} \quad P_2^* = P_2^*(z, q),$$

where $p = [p_1, \dots, p_{n+1}]^{\mathrm{T}} \in \mathbb{R}^{n+1}$, $q = [q_1, \dots, q_{n+1}]^{\mathrm{T}} \in \mathbb{R}^{n+1}$, and p_j and q_j are free parameters of the elements in the j-th column of matrices $P_1^*(z, p)$ and $P_2^*(z, q)$ for $j \in \overline{n+1}$, respectively. Therefore, by using (2), the problem of obtaining the least μ_1 measure of the matrices $P_1^*(z, p)$ and $P_2^*(z, q)$ can be mathematically described by

$$\begin{aligned} &\inf_T \max_{j\in\overline{n+1}} \{\psi_j(P_1^*(z, p)), \psi_j(P_2^*(z, q))\} \\ &\text{s.t. } A_1T = TP_1^*(z, p), \\ &\qquad A_2T = TP_2^*(z, q). \end{aligned} \tag{7}$$

Note that $\mathrm{rank}(T) = n$, so the largest real part of the eigenvalues of $P_1^*(z, p)$ and $P_2^*(z, q)$ is not less than that of A_1 and A_2. Combining this with Theorem 2.11 in [41] indicates that

$$\mu_1(P_1^*(z, p), P_2^*(z, q)) \geqslant \lambda(\mathcal{A}). \tag{8}$$

Hence, the optimal function value for problem (7) exists. An intrinsic difficulty when trying to solve problem (7) is the fact that the function $\mu_1(P_1^*(z,p),P_2^*(z,q))$ is discontinuous and non-convex over the free parameters p_j, q_j, $j\in\overline{n+1}$. To address this problem, we propose an iterative scheme for calculating the optimal value, as shown in Algorithm 1.

Algorithm 1 Iterative procedure for solving problem (7).

Initiation

Step 1. Find a pair $(j_\sharp, r_\sharp)\in(\overline{n},\overline{2})$, such that $\mu_1(A_1,A_2)=\psi_{j_\sharp}(A_{r_\sharp})$. Let $i:=j_\sharp$ and $u_i:=[0,\dots,0]^{\rm T}\in\mathbb{R}^n$.

Recursion

Step 2. Set $k:=\mathrm{mod}(i,n)$. If $k=0$, then $k:=n$. Set

$$b_i(z_i):=[0,\dots,z_i,\dots,0]^{\rm T}\in\mathbb{R}^n, \tag{9}$$

where the k-th entry is z_i and $T_i(z_i):=(I_n,u_i+b_i(z_i))\in\mathbb{R}^{n\times(n+1)}$.

Step 3. Solve the following problem

$$\begin{aligned}&\min_{z_i}\max_{j\in\overline{n+1}}\{\psi_j(P_1^i(z_i,p)),\psi_j(P_2^i(z_i,q))\}\\ &\text{s.t. } A_1T_i(z_i)=T_i(z_i)P_1^i(z_i,p),\\ &\qquad A_2T_i(z_i)=T_i(z_i)P_2^i(z_i,q)\end{aligned} \tag{10}$$

and obtain $z_i=d_i$.

Step 4. Update $u_{i+1}:=u_i+b_i(d_i)$, set $i:=i+1$, and go to Step 2.

Remark 4. According to Step 1, the first transformation is performed on the $j_\sharp$-th column, which satisfies $\mu_1(A_1,A_2)=\max_{r=1,2}\psi_{j_\sharp}(A_r)$.

Remark 5. As discussed in Proposition 1, in some cases, the procedure could produce a repeated output so we need to stop the procedure accordingly. Thus, the procedure is stopped in a finite number of steps; otherwise, let $d_i(k_j)$ be d_i with $k=j$. Based on Proposition 1, we can verify that the sequence $\{h_i\}$ obtained by Procedure 1 is convergent if the sequence $\{d_i(k_j)\}$ is convergent to 0 for each $j\in\{1,\dots,n\}$. Therefore, in the simulation, for a given optimality tolerance, the procedure is stopped when $|d_i(k_j)|\leqslant\epsilon$, $j=1,\dots,n$, where ϵ is a sufficiently small positive constant.

The procedure is designed such that the minimum of the matrix set measure, i.e., the optimal value for problem (10), is obtained at each iteration. In order to solve the critical problem (10) in Step 3, we first state the following two results.

Lemma 4. Let $\varphi(p_1)=p_1+\sum_s|a_s+b_sp_1|+\sum_i|c_i+\frac{e_ip_1}{z_1}|$, where a_s, b_s, c_i, e_i are given constants, z_1 is a constant with unknown value, $z_1\neq0$, and p_1 is a variable. The problem $\inf_{p_1\in\mathbb{R}}\varphi(p_1)$ can be solved by selecting p_1 as non-differentiable points of $\varphi(p_1)$ or $p_1=-\infty$.

Proof. It can be seen that $\varphi(p_1)$ is continuous with a finite number of non-differentiable points for $p_1\in\mathbb{R}$. All the non-differential points of $\varphi(p_1)$ are $p_1=-\frac{a_s}{b_s}$ and $p_1=-\frac{c_iz_1}{e_i}$, denoted by w_r. We rearrange them in order such that $w_1\leqslant w_2\leqslant\cdots\leqslant w_{n_1}$. Then, the minimum value of the function $\varphi(p_1)$ is obained in the intervals $(-\infty,w_1]\cup[w_1,w_{n_1}]\cup[w_{n_1},+\infty)$.

For any z_1 that satisfies $|z_1|\leqslant\frac{\sum_i|e_i|}{1-\sum_s|b_s|}$, it can be seen that $\varphi(p_1)$ is decreasing when $p_1\in(-\infty,w_1]$, and $\varphi(p_1)$ is increasing when $p_1\in[w_{n_1},+\infty)$. In particular, if $1+\sum_s|b_s|\,\mathrm{sgn}(a_s+b_sp_1)+\sum_i\frac{|e_i|}{|z_1|}\,\mathrm{sgn}(c_i+\frac{e_ip_1}{z_1})=0$, there exist $w_{i_0}<w_{i_1}\in\{w_i\}$, or $w_{i_0}=-\infty$ and $w_{i_1}\in\{w_i\}$, or $w_{i_0}\in\{w_i\}$ and $w_{i_1}=+\infty$, such that $\varphi(w_{i_0})=\varphi(p_1)=\varphi(w_{i_1})$, $p_1\in[w_{i_0},w_{i_1}]$. By Fermat's theorem, the set of extreme points of $\varphi(p_1)$ is a subset of $\{w_1,w_2,\dots,w_{n_1}\}$ when $p_1\in[w_1,w_{n_1}]$. Therefore, we have

$$\inf_{p_1\in\mathbb{R}}\varphi(p_1)=\min\{\varphi(w_1),\dots,\varphi(w_{n_1})\}.$$

In addition, for any z_1 that satisfies $|z_1|>\frac{\sum_i|e_i|}{1-\sum_s|b_s|}$, it follows that $\varphi(p_1)$ is increasing when $p_1\in(-\infty,w_1]\cup[w_{n_1},+\infty)$. In this case, we have

$$\inf_{p_1\in\mathbb{R}}\varphi(p_1)=-\infty.$$

Therefore, we can obtain $\inf_{p_1\in\mathbb{R}}\varphi(p_1)$ by selecting $p_1 = w_i$, $i = 1,\dots,n_1$ or $p_1 = -\infty$.

Remark 6. We obtain a similar result for $\varphi(p_1) = p_1 + \sum_s |a_s + b_s p_1| + \sum_i |c_i p_1 z_1 + e_i|$.

For the general case where the $j_\sharp$-th column of A with $\mu_1(A) = \psi_{j_\sharp}(A)$ has at least one nonzero off-diagonal element, we first state a specific result, i.e., the first generalized transformation maintains the least μ_1 measure as unchanged and it corresponds to infinitely many optimal solutions.

Lemma 5. Suppose that $A = (a_{ij})_{n\times n}$ with $\mu_1(A) = \psi_{j_\sharp}(A)$, $j_\sharp \in \overline{n}$ and $T(z_1) = (I_n, b_{j_\sharp}(z_1)) \in \mathbb{R}^{n\times(n+1)}$ with $b_{j_\sharp}(z_1)$ in (9). The following problem:

$$\min_{z_1} \mu_1(P(z_1,p)) \quad \text{s.t. } AT(z_1) = T(z_1)P(z_1,p) \tag{11}$$

has infinitely many solutions, where $p = [p_1,\dots,p_{n+1}]^{\rm T}$ and p_j are free parameters of the elements in the j-th column of $P(z_1,p)$ for $j \in \overline{n+1}$.

Proof. Obviously, $\mu_1(P(0,p)) = \mu_1(A)$. If $z_1 \neq 0$, then according to Lemma 3, the expression of the generalized transformed matrix, $P(z_1,p)$, can be obtained. If $|z_1| > 1$, it can be verified that $\sum_{s=2}^n |a_{s,j_\sharp} z_1| - \sum_{s=2}^n |a_{s,j_\sharp}| > 0$ since at least one $a_{s,j_\sharp} \neq 0$, $(s \neq j_\sharp)$ exists. $\psi_{n+1}(P(z_1,p)) \leqslant \psi_{j_\sharp}(A)$, so this also implies that

$$|z_1(a_{j_\sharp,j_\sharp} - p_{n+1})| + p_{n+1} - a_{j_\sharp,j_\sharp} < 0,$$

which is simply impossible. Therefore, the feasible region is $\{z_1 : |z_1| \leqslant 1\}$. From Lemma 4 and $|z_1| \leqslant 1$, it follows that

$$\inf_{p_1\in\mathbb{R}} \psi_j(P(z_1,p)) = \psi_j(A), \quad j = 1,\dots,n.$$

By combining this with $\mu_1(A) = \psi_{j_\sharp}(A)$, we have

$$\min_{z_1\neq 0} \mu_1(P(z_1,p)) = \mu_1(A).$$

Therefore, the problem has infinitely many solutions $\{z_1 : |z_1| \leqslant 1\}$.

In the following, an iterative optimization procedure, presented as Algorithm 2, is proposed to solve the problem

$$\begin{aligned}
&\min_{z_i} \max_{j\in\overline{n+1}} \{\psi_j(P_1(z_i,p)), \psi_j(P_2(z_i,q))\}, \quad i \in \mathbb{N}^+ \\
&\text{s.t. } A_1T_i(z_i) = T_i(z_i)P_1(z_i,p), \\
&\qquad A_2T_i(z_i) = T_i(z_i)P_2(z_i,q),
\end{aligned} \tag{12}$$

where $T_i(z_i) = (I_n, c + b_i(z_i))$ with $b_i(z_i)$ in (9), and $c = [c_1,\dots,c_n]^{\rm T}$ where c_j are real constants, $j \in \overline{n}$.

We note that $\gamma_i(z_i)$ is non-convex, so in Step 5, we cannot directly find the decreasing direction as the searching direction. Each function $\psi_j(\tilde{P}_r(z_i))$, $r \in \overline{2}$, $j \in \overline{n+1}$, $j \neq i$ may have jumps, and thus the maximum function $\gamma_i(z_i)$ may be discontinuous so we need to obtain all of its non-differentiable points.

Remark 7. The function $\gamma_i(z_i)$ in successive iterations is shown in Figure 1, where we assume that $n = 3$, $\mathcal{B}_1 = \{2,3\}$ and $\mathcal{B}_2 = \{1,2,3\}$. It follows that $\bar{\mathcal{B}}_1 = \{1,4\}$ and $\bar{\mathcal{B}}_2 = \{4\}$. After identifying the free parameters p_1, p_4, q_4, problem (12) is transformed into $\min_{z_i}\max\{\psi_1(\tilde{P}_1(z_i)), \psi_4(\tilde{P}_1(z_i)), \psi_4(\tilde{P}_2(z_i))\}$. After Steps 2 and 3, we obtain the discontinuous point w and the initial point y_0 because $\gamma_i(y_0) < \gamma_i(0)$, as shown in Figure 1. According to Figure 1, $\psi_4(\tilde{P}_1(y_0)) = \gamma_i(y_0)$, and we find the pair $(j_0, r_0) = (4,1)$ in Step 4. After Step 5, we obtain the set $\mathcal{W}_1$ and y_1 can then be found accordingly. $r_0 = 1$, so we have $\bar{\mathcal{B}}_1 = \{1\}$, i.e., the function $\psi_4(\tilde{P}_1(z_i))$ is not considered in the following iterations. The stopping condition in Step 6 is not satisfied, and thus we loop to Step 4. As shown in Figure 1, we find the current pair $(j_0, r_0) = (4,2)$ in Step 4 because $\gamma_i(y_1) = \psi_4(\tilde{P}_2(y_1))$. Similarly, according to Step 5, we obtain y_2. Hence, $\bar{\mathcal{B}}_2 = \emptyset$, which combined with $\bar{\mathcal{B}}_1 = \{1\}$ implies that the procedure stops. Therefore, we can find the optimal solution, $y_{s_0} = \arg\min_{y_s\in\{y_0,y_1,y_2,w\}} \gamma_i(y_s) = y_2$, and the corresponding optimal value, $\gamma_i(y_2)$.

Algorithm 2 Iterative procedure for solving problem (12).

Step 1. Set $r := 1$. Let $\mathcal{B}_1 := \{j : \psi_j(A_1) < \lambda(\mathcal{A})\}$ and $\mathcal{B}_2 := \{j : \psi_j(A_2) < \lambda(\mathcal{A})\}$. Set $\bar{\mathcal{B}_1} := \overline{n+1}\backslash\mathcal{B}_1$ and $\bar{\mathcal{B}_2} := \overline{n+1}\backslash\mathcal{B}_2$.

Step 2. Get

$$\psi_j(\tilde{P}_1(z_i)) := \inf_{p_j} \psi_j(P_1(z_i, p)), \quad j \in \bar{\mathcal{B}_1},$$

and

$$\psi_j(\tilde{P}_2(z_i)) := \inf_{q_j} \psi_j(P_2(z_i, q)), \quad j \in \bar{\mathcal{B}_2}$$

for $z_i \in \mathbb{R}$. Let $\gamma_i(z_i) := \max_{s_1\in\bar{\mathcal{B}_1}, s_2\in\bar{\mathcal{B}_2}}\{\psi_{s_1}(\tilde{P}_1(z_i)), \psi_{s_2}(\tilde{P}_2(z_i))\}$. Define the discontinuous points of $\gamma_i(z_i)$ as w_s. Let $w := \arg\min \gamma_i(w_s)$.

Step 3. Set

$$\mathcal{B}_3 := \{c|\exists(j,l) \in (\bar{\mathcal{B}_l}, \overline{2}), \text{ such that } c \text{ is a nondifferentiable point of } \psi_j(\tilde{P}_l(z_i)), \text{ and } \psi_j(\tilde{P}_l(c)) = \gamma_i(c)\},$$

and

$$\mathcal{B}_4 := \{c|\exists l \in \overline{2}, \text{ such that } \psi'_{n+1}(\tilde{P}_l(c)) = 0, \psi''_{n+1}(\tilde{P}_l(c)) > 0, \text{ and } \psi_{n+1}(\tilde{P}_l(c)) = \gamma_i(c)\}.$$

Let $c_0 := \arg\min_{c\in\mathcal{B}_3\cup\mathcal{B}_4} \gamma_i(c)$. If $\gamma_i(0) < \gamma_i(c_0)$, then set $y_0 := 0$; otherwise, set $y_0 := c_0$.

Recursion

Step 4. Find a $(j_0, r_0) \in (\bar{\mathcal{B}_l}, \overline{2})$, such that $\psi_{j_0}(\tilde{P}_{r_0}(y_{r-1})) = \gamma_i(y_{r-1})$.

Step5. Obain the set

$$\mathcal{W}_r := \{v|\exists(s,l) \in (\bar{\mathcal{B}_l}, \overline{2}), (s,l) \neq (j_0, r_0), \text{ such that } \psi_{j_0}(\tilde{P}_{r_0}(v)) = \psi_s(\tilde{P}_l(v)) = \gamma_i(v)\}. \tag{13}$$

Find $y_r := \arg\min_{v\in\mathcal{W}_r} \psi_{j_0}(\tilde{P}_{r_0}(v))$ and obain the index, (j_0, r_0), of the function $\psi_{j_0}(\tilde{P}_{r_0}(z_i))$. If $r_0 = 1$, then set $\bar{\mathcal{B}_1} := \bar{\mathcal{B}_1}\backslash\{j_0\}$; otherwise, $\bar{\mathcal{B}_2} := \bar{\mathcal{B}_2}\backslash\{j_0\}$.

Step 6. If there is only one element in $\bar{\mathcal{B}_1} \cup \bar{\mathcal{B}_2}$, set $y_{s_0} := \arg\min_{y_s\in\{y_0,\ldots,y_r,w\}} \gamma_i(y_s)$ and then STOP; otherwise, set $r := r+1$, and go to Step 4.

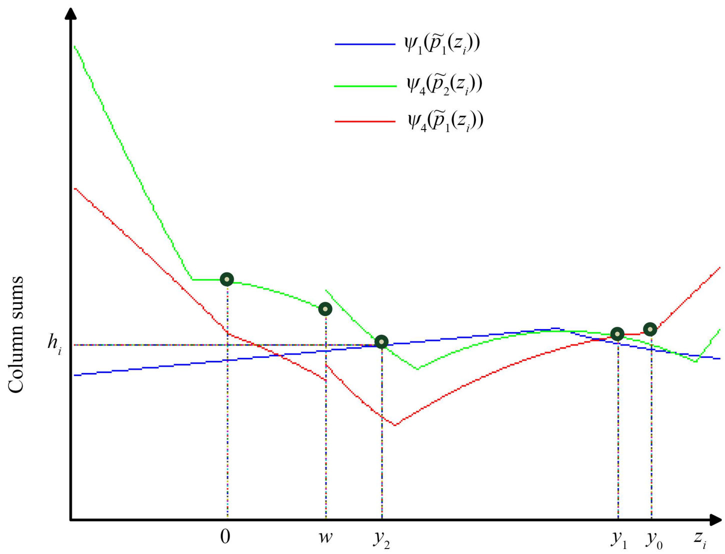

Figure 1 (Color online) The function $\gamma_i(z_i)$ in successive iterations.

Theorem 1. Algorithm 2 finds the optimal value, h_i, and the corresponding optimal solution, d_i, to problem (12) in a finite number of steps.

Proof. Similarly as discussed in (8), we have

$$\min_{z_i} \mu_1(P_1(z_i, p), P_2(z_i, q)) \geqslant \lambda(\mathcal{A}). \tag{14}$$

Hence, each minimum of the matrix set measure, h_i, $i = 1, 2\ldots$, satisfies $h_i \geqslant \lambda(\mathcal{A}) > \psi_j(A_l)$ for any $(j,l) \in (\mathcal{B}_l, \overline{2})$. Therefore, problem (12) is transformed into

$$\min_{z_i} \max_{s_1\in\bar{\mathcal{B}_1}, s_2\in\bar{\mathcal{B}_2}} \{\psi_{s_1}(P_1(z_i, p)), \psi_{s_2}(P_2(z_i, q))\}. \tag{15}$$

As discussed in Lemma 4, we can identify the free parameters p_{s_1} and q_{s_2}, $s_1 \in \bar{\mathcal{B}}_1$, $s_2 \in \bar{\mathcal{B}}_2$ such that the corresponding column sums $\psi_{s_1}(P_1(z_i,p))$ and $\psi_{s_2}(P_2(z_i,q))$ are minimized for any $z_i \in \mathbb{R}$, respectively. Then, we have

$$\psi_{s_1}(\tilde{P}_1(z_i)) = \inf_{p_{s_1}} \psi_{s_1}(P_1(z_i,p)),$$

and

$$\psi_{s_2}(\tilde{P}_2(z_i)) = \inf_{q_{s_2}} \psi_{s_2}(P_2(z_i,q)).$$

We can verify that each function $\psi_j(\tilde{P}_l(z_i))$, $(j,l) \in (\bar{\mathcal{B}}_l, \overline{2})$ may have discontinuous points on $z_i \in \mathbb{R}$. Then, problem (15) is equivalent to

$$\min_{z_i} \max_{s_1\in\bar{\mathcal{B}}_1, s_2\in\bar{\mathcal{B}}_2} \left\{\psi_{s_1}(\tilde{P}_1(z_i)), \psi_{s_2}(\tilde{P}_2(z_i))\right\}. \tag{16}$$

By checking the rules in Steps 4 and 5, we obtain y_r, which satisfies (13). Next, the function $\psi_{j_0}(\tilde{P}_{r_0}(z_i))$ is not considered in subsequent iterations. Therefore, after a maximum of $2n+1$ steps, only one function remains, so we can obtain $y_{s_0} = \arg\min_{y_s\in\{y_0,\ldots,y_r,w\}} \gamma_i(y_s)$. Obviously, the points $\tilde{y}_s$ that satisfy $\gamma_i'(\tilde{y}_s) = 0$ and $\gamma_i''(\tilde{y}_s) > 0$ can only be found in $\mathcal{B}_4$. The set of all the non-differentiable points of $\gamma_i(z_i)$ and the points in $\mathcal{B}_4$, which are denoted by $\tilde{y}_l$, is a subset of $\cup_r \mathcal{W}_r \cup \mathcal{B}_3 \cup \mathcal{B}_4 \cup \{w_i\}$, which further implies that $\gamma_i(y_{s_0}) = \min_l \gamma_i(\tilde{y}_l)$. The maximum function $\gamma_i(z_i)$ is lower semi-continuous on $[\min_l \tilde{y}_l, \max_l \tilde{y}_l]$ and $\gamma_i(w_i) \neq \pm\infty$, so the following equality holds true:

$$\min_{z_i} \gamma_i(z_i) = \gamma_i(y_{s_0}), \quad z_i \in \left[\min_l \tilde{y}_l, \max_l \tilde{y}_l\right].$$

In addition, it can be seen that $\gamma_i(z_i)$ is decreasing when $z_i \in (-\infty, \min_l \tilde{y}_l]$, while it is increasing when $z_i \in [\max_l \tilde{y}_l, +\infty)$. Therefore, we have $y_{s_0} = \arg\min_{z_i} \gamma_i(z_i)$, which completes the proof.

Remark 8. We note that for a given positive number i, the transformation matrix $T_i(z_i)$ in Algorithm 1 is determined according to Step 2. Therefore, for a given matrix set $\mathcal{A}$ and any fixed i-th transformation, problem (10) is exactly problem (12). Hence, it follows that problem (10) is solved exactly by Algorithm 2 for each non-negative integer i.

By setting $P_1(z_i,p)$ and $P_2(z_i,q)$ as equal to the matrices $P_1^i(z_i,p)$ and $P_2^i(z_i,q)$, $i = 0,1,2,\ldots$ in (12), respectively, problem (10) can be solved as discussed in Theorem 1, and the minimum of the matrix set measure is obtained. Hence, Algorithm 1 generalizes a sequence of minimums of matrix set measures $\{h_i\}$ in successive iterations. In the following, we analyze the convergence property of the previous optimal sequence.

Proposition 1. If we suppose that $\{h_i\}$ is a sequence generated by Algorithm 1, it holds that Algorithm 1 terminates finitely at some number i^* such that $d_{i^*} = \cdots = d_{i^*+n-1} = 0$ or the sequence $\{h_i\}$ is convergent.

Proof. Note that

$$\max_{j\in\overline{n+1}} \left\{\min_{p_j} \psi_j(P_1^i(0,p)), \min_{q_j} \psi_j(P_2^i(0,q))\right\} = h_{i-1}.$$

Theorem 1 shows that $h_i \leqslant h_{i-1}$; thus, the sequence $\{h_i\}$ is decreasing. By combining this with (14), we can deduce that the sequence $\{h_i\}$ is convergent.

In addition, if there is a natural number i^* such that $d_{i^*} = \cdots = d_{i^*+n-1} = 0$, we can find $h_{i^*-1} = h_{i^*+n-1}$, which implies that the sequence reaches its minimum, so Algorithm 1 terminates.

Remark 9. As discussed in Lemma 5, the optimal solutions to problem (10) with $i = j_\sharp$ are $\{z_i : |z_i| \leqslant 1\}$. These solutions are designed to ensure the maximum flexibility when selecting $d_{j_\sharp}$ and the finiteness of the solutions to problem (10) with $i > j_\sharp$ if $h_i \neq \psi_j(A_s)$, $(j,s) \in (\bar{\mathcal{B}}_s, \overline{2})$. It should be noted that different choices of $d_{j_\sharp}$ may induce different limits for the sequence $\{h_i\}$.

Remark 10. Proposition 1 provides an upper bound estimate of the spectral abscissa. We note that each maximum function $\gamma_i(z_i)$ in Algorithm 2 is non-convex and it may even be discontinuous, so the optimal solution to problem (11) that satisfies $h_{i+1} \leqslant \alpha^i h_i$ where α^i is any positive step size may not exist. Therefore, the formulation used to characterize the relationship between $h_{i+1} - h^*$ and $h_i - h^*$ is still missing, where h^* is the limit of sequence $\{h_i\}$. Hence, it is not possible to estimate the convergence rate for $\{h_i\}$.

Remark 11. The objective function of problem (7) is non-convex, and thus the convergence of the previous sequence $\{h_i\}$ to the global optimum is not guaranteed. Considering that the objective function is discontinuous and non-convex over the free independent parameters, designing an algorithm to achieve the global minimum is a challenging issue.

4 Numerical example

We consider the third-order switched linear system (1) with

$$A_1 = \begin{bmatrix} -2.5534 & 0.8706 & 1.8128 \\ 2.0876 & -4.7910 & 2.9792 \\ -0.9865 & 1.6241 & -5.5000 \end{bmatrix}$$

and

$$A_2 = \begin{bmatrix} -3.833 & -2.4284 & 2.1042 \\ 0.5480 & -5.7010 & 0.6131 \\ 1.0578 & 1.0753 & -3.8557 \end{bmatrix}.$$

According to the procedures presented in [30], the least μ_1 measures obtained by elementary transformations of types II and III are -0.5179 and -1.5197, respectively.

In addition, we can see that $\lambda(\mathcal{A}) = -1.7763$ and $\mu_1(A_1, A_2) = \psi_1(A_1) = 0.5207$. $\psi_2(A_1) < \lambda(\mathcal{A})$, $\psi_1(A_2) < \lambda(\mathcal{A})$ and $\psi_2(A_2) < \lambda(\mathcal{A})$, so we have $\bar{\mathcal{B}}_1 = \{1,3,4\}$, $\bar{\mathcal{B}}_2 = \{3,4\}$. Therefore, problem (7) is equivalent to

$$\inf_{z} \max_{s_1 \in \bar{\mathcal{B}}_1, s_2 \in \bar{\mathcal{B}}_2} \{\psi_{s_1}(P_1^*(z,p)), \psi_{s_2}(P_2^*(z,q))\}.$$

By Lemma 5, in this example, we choose $d_1 = 1$ and then $h_1 = 0.5207$. Then, we solve the problem (12), where $b_2(z_2)$ in (9) and $c = [1,0,0]^{\mathrm{T}}$. By applying Algorithm 2, we find that $d_2 = 0.5200$ and $h_2 = -1.2701$. After repeating the process, we obtain the sequence $\{h_i\}$ and the corresponding $\{d_i\}$. We find that $d_{12} = d_{13} = d_{14} = 0$. By Proposition 1, Algorithm 1 terminates with $i = 12$, which also shows that the least μ_1 measure is obtained, i.e., $\mu_1(P_1^{12}, P_2^{12}) = -1.6354$, and this is clearly more accurate than that obtained by the elementary coordinate transformations. Therefore, the spectral abscissa lies in the interval $[-1.7763, -1.6354]$. This result establishes the stability of this system as well as indicating that the largest divergence rate for this system is less than -1.6354. The generalized coordinate transformation matrix is

$$T_{12} = \begin{bmatrix} 1.0000 & 0 & 0 & 1.1511 \\ 0 & 1.0000 & 0 & 0.7169 \\ 0 & 0 & 1.0000 & -0.2218 \end{bmatrix}.$$

5 Conclusion

In this study, we presented a computational scheme for approximating the spectral abscissa for continuous-time switched linear systems. The scheme is based on the generalized coordinate transformation method, where the free parameters in the $n \times (n+1)$ transformation matrix were utilized to decrease the μ_1 measure of the transformed system. We proposed detailed procedures to obtain decreasing sequences of the minimums of matrix set μ_1 measures in an iterative manner. The limits of the sequences are upper bound estimates of the spectral abscissa. We also presented a numerical simulation to illustrate the performance of the procedures.

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