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Controllability analysis of multi-agent systems with switching topology over finite fields

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Abstract In this paper, we investigate the controllability problem of multi-agent systems with switching topology over finite fields. The multi-agent system is defined over finite fields, where agents process only values from a finite alphabet. Under leader-follower structure, one agent is selected as a leader for each subsystem. First, we prove that a multi-agent system with switching topology is controllable over a finite field if the graph of the subsystem is a spanning forest, and the size of the field is sufficiently large. Second, we show that, by appropriately selecting leaders, the multi-agent system with switching topology can be controllable over a finite field even if each of its subsystems is not controllable. Specifically, we show that the number of leaders for ensuring controllability of the switched multi-agent system is less than the minimum number of leaders for ensuring the controllability of all subsystems. Finally, it is proved that the multi-agent system is controllable over a finite field if the union of the graphs is a directed path graph or a star graph.

Keywords multi-agent systems, leader-follower structure, controllability, finite fields, switching topology

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1 Introduction

Multi-agent systems consist of many communicating agents, which are governed by neighbor-based protocols. In recent years, decentralized coordination of multi-agent systems has gained more and more attention [1–5]. This is partly due to its broad applications ranging from self-organized sensor networks to formation control of mobile robots. Many critical problems are investigated in cooperative control, such as consensus [6–13], stabilizability [14,15], and controllability [16–25]. Among all the aforementioned research issues, controllability is a critical problem of multi-agent systems. Multi-agent controllability was first studied by [16] under a leader-follower framework. It is proved that the controllability is equivalent to that of a pair of submatrices of Laplacian matrix. Following that, the controllability was investigated from a graph-theoretic perspective [17–20]. There are some other studies, e.g., results on structural controllability [21], and target control [22].

In majority of the existing studies, the interaction values between agents are real numbers or quantized values [7–11]. In many digital control applications of networked systems, the communication bandwidth from sensors to controllers is often restricted and the limitations also occur in multi-agent systems. Due to the development of digital communication and memory constraints, more realistic case of finite communication bandwidth in the communication channels is under investigation, where each agent can

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only take a fixed number of states and can only update its states by a finite number of bits with its neighbors. Compared with the system defined over complex fields, such a multi-agent system has a considerable advantage in their convergence time and resilience to communication noises, which has many practical applications, e.g., quantized control and distributed estimation. In this paper, we model this case with the formalism of finite fields, where each agent is assumed to have the ability to store, process, and transfer solely elements from a finite field, and operations are performed according to modular arithmetic [3, 4].

Structural controllability and observability of multi-agent systems with fixed topology over finite fields was first studied in [26, 27]. It is well known that if the networks are time-varying or switching topology, the controllability or observability becomes complicated [20, 28]. Switched systems are hybrid systems that consist of two or more subsystems. Therefore, switched systems deserve investigation for theoretical interest as well as for practical applications [20, 28]. As a result, it is important to address the controllability problem over finite fields in directed information interaction under link failure or creation (i.e., switching topology). In [29], algebraic conditions for the controllability of switched multi-agent systems over finite fields were proposed. To the best of our knowledge, however, the controllability of a multi-agent system is solely decided by its communication topology structure [18]. Therefore, it is necessary to study the graph-theoretic conditions for the controllability of switched multi-agent systems over finite fields. Particularly, we investigate the controllability of multi-agent systems with switching topology over finite fields. That is, we prove that (a) a multi-agent system with switching topology is controllable over a finite field if each graph of the subsystem is a spanning forest, and that (b) the number of leaders for ensuring controllability of the switched multi-agent system is less than that of ensuring any of the subsystems if the graphs of the subsystems satisfy certain properties, and that (c) the switched multi-agent system is controllable over a finite field if the union of the graphs is a directed path graph or a star graph.

Notation. The following notations will be used throughout this paper. $\mathbb{N}$ denotes the set of natural numbers. $e_{l_1,n}$ is a column-vector of length n with a 1 in its l_1th position and zeros elsewhere. The transpose of matrix A is denoted by A^{T}. $\emptyset$ represents the empty set. $\mathbf{1}_n$ denotes $[1,\ldots,1]^{\mathrm{T}}$ with dimension n. $\mathcal{F}_q^{m\times n}$ is the set of $m\times n$ dimensional matrices defining over the finite field $\mathcal{F}_q$. $\mathcal{F}_q^n$ is the vector space of dimension n over the field $\mathcal{F}_q$. $\mathbf{0}$ denotes an all-zero column vector or matrix with a compatible dimension. $\Lambda(A)$ represents the set of all eigenvalues of matrix A. $\mathrm{diag}\{a_1,a_2,\ldots,a_n\}$ represents a diagonal matrix with entries $a_i, i=1,2,\ldots,n$.

2 Preliminaries

2.1 Graph theory

In this paper, a directed graph will be used to model the interaction topology among agents. Let $\mathcal{G}=(\mathcal{N},\mathcal{E},A)$ be a weighted directed graph consisting of a node set $\mathcal{N}=\{1,2,\ldots,n\}$, an edge set $\mathcal{E}=\{(i,j)\in\mathcal{N}\times\mathcal{N}\}$ and a weighted adjacency matrix $A=[a_{ij}]\in\mathcal{F}_q^{n\times n}$. An edge $(j,i)\in\mathcal{E}$ if the agent i can access the information of the agent j. The weighted adjacency matrix A is defined by $a_{ij}\in\mathcal{F}_q$ if $(j,i)\in\mathcal{E}$ and $a_{ij}=0$ otherwise. The set of neighbors of node i in $\mathcal{G}$ is denoted by $\mathscr{N}_i=\{j\in\mathcal{N}:(j,i)\in\mathcal{E}\}$. A directed path in a directed graph is a sequence $j_1,j_2,\ldots,j_t$ of nodes such that $(j_{i-1},j_i)\in\mathcal{E}$ for $i=2,\ldots,t$. A graph is a spanning tree rooted at i if it is a directed graph where every node in the graph can be reached by a path starting from i, and every node except i has in-degree exactly equal to 1. The node with no outgoing edges is called a leaf node of the tree. A spanning forest is a directed graph consisting of one or more spanning trees, every two of which have no node in common. Throughout this paper, all the graph topologies are assumed to be simple.

2.2 Finite field

An algebraic field $\mathcal{F}$, together with two operations of addition $(+)$ and multiplication $(\cdot)$, satisfies the following properties:

(1) Closure of addition and multiplication. $a+b \in \mathcal{F}$ and $a \cdot b \in \mathcal{F}$ for all $a, b \in \mathcal{F}$.

(2) Commutativity of addition and multiplication. $a+b=b+a$ and $a \cdot b = b \cdot a$ for all $a, b \in \mathcal{F}$.

(3) Associativity of addition and multiplication. $a+(b+c)=(a+b)+c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in \mathcal{F}$.

(4) The field includes an additive identity and a multiplicative identity, denoted by 0 and 1, respectively. $a+0=a$ and $1 \cdot a = 1$ for all $a \in \mathcal{F}$.

(5) Distributivity of multiplication over addition. $a \cdot (b+c) = (a \cdot b) + (a \cdot c)$ for all $a, b, c \in \mathcal{F}$.

(6) Existence of additive and multiplicative inverse elements. For each element $a \in F_q$, there are additive inverse and multiplicative inverse denoted by $\hat{a} \in \mathcal{F}$, $\bar{a} \in \mathcal{F}$, respectively, such that $a + \hat{a} = 0$, and $a \cdot \bar{a} = 1$.

For example, the addition and multiplication tables for $\mathcal{F}_7 = \{0,1,2,3,4,5,6\}$ are given by

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

×	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

In this paper, the finite field is referred to as $\mathcal{F}_q = \{0,1,2,\ldots,q-1\}$, together with the addition and the multiplication modulo q, where q is a prime number.

3 Main results

In this paper, the controllability problem is studied under leader-follower structure. Let us start by dividing agents into leaders and followers. For a given multi-agent system, an agent is a leader if the agent is actuated by some exogenous control inputs; otherwise, the agent is called a follower. In this paper, the set of followers is denoted as $\mathcal{N}_f$, the set of leaders is denoted as $\mathcal{N}_l$ and $\mathcal{N} = \mathcal{N}_f \cup \mathcal{N}_l = \{1,2,\ldots,n\}$. In this section, we consider a multi-agent system composed of n agents, which are labeled from 1 through n. Consider the following multi-agent system over $\mathcal{F}_q$,

$$x_i(k+1) = a_{ii}x_i(k) + \textstyle\sum_{j\in\mathcal{N}_i} a_{ij}x_j(k), \qquad i \in \mathcal{N}_f, \tag{1}$$

$$x_i(k+1) = a_{ii}x_i(k) + \textstyle\sum_{j\in\mathcal{N}_i} a_{ij}x_j(k) + u_i^{\rm ex}(k), \qquad i \in \mathcal{N}_l, \tag{2}$$

where $x_i(k) \in \mathcal{F}_q$ is the state of agent i and $u_i^{\rm ex}(k) \in \mathcal{F}_q$ is the exogenous control input applied to the agent i. Then, we can write the system model into a compact form

$$x(k+1) = Ax(k) + BU^{\rm ex}(k), \tag{3}$$

where $x(k) = [x_1(k), x_2(k), \ldots, x_n(k)]^{\rm T}$, $B = [e_{l_1,n}, e_{l_2,n}, \ldots, e_{l_{|\mathcal{N}_l|},n}]$, $U^{\rm ex}(k) = [u_{l_1}^{\rm ex}(k), u_{l_2}^{\rm ex}(k), \ldots, u_{l_{|\mathcal{N}_l|}}^{\rm ex}(k)]^{\rm T}$ is the stacked vector of the exogenous control inputs which are applied to the leaders and $1 \leqslant l_1 \leqslant l_2 \leqslant \cdots \leqslant l_{|\mathcal{N}_l|} \leqslant n$. The communication topology of representing the information flow among agents may vary due to link failure or creation. This case is usually described by switching topology. Under m ($m \in \mathbb{N}$) switching topologies, a switched multi-agent system over $\mathcal{F}_q$ is given by

$$x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}U^{\rm ex}(k). \tag{4}$$

The constant scalar function $\sigma(k) : \{0,1,\ldots\} \to \underline{M} \triangleq \{1,2,\ldots,m\}$ is the switching signal/path to be designed. Moreover, $\sigma(k) = i_k \in \underline{M}$, ($k \in \{0,1,2,\ldots,\}$) implies that matrix pairs (A_{i_k}, B_{i_k}) are selected as the subsystem realization, that is, matrix pairs (A_{i_k}, B_{i_k}) for $i_k \in \underline{M}$ are referred to as the subsystems of (4). Let $\{\mathcal{G}^p : p \in \underline{M}\}$ be the set of possible directed graphs and $A_{\sigma(k)} = [a_{ij}(\sigma(k))]$ be weighted adjacency matrices which are associated with the graphs $\mathcal{G}^{\sigma(k)}(\mathcal{N}^{\sigma(k)}, \mathcal{E}^{\sigma(k)})$, where $\mathcal{N}^{\sigma(k)} = \mathcal{N} = \{1, 2, \ldots, n\}$.

Definition 1 (Controllability under switching topology)**.** Given m possible network topologies, the switched system (4) is said to be p-controllable if for any arbitrary state $x_f \in \mathcal{F}_q^n$, there exist a p ($p \in \mathbb{N}$), a switching signal $\sigma : \underline{P} = \{0, 1, \ldots, p-1\} \rightarrow \{1, 2, \ldots, m\}$ and input sequence $U^{\rm ex}(0), U^{\rm ex}(1), \ldots, U^{\rm ex}(p-1) \in \mathcal{F}_q^{|\mathcal{N}_l|}$ such that the switched system (4) can be driven from state $x(0) \in \mathcal{F}_q^n$ to x_f in p steps. The switched system (4) is said to be controllable if it is p-controllable for some p.

A switching sequence is a finite or infinite set of pairs $\pi = \{(i_0, h_0), (i_1, h_1), \ldots, (i_{m-1}, h_{m-1})\}$, where the positive integer m is the length of π, $0, h_0, h_0 + h_1, \ldots, \sum_{j=0}^{m-2} h_j$ is the switching instant sequence, $i_0 = \sigma(0), i_1 = \sigma(h_0), \ldots, i_{m-1} = \sigma(\sum_{j=0}^{m-2} h_j)$ is the switching index sequence.

Definition 2. Given a switching sequence $\pi = \{(i_0, h_0), \ldots, (i_{m-1}, h_{m-1})\}$, the controllable state set of π is defined by $\mathcal{T}(\pi) = \{x(\sum_{j=0}^{m-1} h_j) \in \mathcal{F}_q^n \mid$ there are inputs $U^{\rm ext}(i) \in \mathcal{F}_q^{|\mathcal{N}_l|}$such that $x(0) = 0$ and $x(\sum_{j=0}^{m-1} h_j) = x_f$, where $x_f \in \mathcal{F}_q^n\}$.

In order to establish a criterion for checking controllability, we resolve the recursion within (4). Then

$$x(p) = \left(\prod_{k=0}^{p-1} A_{i_k}\right) x_0 + \left(\prod_{k=1}^{p-1} A_{i_k}\right) B_{i_0} U^{\rm ex}(0) + \cdots + A_{i_{p-1}} B_{i_{p-2}} U^{\rm ex}(p-2) + B_{i_{p-1}} U^{\rm ex}(p-1),$$

where $\sigma(k) = i_k$ for $k, p \in \mathbb{N}$, $i_k \in \{1, 2, \ldots, m\}$ and $\prod_{k=0}^{p-1} A_{i_k} = A_{i_{p-1}} A_{i_{p-2}} \cdots A_{i_0}$. Then we get

$$x(p) - \left(\prod_{k=0}^{p-1} A_{i_k}\right) x_0 = \left[B_{i_{p-1}}, A_{i_{p-1}} B_{i_{p-2}}, \ldots, \left(\prod_{k=1}^{p-1} A_{i_k}\right) B_{i_0}\right] \cdot [(U^{\rm ex}(p-1))^{\rm T}, (U^{\rm ex}(p-2))^{\rm T}, \ldots, (U^{\rm ex}(0))^{\rm T}]^{\rm T}.$$

Suppose μ is the minimum integer for which $\operatorname{rank}[B_{i_{\mu-1}}, A_{i_{\mu-1}} B_{i_{\mu-2}}, \ldots, (\prod_{k=1}^{\mu-1} A_{i_k}) B_{i_0}] = n$. The integer μ is called the controllability index of the switched multi-agent system (4).

Theorem 1. The switched multi-agent system (4) is p-controllable over the field $\mathcal{F}_q$ if the matrix $[B_{i_{p-1}}, A_{i_{p-1}} B_{i_{p-2}}, \ldots, (\prod_{k=1}^{p-1} A_{i_k}) B_{i_0}] \in \mathcal{F}_q^{n \times |\mathcal{N}_l| p}$ has (full) rank n.

It is known that controllability over finite-fields requires more strict conditions than that of controllability over the field of complex numbers [27]. If a graph is a spanning tree. It is shown that the graph is controllable over the field $\mathcal{F}_q$, with controllability index equal to n. If all the spanning trees are controllable and all the controllability index are equal to n, whether the system with switching topology reaches the target state at time-step n (the definition of controllability index for fixed topology case is similar to that of the switching case)? We show that this is not true by providing the following Example 1.

Example 1. Consider a multi-agent system (4) over $\mathcal{F}_7 = \{0, 1, 2, 3, 4, 5, 6\}$, where $n = 4$. Three switching topologies of system (4) is illustrated in Figure 1. Let $\sigma(0) = i_0, \sigma(1) = i_1, \sigma(2) = i_2, \sigma(3) = i_3$ with $i_0 = i_1 = 1, i_2 = 2, i_3 = 3$ and $B_1 = B_2 = B_3 = e_{1,4}$. A_1, A_2, A_3 are, respectively, the corresponding adjacency matrices of graphs $\mathcal{G}^1$, $\mathcal{G}^2$ and $\mathcal{G}^3$. Then the first switching path (a) is $\mathcal{G}^1 \rightarrow \mathcal{G}^1 \rightarrow \mathcal{G}^2 \rightarrow \mathcal{G}^3$, that is,

$$x(1) = A_1 x(0) + B_1 U^{\rm ex}(0), \quad x(2) = A_1 x(1) + B_1 U^{\rm ex}(1),$$
$$x(3) = A_2 x(2) + B_2 U^{\rm ex}(2), \quad x(4) = A_3 x(3) + B_3 U^{\rm ex}(3).$$

The weights on the self-loops are $a_{11}(i_k) = 2$, $a_{22}(i_k) = 3$, $a_{33}(i_k) = 4$ and $a_{44}(i_k) = 1, k = 0, 1, 2, 3$. The adjacency matrices A_1, A_2, A_3 are, respectively, as follows:

$$A_1 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

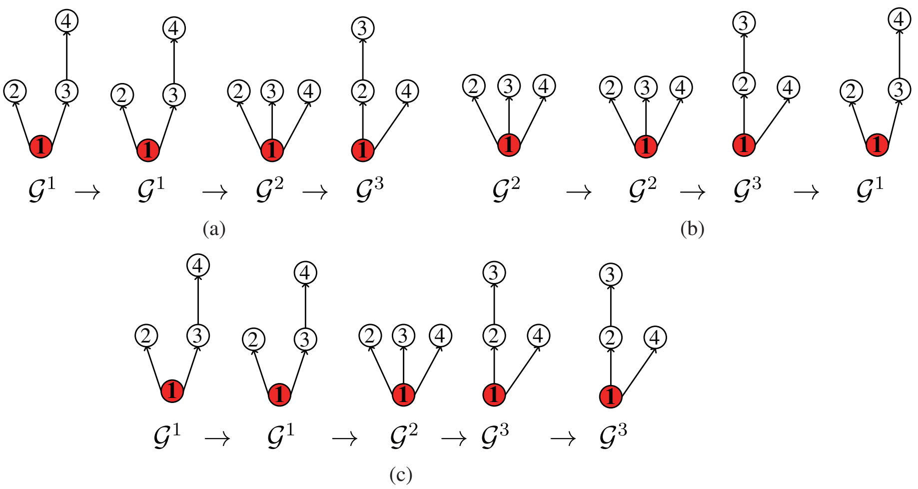

Figure 1 (Color online) Three different switching paths. (a) $\mathcal{G}^1\rightarrow\mathcal{G}^1\rightarrow\mathcal{G}^2\rightarrow\mathcal{G}^3$; (b) $\mathcal{G}^2\rightarrow\mathcal{G}^2\rightarrow\mathcal{G}^3\rightarrow\mathcal{G}^1$; (c) $\mathcal{G}^1\rightarrow\mathcal{G}^1\rightarrow\mathcal{G}^2\rightarrow\mathcal{G}^3\rightarrow\mathcal{G}^3$.

Define $Q_1 = [B_3, A_3B_2, A_3A_2B_1, A_3A_2A_1B_1]$. It is straightforward to see that $\operatorname{rank}(Q_1) = 3$ over $\mathcal{F}_7$ and $\operatorname{rank}(Q_1) = 4$ over $\mathcal{F}_{11}$. However, let $\sigma(0) = i_0, \sigma(1) = i_1, \sigma(2) = i_2, \sigma(3) = i_3$ with $i_0 = i_1 = 2, i_2 = 3, i_3 = 1$. Then the second switching path (b) is $\mathcal{G}^2 \rightarrow \mathcal{G}^2 \rightarrow \mathcal{G}^3 \rightarrow \mathcal{G}^1$, that is,

$$x(1) = A_2x(0) + B_2U^{\rm ex}(0), \quad x(2) = A_2x(1) + B_2U^{\rm ex}(1),$$
$$x(3) = A_3x(2) + B_3U^{\rm ex}(2), \quad x(4) = A_1x(3) + B_1U^{\rm ex}(3).$$

The matrices A_j and B_j, $j = 1, 2, 3$ are the same as above. Define $Q_2 = [B_1, A_1B_3, A_1A_3B_2, A_1A_3A_2B_2]$. It is straightforward to see that $\operatorname{rank}(Q_2) = 4$ over $\mathcal{F}_7$. Consider graph (c) in Figure 1. Let $a_{11}(i_k) = 1$, $a_{22}(i_k) = 5$, $a_{33}(i_k) = 2$ and $a_{44}(i_k) = 6, k = 0, 1, 2, 3, 4$ with $i_0 = i_1 = 1, i_2 = 2, i_3 = 3, i_4 = 3$, $B_1 = B_2 = B_3 = e_{1,4}$. Then the third switching path (c) is $\mathcal{G}^1 \rightarrow \mathcal{G}^1 \rightarrow \mathcal{G}^2 \rightarrow \mathcal{G}^3 \rightarrow \mathcal{G}^3$, that is,

$$x(1) = A_{i_0}x(0) + B_{i_0}U^{\rm ex}(0), \quad x(2) = A_{i_1}x(1) + B_{i_1}U^{\rm ex}(1),$$
$$x(3) = A_{i_2}x(2) + B_{i_2}U^{\rm ex}(2), \quad x(4) = A_{i_3}x(3) + B_{i_3}U^{\rm ex}(3), \quad x(5) = A_{i_4}x(4) + B_{i_4}U^{\rm ex}(4).$$

Define $Q_3 = [B_3, A_3B_3, A_3^2B_2, A_3^2A_2B_1, A_3^2A_2A_1B_1]$. It can be be calculated that $\operatorname{rank}(Q_3) = 3$ over $\mathcal{F}_7$.

It follows that the switched multi-agent system (4) is not necessarily n-controllable through arbitrary switching when all the subsystems (A_{i_k}, B_{i_k}) are controllable at time-step n. In order to realize the controllability, the matrix $[B_{i_{p-1}}, A_{i_{p-1}}B_{i_{p-2}}, \ldots, (\prod_{k=1}^{p-1} A_{i_k})B_{i_0}]$ must be full rank. However, there are too many possible switching paths (switching orders) when the the number of nodes becomes large, and it is computationally expensive to calculate the above matrix under all possible switching paths. In this paper, we are interested in a particular path which is proposed by us. This path facilitates us to figure out an effective way to ensure the controllability. Furthermore, in the every subsystem, there is a single leader. That is, only one agent is selected as the leader for each subsystem and $B_{i_k} \in \mathcal{F}_q^{n\times 1}, U^{\rm ex}(k) \in \mathcal{F}_q$.

In the following, the proposed switching path is given by

$$\begin{aligned}
&\sigma(0) = \sigma(1) = \cdots = \sigma(h_0 - 1) = i_0, \quad \sigma(h_0) = \sigma(h_0 + 1) = \cdots = \sigma(h_0 + h_1 - 1) = i_1,\\
&\cdots\\
&\sigma\left(\sum_{j=0}^{m-2} h_j\right) = \sigma\left(\sum_{j=0}^{m-2} h_j + 1\right) = \cdots = \sigma\left(\sum_{j=0}^{m-1} h_j - 1\right) = i_{m-1},
\end{aligned} \tag{5}$$

where $h_0, h_1, \ldots, h_{m-1} \in \mathbb{N}$, $\sigma(k) = i_k = i_0, k = 0, 1, \ldots, h_0 - 1, \ldots, \sigma(k) = i_k = i_{m-1}, k = h_0 + \cdots + h_{m-2}, \ldots, h_0 + \cdots + h_{m-2} + h_{m-1} - 1$ and $i_s \in \{1, 2, \ldots, m\}, s = 0, 1, \ldots, m - 1$.

Given m graphs $\mathcal{G}^{i_s}(\mathcal{N}^{i_s}, \mathcal{E}^{i_s}, A_{i_s})$, where $s \in \{0, 1, \cdots, m - 1\}$ and $\|\mathcal{N}^{i_s}\| = n$, each graph $\mathcal{G}^{i_s}$ is a spanning forest and each spanning forest $\mathcal{G}^{i_s}$ has $\nu_s(\nu_s \in \mathbb{N}), s \in \{0, 1, \ldots, m - 1\}$ trees, denoted by

$\mathcal{G}_j^{i_s}, j \in \{1,2,\ldots,\nu_s\}$. A subset $\mathcal{C}$ of $\mathcal{N}$ is called a cell. A collection of cells $\{\mathcal{G}_1^{i_s}, \mathcal{G}_2^{i_s}, \ldots, \mathcal{G}_{\nu_s}^{i_s}\}, \nu_s \in \mathbb{N}$ is called a partition of graph $\mathcal{G}^{i_s}$ if the cells are mutually disjoint and $\sum_{j=1}^{\nu_s} \mathcal{N}(\mathcal{G}_j^{i_s}) = \mathcal{N}$, where $\mathcal{N}(\mathcal{G}_j^{i_s})$ are the node sets of spanning trees $\mathcal{G}_j^{i_s}$. In the following, we assume that there is some consistent topological ordering throughout all of graphs $\mathcal{G}^{i_s}$, that is, $a_{ij}(\sigma(k)) = 0$ for $j > i, k = 0, 1, \ldots, n$. In Theorem 2 and Corollary 1, for $s = 0, \ldots, m-1$, we assume that there is a spanning tree in each $\mathcal{G}^{i_s}$, where the number of nodes of these spanning trees is n_{s+1}, satisfying $n_1 + n_2 + \cdots + n_m = n$. Subsequently, we renumber the nodes so that the nodes $1, n_1+1, n_1+n_2+1, \ldots, n_1+n_2+\cdots+n_{m-1}+1$ represent, respectively, the root nodes of those above spanning trees. Those specified spanning trees are denoted, respectively, by $\mathcal{G}_{\tau_0}^{i_0}, \mathcal{G}_{\tau_1}^{i_1}, \ldots, \mathcal{G}_{\tau_{m-2}}^{i_{m-2}}$ and $\mathcal{G}_{\tau_{m-1}}^{i_{m-1}}$, where $\tau_0 = 1$, $\tau_{m-1} = \nu_{m-1}$ and $\tau_l \in \{1,2,\ldots,\nu_l\}, l = 1,2,\ldots,m-2$. Let

$$
\begin{aligned}
&\mathcal{N}(\mathcal{G}_{\tau_0}^{i_0}) = \{1,2,\ldots,n_1\}, \quad \mathcal{N}\backslash\mathcal{N}(\mathcal{G}_{\tau_0}^{i_0}) = \{n_1+1, n_1+2, \ldots, n\},\\
&\mathcal{N}(\mathcal{G}_{\tau_1}^{i_1}) = \{n_1+1, n_1+2, \ldots, n_1+n_2\}, \quad \mathcal{N}\backslash\mathcal{N}(\mathcal{G}_{\tau_1}^{i_1}) = \{1,2,\ldots,n_1, n_1+n_2+1, \ldots, n\},\\
&\mathcal{N}(\mathcal{G}_{\tau_2}^{i_2}) = \{n_1+n_2+1, n_1+n_2+2, \ldots, n_1+n_2+n_3\},\\
&\mathcal{N}\backslash\mathcal{N}(\mathcal{G}_{\tau_2}^{i_2}) = \{1,2,\ldots,n_1, n_1+1, n_1+2, \ldots, n_1+n_2, n_1+n_2+n_3+1, \ldots, n\},\\
&\cdots\\
&\mathcal{N}(\mathcal{G}_{\tau_{m-2}}^{i_{m-2}}) = \{n_1+\cdots+n_{m-2}+1, n_1+\cdots+n_{m-2}+2, \ldots, n_1+\cdots+n_{m-1}\},\\
&\mathcal{N}\backslash\mathcal{N}(\mathcal{G}_{\tau_{m-2}}^{i_{m-2}}) = \{1,2,\ldots,n_1+\cdots+n_{m-2}, n_1+\cdots+n_{m-1}+1, \ldots, n\},\\
&\mathcal{N}(\mathcal{G}_{\tau_{m-1}}^{i_{m-1}}) = \{n_1+\cdots+n_{m-1}+1, n_1+\cdots+n_{m-1}+2, \ldots, n\},\\
&\mathcal{N}\backslash\mathcal{N}(\mathcal{G}_{\tau_{m-1}}^{i_{m-1}}) = \{1,2,\ldots,n_1+n_2+\cdots+n_{m-1}\}.
\end{aligned}
$$

For the spanning tree with the root node $\sum_{s=0}^{t} n_s + 1, t \in \{0,1,\ldots,m-1\}$, we renumber all the other nodes besides $\sum_{s=0}^{t} n_s + 1$, i.e., leaf nodes, by $\sum_{s=0}^{t} n_s + 2, \ldots, \sum_{s=0}^{t+1} n_s$, where $n_0 = 0$.

Theorem 2. Consider the matrix pairs $(A_{i_s}, B_{i_s}), s = 0, 1, \ldots, n-1$, where $B_{i_0} = e_{1,n}$, $B_{i_{n_1}} = e_{n_1+1,n}, \ldots, B_{i_{n_1+\cdots+n_{m-1}}} = e_{n_1+n_2+\cdots+n_{m-1}+1,n}$ and A_{i_s} is an $n \times n$ matrix with elements from a field $\mathcal{F}_q$ of size at least $n+1$. Suppose that the following four conditions hold:

(i) Each graph $\mathcal{G}^{i_s}$ is a spanning forest, augmented with self-loops on each node, where $a_{jj}(i_s) \neq 0$ for $s, j = 1, 2, \ldots, n$.

(ii) For each spanning tree, neither of two nodes in the same tree has the same weight on their self-loops and $a_{ij}(i_s) = 1$ if $(j,i) \in \mathcal{E}^{i_s}$, where $i \neq j$.

(iii) Each spanning forest $\mathcal{G}^{i_s}$ has a spanning tree $\mathcal{G}_{\tau_s}^{i_s}$, with the number of nodes of the spanning tree $\mathcal{G}_{\tau_s}^{i_s}$ equal to $n_{s+1}, s = 0, \ldots, m-1$, where $n_1 + \cdots + n_m = n$.

(iv) The switching path is given by (5), where $h_j = n_{j+1}, j = 0, 1, 2, \ldots, m-1$.

Then the multi-agent system (4) with the communication topologies $\mathcal{G}^{i_s}$, $s \in \{0, 1, \ldots, n-1\}$ is controllable over $\mathcal{F}_q$, with controllability index equal to n.

Proof. Because each graph $\mathcal{G}^{i_s}$ is a spanning forest consisting of ν_s trees, the corresponding adjacency matrix A_{i_s} is $A_{i_s} = \operatorname{diag}\{A_1(i_s), A_2(i_s), \ldots, A_{\nu_s}(i_s)\}$, where $A_j(i_s)$ are adjacency matrices corresponding to the spanning trees $\mathcal{G}_j^{i_s}$ and $A_{\tau_s}(i_s) \in \mathcal{F}_q^{n_{s+1} \times n_{s+1}}, s = 0, 1, \ldots, n-1$, $j = 1, 2, \ldots, \nu_s$. Note that the matrices A_{i_s} are lower-triangular. Then we have that $V_{i_s}^{-1} A_{i_s} V_{i_s} = J(A_{i_s})$, where $V_{i_s} = \operatorname{diag}\{V_1(i_s), V_2(i_s), \ldots, V_{\nu_s}(i_s)\}$, $J(A_{i_s}) = \operatorname{diag}\{\lambda_1(i_s), \lambda_2(i_s), \ldots, \lambda_n(i_s)\}, \lambda_k(i_s)$ is the kth eigenvalue of matrix A_{i_s}, $\lambda_k(i_s) \in \Lambda(A_{i_s}), k = 1, 2, \ldots, n$ and $V_j(i_s)$ is the same size as $A_j(i_s)$. It follows that the left eigenvectors of the matrix A_{i_s} contain the left eigenvectors for each of the matrices $A_j(i_s)$ (augmented with zeros). Because each $A_j(i_s)$ corresponds to a spanning tree, we have that the first column vector of $(V_{\tau_s}(i_s))^{-1} \in \mathcal{F}_q^{n_{s+1} \times n_{s+1}}$ is $\mathbf{1}_{n_{s+1}}$, where $s = 0, 1, \ldots, n-1$.

Let

$$
\begin{aligned}
&\mu_{i_0} = [\mathbf{1}_{n_1}^{\mathrm{T}}, \mathbf{0}_{n_2+\cdots+n_m}^{\mathrm{T}}]^{\mathrm{T}}, \quad \mu_{i_1} = [\mathbf{0}_{n_1}^{\mathrm{T}} \mathbf{1}_{n_2}^{\mathrm{T}}, \mathbf{0}_{n_3+\cdots+n_m}^{\mathrm{T}}]^{\mathrm{T}},\\
&\cdots\\
&\mu_{i_{m-2}} = [\mathbf{0}_{n_1+n_2+\cdots+n_{m-2}}^{\mathrm{T}}, \mathbf{1}_{n_{m-1}}^{\mathrm{T}}, \mathbf{0}_{n_m}^{\mathrm{T}}]^{\mathrm{T}}, \quad \mu_{i_{m-1}} = [\mathbf{0}_{n_1+n_2+\cdots+n_{m-1}}^{\mathrm{T}}, \mathbf{1}_{n_m}^{\mathrm{T}}]^{\mathrm{T}},
\end{aligned}
$$

then we get $V_{i_s}^{-1}B_{i_s}=\mu_{i_s}$. From condition (iv) and $x(0)=\mathbf{0}$, we get

$$\begin{aligned}
x(n)={}&A_{i_{m-1}}^{n_m}A_{i_{m-2}}^{n_{m-1}}\cdots A_{i_2}^{n_3}A_{i_1}^{n_2}A_{i_0}^{n_1}x(0)+\Big[\underbrace{B_{i_{m-1}},A_{i_{m-1}}B_{i_{m-1}},\dots,A_{i_{m-1}}^{n_m-1}B_{i_{m-1}}}_{Q_{m-1}},\\
&\underbrace{A_{i_{m-1}}^{n_m}B_{i_{m-2}},A_{i_{m-1}}^{n_m}A_{i_{m-2}}B_{i_{m-2}},\dots,A_{i_{m-1}}^{n_m}A_{i_{m-2}}^{n_{m-1}-1}B_{i_{m-2}}}_{Q_{m-2}},\dots,\\
&\underbrace{A_{i_{m-1}}^{n_m}A_{i_{m-2}}^{n_{m-1}}\cdots A_{i_2}^{n_3}B_{i_1},A_{i_{m-1}}^{n_m}A_{i_{m-2}}^{n_{m-1}}\cdots A_{i_2}^{n_3}A_{i_1}B_{i_1},\dots,A_{i_{m-1}}^{n_m}A_{i_{m-2}}^{n_{m-1}}\cdots A_{i_2}^{n_3}A_{i_1}^{n_2-1}B_{i_1}}_{Q_1},\\
&\underbrace{A_{i_{m-1}}^{n_m}A_{i_{m-2}}^{n_{m-1}}\cdots A_{i_2}^{n_3}A_{i_1}^{n_2}B_{i_0},A_{i_{m-1}}^{n_m}A_{i_{m-2}}^{n_{m-1}}\cdots A_{i_1}^{n_2}A_{i_0}B_{i_0},\dots,A_{i_{m-1}}^{n_m}A_{i_{m-2}}^{n_{m-1}}\cdots A_{i_1}^{n_2}A_{i_0}^{n_1-1}B_{i_0}}_{Q_0}\Big]\cdot\\
&[U^{\rm ex}(n_1+\cdots+n_m-1),U^{\rm ex}(n_1+\cdots+n_m-2),\dots,U^{\rm ex}(n_1+\cdots+n_{m-1}),\dots,U^{\rm ex}(n_1-1),\\
&\quad U^{\rm ex}(n_1-2),\dots,U^{\rm ex}(0)]^{\rm T}\\
&=[Q_{m-1},Q_{m-2},\dots,Q_0][U^{\rm ex}(n_1+\cdots+n_m-1),U^{\rm ex}(n_1+\cdots+n_m-2),\\
&\quad\dots,U^{\rm ex}(n_1+\cdots+n_{m-1}),\dots,U^{\rm ex}(n_1-1),U^{\rm ex}(n_1-2),\dots,U^{\rm ex}(0)]^{\rm T}.
\end{aligned}$$

Define

$$\begin{aligned}
&\overline{V}(i_{m-1})=\mathrm{diag}\left\{V_1(i_{m-1}),V_2(i_{m-1}),\dots,V_{\nu_{m-1}-1}(i_{m-1}),V_{\nu_{m-1}}(i_{m-1})\right\},\\
&\overline{V}(i_{m-2})=\mathrm{diag}\left\{V_1(i_{m-2}),V_2(i_{m-2}),\dots,V_{\tau_{m-2}}(i_{m-2}),\underbrace{V_{\tau_{m-2}+1}(i_{m-2}),\dots,V_{\nu_{m-2}}(i_{m-2})}_{W(i_{m-2})}\right\},\\
&\overline{V}(i_j)=\mathrm{diag}\left\{V_1(i_j),V_2(i_j),\dots,V_{\nu_j-1}(i_j),V_{\nu_j}(i_j)\right\},\quad j=0,1,\dots,m-3,\\
&A_{i_{m-1}}=\mathrm{diag}\left\{A_1(i_{m-1}),A_2(i_{m-1}),\dots,A_{\nu_{m-1}-1}(i_{m-1}),A_{\nu_{m-1}}(i_{m-1})\right\},\\
&A_{i_{m-2}}=\mathrm{diag}\left\{A_1(i_{m-2}),\dots,A_{\tau_{m-2}-1}(i_{m-2}),A_{\tau_{m-2}}(i_{m-2}),A_{\tau_{m-2}+1}(i_{m-2}),\dots,A_{\nu_{m-2}}(i_{m-2})\right\},\\
&Q_{m-1,m-1}=\Big[\mathbf{1}_{n_m},\left(V_{\nu_{m-1}}(i_{m-1})\right)^{-1}A_{\nu_{m-1}}(i_{m-1})V_{\nu_{m-1}}(i_{m-1})\mathbf{1}_{n_m},\dots,\\
&\qquad\left(V_{\nu_{m-1}}(i_{m-1})\right)^{-1}\left(A_{\nu_{m-1}}(i_{m-1})\right)^{n_m-1}V_{\nu_{m-1}}(i_{m-1})\mathbf{1}_{n_m}\Big],\\
&Q_{m-2,m-2}=\Big[\mathbf{1}_{n_{m-1}},\left(V_{\tau_{m-2}}(i_{m-2})\right)^{-1}A_{\tau_{m-2}}(i_{m-2})V_{\tau_{m-2}}(i_{m-2})\mathbf{1}_{n_{m-1}},\dots,\\
&\qquad\left(V_{\tau_{m-2}}(i_{m-2})\right)^{-1}\left(A_{\tau_{m-2}}(i_{m-2})\right)^{n_{m-1}-1}V_{\tau_{m-2}}(i_{m-2})\mathbf{1}_{n_{m-1}}\Big],\\
&Q_{j,j}=\Big[\mathbf{1}_{n_{j+1}},\left(V_{\tau_j}(i_j)\right)^{-1}A_{\tau_j}(i_j)V_{\tau_j}(i_j)\mathbf{1}_{n_{j+1}},\dots,\\
&\qquad\left(V_{\tau_j}(i_j)\right)^{-1}\left(A_{\tau_j}(i_j)\right)^{n_{j+1}-1}V_{\tau_j}(i_j)\mathbf{1}_{n_{j+1}}\Big],\quad j=0,1,\dots,m-3,\\
&D_{m-1,m-1}=\begin{bmatrix}
1 & \lambda_{n_1+\cdots+n_{m-1}+1}(i_{m-1}) & (\lambda_{n_1+\cdots+n_{m-1}+1}(i_{m-1}))^2 & \cdots & (\lambda_{n_1+\cdots+n_{m-1}+1}(i_{m-1}))^{n_m-1}\\
1 & \lambda_{n_1+\cdots+n_{m-1}+2}(i_{m-1}) & (\lambda_{n_1+\cdots+n_{m-1}+2}(i_{m-1}))^2 & \cdots & (\lambda_{n_1+\cdots+n_{m-1}+2}(i_{m-1}))^{n_m-1}\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
1 & \lambda_{n_1+\cdots+n_m-1}(i_{m-1}) & (\lambda_{n_1+\cdots+n_m-1}(i_{m-1}))^2 & \cdots & (\lambda_{n_1+\cdots+n_m-1}(i_{m-1}))^{n_m-1}\\
1 & \lambda_{n_1+\cdots+n_m}(i_{m-1}) & (\lambda_{n_1+\cdots+n_m}(i_{m-1}))^2 & \cdots & (\lambda_{n_1+\cdots+n_m}(i_{m-1}))^{n_m-1}
\end{bmatrix},
\end{aligned}$$

$$D_{m-2,m-2}=\begin{bmatrix} 1 & \lambda_{n_1+\cdots+n_{m-2}+1}(i_{m-2}) & (\lambda_{n_1+\cdots+n_{m-2}+1}(i_{m-2}))^2 & \cdots & (\lambda_{n_1+\cdots+n_{m-2}+1}(i_{m-2}))^{n_{m-1}-1} \\ 1 & \lambda_{n_1+\cdots+n_{m-2}+2}(i_{m-2}) & (\lambda_{n_1+\cdots+n_{m-2}+2}(i_{m-2}))^2 & \cdots & (\lambda_{n_1+\cdots+n_{m-2}+2}(i_{m-2}))^{n_{m-1}-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_{n_1+\cdots+n_{m-1}-1}(i_{m-2}) & (\lambda_{n_1+\cdots+n_{m-1}-1}(i_{m-2}))^2 & \cdots & (\lambda_{n_1+\cdots+n_{m-1}-1}(i_{m-2}))^{n_{m-1}-1} \\ 1 & \lambda_{n_1+\cdots+n_{m-1}}(i_{m-2}) & (\lambda_{n_1+\cdots+n_{m-1}}(i_{m-2}))^2 & \cdots & (\lambda_{n_1+\cdots+n_{m-1}}(i_{m-2}))^{n_{m-1}-1} \end{bmatrix}.$$

Then we get

$$\begin{aligned} Q_{m-1,m-1} &= [\, B_{i_{m-1}}, A_{i_{m-1}}B_{i_{m-1}}, \ldots, A_{i_{m-1}}^{n_m-1}B_{i_{m-1}} \,] \\ &= \overline{V}(i_{m-1})\cdot\Big[\big(\overline{V}(i_{m-1})\big)^{-1}\cdot e_{n_1+n_2+\cdots+n_{m-1}+1,n}, \big(\overline{V}(i_{m-1})\big)^{-1}\cdot A_{i_{m-1}}\cdot\overline{V}(i_{m-1})\cdot\big(\overline{V}(i_{m-1})\big)^{-1} \\ &\quad \cdot e_{n_1+n_2+\cdots+n_{m-1}+1,n}, \ldots, \big(\overline{V}(i_{m-1})\big)^{-1}\cdot A_{i_{m-1}}^{n_m-1}\cdot\overline{V}(i_{m-1})\cdot\big(\overline{V}(i_{m-1})\big)^{-1}\cdot e_{n_1+n_2+\cdots+n_{m-1}+1,n}\Big] \\ &= \overline{V}(i_{m-1})\cdot[\mathbf{0}^{\mathrm{T}},\mathbf{0}^{\mathrm{T}},\ldots,\mathbf{0}^{\mathrm{T}},Q_{m-1,m-1}^{\mathrm{T}}]^{\mathrm{T}} = \overline{V}(i_{m-1})\cdot[\mathbf{0}^{\mathrm{T}},\mathbf{0}^{\mathrm{T}},\ldots,\mathbf{0}^{\mathrm{T}},D_{m-1,m-1}^{\mathrm{T}}]^{\mathrm{T}}. \end{aligned}$$

Note that matrix $A_{i_{m-1}}$ is lower-triangular and matrix $V_{\nu_{m-1}}(i_{m-1})$ is nonsingular over $\mathcal{F}_q$. Because the self-loop weights on the diagonal entries of the matrix $A_{i_{m-1}}$ are different and $\lambda_{n_1+\cdots+n_{m-1}+1}(i_{m-1}),\ldots,$ $\lambda_{n_1+\cdots+n_m}(i_{m-1})$ are all different nonzero elements from the field $\mathcal{F}_q$. From conditions (i) and (ii), we have that $\mathrm{rank}(V_{\nu_{m-1}}(i_{m-1})D_{m-1,m-1})=n_m$.

Note that matrices $A_{i_{m-1}}, A_{i_{m-2}}$ are lower-triangular and matrix $V_{\tau_{m-2}}(i_{m-2})$ is nonsingular over $\mathcal{F}_q$, where $V_{\tau_{m-2}}(i_{m-2})\in\mathcal{F}_q^{n_{m-1}\times n_{m-1}}$ and $W(i_{m-2})\in\mathcal{F}_q^{n_m\times n_m}$. Then the matrix $A_{i_{m-1}}^{n_m}$ can be written as follows:

$$A_{i_{m-1}}^{n_m}=\begin{bmatrix} C_1 & \mathbf{0} & \mathbf{0} \\ * & C_2 & \mathbf{0} \\ * & * & C_3 \end{bmatrix},$$

where $C_1\in\mathcal{F}_q^{(n_1+n_2+\cdots+n_{m-2})\times(n_1+n_2+\cdots+n_{m-2})}$, $C_2\in\mathcal{F}_q^{n_{m-1}\times n_{m-1}}$ and $C_3\in\mathcal{F}_q^{n_m\times n_m}$. Because the matrix $A_{i_{m-1}}^{n_m}$ is lower-triangular and the diagonal entries of the matrix $A_{i_{m-1}}^{n_m}$ are the form of $a_{jj}^{n_m}(i_{m-1}), j=1,2,\ldots,n$, we have that

$$\begin{aligned} Q_{m-2,m-2} &= A_{i_{m-1}}^{n_m}\cdot\Big[B_{i_{m-2}}, A_{i_{m-2}}B_{i_{m-2}},\ldots,A_{i_{m-2}}^{n_{m-1}-1}B_{i_{m-2}}\Big] \\ &= A_{i_{m-1}}^{n_m}\cdot\overline{V}(i_{m-2})\cdot\Big[\big(\overline{V}(i_{m-2})\big)^{-1}\cdot e_{n_1+n_2+\cdots+n_{m-1}+1,n}, \big(\overline{V}(i_{m-2})\big)^{-1}\cdot A_{i_{m-2}}\cdot\overline{V}(i_{m-2}) \\ &\quad \cdot e_{n_1+n_2+\cdots+n_{m-1}+1,n},\ldots,\big(\overline{V}(i_{m-2})\big)^{-1}\cdot(A_{i_{m-2}})^{n_{m-1}-1}\cdot\overline{V}(i_{m-2})\cdot e_{n_1+n_2+\cdots+n_{m-1}+1,n}\Big] \\ &= A_{i_{m-1}}^{n_m}\overline{V}(i_{m-2})\cdot\left[\mathbf{0}^{\mathrm{T}},\mathbf{0}^{\mathrm{T}},\ldots,\mathbf{0}^{\mathrm{T}},Q_{m-2,m-2}^{\mathrm{T}},\mathbf{0}^{\mathrm{T}}\right]^{\mathrm{T}} \\ &= A_{i_{m-1}}^{n_m}\overline{V}(i_{m-2})\cdot\left[\mathbf{0}^{\mathrm{T}},\mathbf{0}^{\mathrm{T}},\ldots,\mathbf{0}^{\mathrm{T}},D_{m-2,m-2}^{\mathrm{T}},\mathbf{0}^{\mathrm{T}}\right]^{\mathrm{T}}. \end{aligned}$$

Because $\lambda_{n_1+\cdots+n_{m-2}+1}(i_{m-2}),\ \ldots,\ \lambda_{n_1+\cdots+n_{m-1}}(i_{m-2})$ are all different nonzero elements from the field $\mathcal{F}_q$, we have $\mathrm{rank}(D_{m-2,m-2}^{\mathrm{T}})=n_{m-1}$ and $\mathrm{rank}(Q_{m-2,m-2})=n_{m-1}$.

Definition 3 ([30]). For a prime p, let $\mathcal{F}_p$ be the set of $\{0,1,\ldots,p-1\}$ of the integers and let $\varphi:\mathbb{Z}/(p)\rightarrow\mathcal{F}_p$ be the mapping defined by $\varphi([a])=a$ for $a=0,1,\ldots,p-1$. Then $\mathcal{F}_p$, endowed with field structure induced by φ, is a finite field, called the Galois field of order p.

In the ring $\mathbb{Z}/(p)$, p prime, the additive order of every nonzero element b is p; that is, $pb=0$, and p is the least positive interger for which this holds. It then follows that $a_{jj}^{n_m}(i_{m-1})a_{jj}^{n_{m-1}}(i_{m-2})\cdots a_{jj}^{n_2}(i_1), j=1,2,\ldots,n$ are nonzero elements over $\mathcal{F}_q$. Then we have $\mathrm{rank}(Q_{j,j})=n_{j+1}, j=0,1,2,\ldots,m-1$. It then follows that $\mathrm{rank}[Q_{m-1},Q_{m-2},\ldots,Q_0]=n$.

Remark 1. While this controllability under the proposed switching path is closely related to the physical system with temporal networks ([31]), it extends the controllability with first-order dynamics in four directions. Firstly, the controllability in our paper considers systems defined over finite fields, rather than

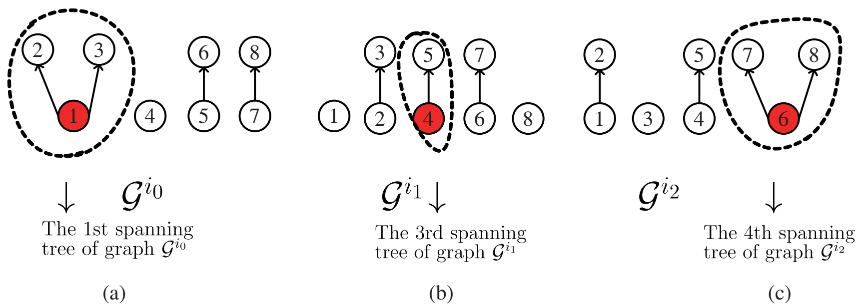

Figure 2 (Color online) Three different topologies (a) $\mathcal{G}^{i_0}$, (b) $\mathcal{G}^{i_1}$ and (c) $\mathcal{G}^{i_2}$, where $\mathcal{N}(\mathcal{G}_{\tau 0}^{i_0}) = \{1,2,3\}$, $\mathcal{N}(\mathcal{G}_{\tau 1}^{i_1}) = \{4,5\}$, $\mathcal{N}(\mathcal{G}_{\tau 2}^{i_2}) = \{6,7,8\}$, and $\tau_0 = 1$, $\tau_1 = 3$, $\tau_2 = 4$.

the field of the complex numbers. Secondly, we can propose a graphical condition for the controllability over finite fields even if each subsystem is not controllable. Thirdly, we show the controllability index for the switched system (4). Theorem 2 facilitates us to make use of less leaders to reach the target of controllability. Actually, this can be achieved provided that the number of the switching graphs is less than the minimum number of leaders required to make the corresponding static topology to be controllable, that is, $m < \min\{\nu_0, \nu_1, \ldots, \nu_{m-1}\}$.

Corollary 1. Consider the matrix pairs $(A_{i_s}, B_{i_s}), s = 0, 1, \ldots, m-1$, where $B_{i_0} = e_{1,n}$, $B_{i_{n_1}} = e_{n_1+1,n}, \ldots, B_{i_{n_1+\cdots+n_{m-1}}} = e_{n_1+n_2+\cdots+n_{m-1}+1,n}$. Suppose that the following four conditions hold:

(i) Each graph $\mathcal{G}^{i_s}$ is a spanning forest, augmented with self-loops on each node, where $a_{ij}(i_s) = 1$ if $(j, i) \in \mathcal{E}^{i_s}, i \neq j$, $s = 0, 1, \ldots, n-1$.

(ii) Each spanning forest $\mathcal{G}^{i_s}$ has a spanning tree $\mathcal{G}_{\tau_s}^{i_s}$, with the number of nodes of the spanning tree $\mathcal{G}_{\tau_s}^{i_s}$ equal to $n_{s+1}, s = 0, 1, \ldots, n-1$, where $n_1 + n_2 + \cdots + n_m = n$.

(iii) Each spanning tree $\mathcal{G}_{\tau_s}^{i_s}$ has at most two branches and each branch is a path. The weights on the self-loops for all nodes in the first branch are 0, and the weights on the self-loops for all nodes in the second branch are 1. Except for the nodes of the spanning trees of $\mathcal{G}_{\tau_s}^{i_s}$, $s = 0, 1, \ldots, n-1$, the self-weights for all the rest of the nodes are nonzero elements from $\mathcal{F}_q$.

(iv) The switching path is given by (5), where $h_j = n_{j+1}, j = 0, 1, 2, \ldots, m-1$.

Then the multi-agent system (4) with the communication topologies $\mathcal{G}^{i_s}$, $s \in \{0, 1, \ldots, n-1\}$ is controllable over any field $\mathcal{F}_q$, with controllability index equal to n.

In what follows, an illustrative example is given.

Example 2. Consider a multi-agent system (4) over $\mathcal{F}_{13}^8$, where $\mathcal{F}_{13} = \{0,1,2,3,4,5,6,7,8,9,10,11,12\}$ and $n = 8$. Three switching topologies of system (4) are illustrated by graph (a) of Figure 2. A_{i_0}, A_{i_1} and A_{i_2} are, respectively, the corresponding adjacency matrices of graphs $\mathcal{G}^{i_0}$, $\mathcal{G}^{i_1}$ and $\mathcal{G}^{i_2}$. The switching order is $\mathcal{G}^{i_0} \rightarrow \mathcal{G}^{i_1} \rightarrow \mathcal{G}^{i_2}$. Let $B_{i_0} = e_{1,8}$, $B_{i_1} = e_{4,8}$, $B_{i_2} = e_{6,8}$, $x(0) = [0,1,2,3,4,5,6,7]^{\mathrm{T}}$, $x(8) = [1,1,2,2,3,3,4,4]^{\mathrm{T}}$. The weights on the self-loops are $a_{11}(i_s) = 1$, $a_{22}(i_s) = 2$, $a_{33}(i_s) = 3$, $a_{44}(i_s) = 4$, $a_{55}(i_s) = 5$, $a_{66}(i_s) = 6$, $a_{77}(i_s) = 7$, $a_{88}(i_s) = 8$, $a_{r,t}(i_s) = 1, r, t = 1, 2, \ldots, n$, where $r \neq t, s = 0, 1, 2$. It follows that $\sigma(0) = \sigma(1) = \sigma(2) = i_0$, $\sigma(3) = \sigma(4) = i_1$ and $\sigma(5) = \sigma(6) = \sigma(7) = i_2$. Define $Q = [B_{i_2}, A_{i_2}B_{i_2}, A_{i_2}^2 B_{i_2}, A_{i_2}^3 B_{i_1}, A_{i_2}^3 A_{i_1} B_{i_1}, A_{i_2}^3 A_{i_1}^2 B_{i_0}, A_{i_2}^3 A_{i_1}^2 A_{i_0} B_{i_0}, A_{i_2}^3 A_{i_1}^2 A_{i_0}^2 B_{i_0}]$. Thus we have that $U^{\mathrm{ex}}(0) = 2, U^{\mathrm{ex}}(1) = 11, U^{\mathrm{ex}}(2) = 1, U^{\mathrm{ex}}(3) = 8, U^{\mathrm{ex}}(4) = 1$, $U^{\mathrm{ex}}(5) = 9, U^{\mathrm{ex}}(6) = 3, U^{\mathrm{ex}}(7) = 10$, and $\mathrm{rank}(Q) = 8$. Then the trajectories of the 8 agents are given in Figures 3 and 4. This is in agreement with the theoretical analysis results.

Let

$$
\begin{aligned}
&\nu_0 = \nu_1 = \cdots = \nu_{m-1}, \quad \mathcal{N}(\mathcal{G}_1^{i_k}) = \{1, 2, \ldots, n_1\}, \quad \mathcal{N}(\mathcal{G}_2^{i_k}) = \{n_1+1, n_1+2, \ldots, n_1+n_2\},\\
&\mathcal{N}(\mathcal{G}_3^{i_k}) = \{n_1+n_2+1, n_1+n_2+2, \ldots, n_1+n_2+n_3\},\\
&\cdots\\
&\mathcal{N}(\mathcal{G}_{m-1}^{i_k}) = \{n_1+n_2+\cdots+n_{m-1}+1, n_1+n_2+\cdots+n_{m-1}+2, \ldots, n\}, k = 1, 2, \ldots, p-1,
\end{aligned} \tag{6}
$$

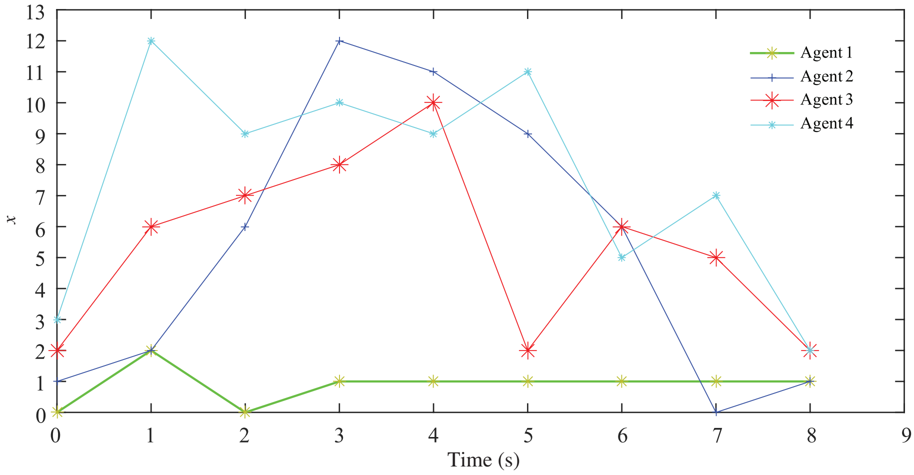

Figure 3 (Color online) The trajectories of the agents.

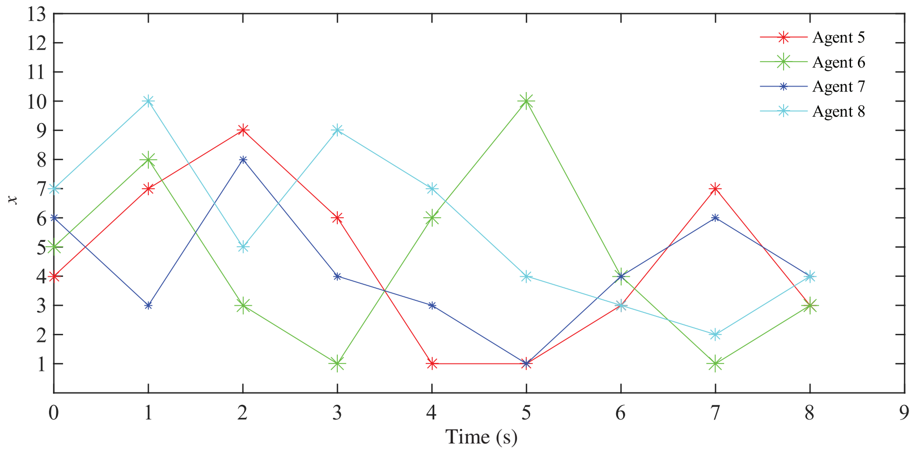

Figure 4 (Color online) The trajectories of the agents.

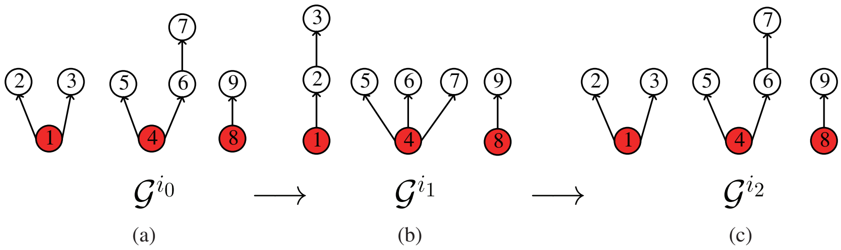

Figure 5 (Color online) Three different topologies $\mathcal{G}^{i_0}$, $\mathcal{G}^{i_1}$ and $\mathcal{G}^{i_2}$, where $\nu_0 = \nu_1 = \nu_2 = 3$, $\mathcal{N}(\mathcal{G}_1^{i_0}) = \mathcal{N}(\mathcal{G}_1^{i_1}) = \mathcal{N}(\mathcal{G}_1^{i_2}) = \{1, 2, 3\}$, $\mathcal{N}(\mathcal{G}_2^{i_0}) = \mathcal{N}(\mathcal{G}_2^{i_1}) = \mathcal{N}(\mathcal{G}_2^{i_2}) = \{4, 5, 6, 7\}$, $\mathcal{N}(\mathcal{G}_3^{i_0}) = \mathcal{N}(\mathcal{G}_3^{i_1}) = \mathcal{N}(\mathcal{G}_3^{i_2}) = \{8, 9\}$, and $\tau_0 = 1$, $\tau_1 = 3$, $\tau_2 = 4$.

Remark 2. In [27], the structural controllability was studied for fixed topology. The differences between the structural controllability problem studied in [27] and this paper are as follows. (i) In this paper, the results are derived for switching topology instead of fixed topology. Subsequent arguments show that the multi-agent system with switching topology brings new features for the study of the controllability problem. (ii) Theorem 3 of [27] can be viewed as a special case of Theorem 2 if (6) holds (e.g., Figure 5). Furthermore, it is not hard to find that Theorem 2 also holds if the condition (iii) of Theorem 2 is replaced by (6).

Consider the system (10) of [32] with a single leader, Theorem 6.1 of [32] is structurally controllable if $\bar{\mathcal{G}}$ has a connected partition. In the following, we present the difference between the graphic conditions in real fields (e.g., [32]) and finite fields.

Firstly, graphic conditions for the controllability over real fields were proposed in [32] by the Popov-Belevich-Hautus (PBH) lemma. However, the PBH lemma is not sufficient to check the controllability of linear systems over finite fields. The reason originates from the fact that finite fields are not algebraically closed ([27]).

Secondly, we prove that the PBH Lemma is valid if the $A \in \mathcal{F}_q^{n\times n}$ is split over $\mathcal{F}_q$ (i.e., A has n eigenvalues). Before proceeding, we give Definition 4 and Lemma 1.

Definition 4 (Split matrix [33])**.** Let R be an arbitrary ring and the polynomial ring over R is denoted by $R[x]$. If F is considered as a finite-dimensional vector space over a field of K, F is called a finite extension of K. Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \in K[x]$ be a positive degree and F be an extension field of K. Then f is said to split in F if f can be written as a product of linear factors in $F[x]$, that is, if there exist elements $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$ such that $f(x) = a(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$, where a is the leading coefficient of $f(x)$. A matrix over $\mathcal{O}_1$ is called split if its characteristic polynomial splits over $\mathcal{F}_q$, where $\mathcal{O}_1$ is a ring of integers.

Lemma 1. Suppose that A is split over $\mathcal{F}_q$. Then system (3) is controllable over $\mathcal{F}_q$ if and only if $\mathrm{rank}[A-\lambda_i I, B] = n$ for all $\lambda_i \in \Lambda(A)$.

Proof. Because $A \in \mathcal{F}_q^{n\times n}$ is split over $\mathcal{F}_q$, it follows from Theorem 3.5 of [34] that the characteristic polynomial of matrix A has n eigenvalues over $\mathcal{F}_q$. Define $Q = [B, AB, \ldots, A^{n-1}B]$.

(Sufficiency) Suppose that $\mathrm{rank}[A-\lambda_i I, B] = n$ for all $\lambda_i \in \Lambda(A)$. Now suppose by contradiction that system (3) is not controllable, that is, $\mathrm{rank}Q = n_1 < n$ and $n - n_1 = k$. Based on the (vii) of Theorem 2.6.3 and Theorem 2.6.5 of [33] (pages 39 and 42), the following statement is valid after some elementary row operations, where the elementary row operations over finite fields are given in [33] (pages 40 and 41). Then there exists a nonsingular matrix T, such that the jth row of matrix TQ equals to $\mathbf{0}$, where $j = n_1+1, n_1+2, \ldots, n$. Let $\bar{B} = TB$, $\bar{A} = TAT^{-1}$ and $\bar{Q} = TQ = T[B, AB, \ldots, A^{n-1}B]$ $= [\bar{B}, \bar{A}\bar{B}, \ldots, \bar{A}^{n-1}\bar{B}]$, then we get $\bar{B} = T\cdot B = \left[\begin{smallmatrix} B_1 \\ \mathbf{0}\end{smallmatrix}\right]$, $\bar{A} = \left[\begin{smallmatrix} A_1 & A_2 \\ A_3 & A_4\end{smallmatrix}\right]$, where $A_1 \in \mathcal{F}_q^{n_1\times n_1}$, $B_1 \in \mathcal{F}_q^{n_1\times m}$. It follows that

$$\bar{A}\bar{B} = \begin{bmatrix} A_1B_1 \\ A_3B_1 \end{bmatrix} \Rightarrow A_3B_1 = \mathbf{0}, \quad \bar{A}^2\bar{B} = \bar{A}\bar{A}\bar{B} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\begin{bmatrix} A_1B_1 \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} A_1^2B_1 \\ A_3A_1B_1 \end{bmatrix} \Rightarrow A_3A_1B_1 = \mathbf{0},$$

$$\vdots$$

$$\bar{A}^{n-1}\bar{B} = \begin{bmatrix} A_1^{n-1}B_1 \\ A_3A_1^{n-2}B_1 \end{bmatrix} = 0 \Rightarrow A_3A_1^{n-2}B_1 = \mathbf{0}, \quad \bar{Q} = \begin{bmatrix} B_1 & A_1B_1 & A_1^2B_1 & \cdots & A_1^{n-1}B_1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}.$$

Then we get $\mathrm{rank}\bar{Q} = n_1$ and $\mathrm{rank}[B_1, A_1B_1, \ldots, A_1^{n-1}B_1] = n_1$. Combing with Cayleigh-Hamilton theorem over $\mathcal{F}_q$ (Theorem 4.1 of [35]) and $n_1 - 1 < n - 1$, we get $\mathrm{rank}[B_1, A_1B_1, \ldots, A_1^{n-2}B_1] = n_1$ and the rows of matrix $[B_1, A_1B_1, \ldots, A_1^{n-2}B_1]$ are linearly independent, where the definition of linear independence is given in [33] (page 33). From $A_3[B_1, A_1B_1, \ldots, A_1^{n-2}B_1] = 0$, we have

$$A_3 = 0, \quad \bar{A} = \begin{bmatrix} A_1 & A_2 \\ \mathbf{0} & A_4 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B_1 \\ \mathbf{0} \end{bmatrix},$$

$$T\begin{bmatrix} A-\lambda I & B \end{bmatrix}\begin{bmatrix} T^{-1} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} = \begin{bmatrix} TA-\lambda T & TB \end{bmatrix}\begin{bmatrix} T^{-1} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} = \begin{bmatrix} TAT^{-1}-\lambda I & TB \end{bmatrix}$$

$$= \begin{bmatrix} \bar{A}-\lambda I & \bar{B} \end{bmatrix} = \begin{bmatrix} A_1-\lambda I_{n_1} & A_2 & B_1 \\ \mathbf{0} & A_4-\lambda I_k & \mathbf{0} \end{bmatrix}.$$

Then we get $\mathrm{rank}\left[\begin{smallmatrix} A_1-\lambda I_{n_1} & A_2 & B_1 \\ \mathbf{0} & A_4-\lambda I_k & \mathbf{0}\end{smallmatrix}\right] < n$ if $\lambda \in \Lambda(A_4)$. It then follows from [36] (page 38, Problems 5) that $\lambda \in \Lambda(A)$ if $\lambda \in \Lambda(A_4)$. This is a contradiction.

(Necessity) Suppose that $\mathrm{rank}[B, AB, \ldots, A^{n-1}B] = n$. Now suppose by contradiction that there exists a $\lambda_0 \in \Lambda(A)$, such that $\mathrm{rank}[A-\lambda_0 I, B] < n$. It follows that the rows of matrix $[A-\lambda_0 I, B]$ are

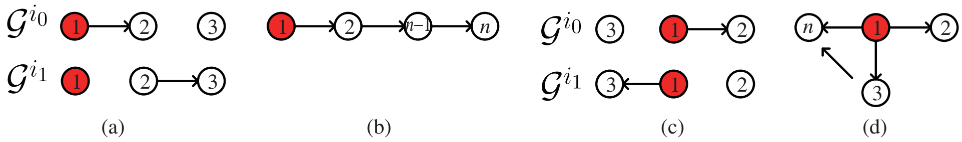

Figure 6 (Color online) (a) Two possible graphs $\mathcal{G}^{i_0}$ and $\mathcal{G}^{i_1}$; (b) a directed path graph; (c) two possible graphs $\mathcal{G}^{i_0}$ and $\mathcal{G}^{i_1}$; (d) a directed star graph.

linearly dependent. Then there exists $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_n] \neq \mathbf{0}$, such that $\alpha[A - \lambda_0 I, B] = \mathbf{0}$. Then we have $\alpha[A - \lambda_0 I] = 0, \alpha B = 0 \Rightarrow \alpha AB = \lambda_0 \alpha B = 0, \ldots, \alpha A^{n-1}B = 0 \Rightarrow \alpha[\, B \ AB \ \cdots \ A^{n-1}B\,] = 0$. It follows that the rows of matrix $[B, AB, \ldots, A^{n-1}B]$ are linearly dependent. This is a contradiction. This completes the proof.

Thirdly, suppose that the matrix A is split. Let $n = 6$ and $\det\overline{A}^2 = 2\lambda^2 + \lambda^5$, and we show that $\det\overline{A}^2 = 0$ over $\mathcal{F}_3 = \{0, 1, 2\}$, where $\widehat{c}_{2,i+1}, \beta_i$, $\det\overline{A}^1$ and $\det\overline{A}^2$ are defined as in Theorem 6.1 of [32]. It is easy to find that $\det\overline{A}^2 \mid_{\lambda} = 0$ for $\lambda = 0, 1, 2$. Consider the field $\mathcal{F}_2 = \{0, 1\}$. Suppose that $\lambda - \sum_{i=1}^{n-1} \widehat{c}_{2,i+1} \cdot \beta_i = 1$. It should be noted that $\det\overline{A}^1 \neq 0$ if $\bar{a}_{n+1,n+1} = 0$ over $\mathcal{F}_2$. Because Theorem 6.1 of [32] is valid for $n \in \mathbb{N}$, Theorem 6.1 of [32] will be hold if the adjacency matrix A is split and $\bar{a}_{n+1,n+1} = 0$ for $n \geqslant 4$, which can be viewed as a strong assumption.

For any matrix $B_{i_s} \in \mathcal{F}_q^{n \times |\mathcal{N}_l|}$, define $\mathcal{B}_{i_s} = \mathrm{Im} B_{i_s}$, where $\mathrm{Im}(\cdot)$ denotes the image space of a matrix. In the following, we investigate the controllability conditions when the union of the graphs is a directed path or star.

Theorem 3. Suppose that each graph $\mathcal{G}^{i_s}$ is augmented with self-loops on every node, where $a_{ii}(i_s) \in \mathcal{F}_q \backslash \{0\}, i = 1, 2, \ldots, n$. Then system (4) with switching topology $\mathcal{G}^{i_s}, s \in \{0, 1, \ldots, n-1\}$ is controllable over $\mathcal{F}_q$ if one of the following two statements holds:

(i) The union of the graphs $\bigcup_{s=0}^{n-1} \mathcal{G}^{i_s}$ is a directed path.

(ii) The union of the graphs $\bigcup_{s=0}^{n-1} \mathcal{G}^{i_s}$ is a directed star.

Proof. (i) We first consider the case of $n = 3$. If $\mathcal{G}^{i_0}$ and $\mathcal{G}^{i_1}$ are directed paths, $(A_{i_0}, e_{1,n})$ and $(A_{i_1}, e_{1,n})$ are controllable. Therefore, we only consider the case that $\mathcal{G}^{i_s}$ is not a directed path. Consider the graph (a) of Figure 6. The adjacency matrix of $\mathcal{G}^{i_s}$ is denoted as A_{i_s}. Let $\pi_1 = \{(i_0, 1), (i_1, 2)\}$, $B_{i_0} = e_{1,3}$ and $B_{i_1} = B_{i_2} = e_{2,3}$, then we get $\mathcal{T}(\pi_1) = \mathrm{Im}[e_{2,3}, A_{i_1} e_{2,3}] + A_{i_1}^2 \,\mathrm{Im}[e_{1,3}] = \{\delta_1, \delta_2, \delta_3\}$, where $\delta_1 = [0, 1, 0]^{\mathrm{T}}$, $\delta_2 = [0, \gamma_2^{i_1}, 1]^{\mathrm{T}}$, $\delta_3 = [(\gamma_1^{i_1})^2, 0, 0]^{\mathrm{T}}$ and $\mathcal{T}(\pi_1) = \mathcal{F}_q^3$, where $\gamma_2^{i_1}, (\gamma_1^{i_1})^2 \in \mathcal{F}_q \backslash \{0\}$. Suppose that system (4) is controllable with $\pi_2 = \{(i_0, h_0), \ldots, (i_{m_1 - 1}, h_{m_1 - 1})\}$ if $\bigcup_{s=0}^{n-1} \mathcal{G}^{i_s}$ is a directed path for $n = k$. We will prove that system (4) is controllable with $\pi_3 = \{(j_0, h_0), \ldots, (j_{m_1-1}, h_{m_1-1}), (j_{m_1}, h_{m_1})\}$ if $\bigcup_{s=0}^{n-1} \mathcal{G}^{j_s}$ is a directed path for $n = k + 1$. It follows that

$$
\begin{aligned}
\mathcal{T}(\pi_2) = {} & A_{i_{m_1-1}}^{h_{m_1-1}} \cdots A_{i_1}^{h_1} \,\mathrm{Im}\left[B_{i_0}, A_{i_0} B_{i_0}, \ldots, A_{i_0}^{h_0 - 1} B_{i_0}\right] + A_{i_{m_1-1}}^{h_{m_1-1}} \cdots A_{i_2}^{h_2} \,\mathrm{Im}[B_{i_1}, A_{i_1} B_{i_1}, \ldots, A_{i_1}^{h_1 - 1} B_{i_1}] \\
& + \cdots + A_{i_{m_1-1}}^{h_{m_1-1}} \,\mathrm{Im}\left[B_{i_{m_1-2}}, A_{i_{m_1-2}} B_{i_{m_1-2}}, \ldots, A_{i_{m_1-2}}^{h_{m_1-2}-1} B_{i_{m_1-2}}\right] \\
& + \mathrm{Im}\left[B_{i_{m_1-1}}, A_{i_{m_1-1}} B_{i_{m_1-1}}, \ldots, A_{i_{m_1-1}}^{h_{m_1-1}-1} B_{i_{m_1-1}}\right],
\end{aligned}
$$

where $\mathcal{T}(\pi_2) = \mathcal{F}_q^k$. According to the structure of the directed path, we have $\mathcal{G}^{j_s} = (\mathcal{V}^{i_s} \bigcup \{k+1\}, \mathcal{E}^{i_s} \bigcup (k, k+1))$ or $\mathcal{G}^{j_s} = (\mathcal{V}^{i_s} \bigcup \{k+1\}, \mathcal{E}^{i_s} \bigcup \emptyset)$, where $\mathcal{V}^{i_s}$ is a set of vertices of $\mathcal{G}^{i_s}$, $\mathcal{E}^{i_s}$ is a set of edges of $\mathcal{G}^{i_s}$, an edge $(j, i) \in \mathcal{E}^{i_s}$ implies that the agent i can access the information of the agent j, and $\emptyset$ is an empty set. It follows that $A_{j_s} = \left[\begin{smallmatrix} A_{i_s} & \mathbf{0} \\ e_{k,k}^{\mathrm{T}} & \gamma_{k+1}^{i_s} \end{smallmatrix}\right]$ or $A_{j_s} = \left[\begin{smallmatrix} A_{i_s} & \mathbf{0} \\ \mathbf{0} & \gamma_{k+1}^{i_s} \end{smallmatrix}\right]$, where $s = 1, 2, \ldots, m_1 - 1$. Let $B_{j_s} = \left[\begin{smallmatrix} B_{i_s} \\ 0 \end{smallmatrix}\right], s = 0, 1, \ldots, m_1 - 1$, $B_{j_{m_1}} = e_{k+1,k+1}$, $A_{j_{m_1}} = \left[\begin{smallmatrix} A_{i_w} & \mathbf{0} \\ \mathbf{0} & \gamma_{k+1}^{i_w} \end{smallmatrix}\right]$ for some w and $h_{m_1} = 1$, where $w = 0, 1, \ldots, m_1 - 1$. Then we have

$$
\begin{aligned}
\mathcal{T}(\pi_3) = A_{j_{m_1}} \Bigg(& A_{j_{m_1-1}}^{h_{m_1-1}} \cdots A_{j_1}^{h_1} \,\mathrm{Im}\left[B_{j_0}, A_{j_0} B_{j_0}, \ldots, A_{j_0}^{h_0-1} B_{j_0}\right] + A_{j_{m_1-1}}^{h_{m_1-1}} \cdots A_{j_2}^{h_2} \\
& \cdot \mathrm{Im}\left[B_{j_1}, A_{j_1} B_{j_1}, \ldots, A_{j_1}^{h_1-1} B_{j_1}\right] + \cdots + A_{j_{m_1-1}}^{h_{m_1-1}} \,\mathrm{Im}\left[B_{j_{m_1-2}}, A_{j_{m_1-2}} B_{j_{m_1-2}}, \ldots, A_{j_{m_1-2}}^{h_{m_1-2}-1} B_{j_{m_1-2}}\right]
\end{aligned}
$$

$$+\operatorname{Im}\left[B_{j_{m_1-1}}, A_{j_{m_1-1}}B_{i_{m_1-1}}, \ldots, A_{j_{m_1-1}}^{h_{m_1-1}-1}B_{j_{m_1-1}}\right]\Bigg) + \operatorname{Im}(B_{j_{m_1}})$$

$$= \operatorname{Im}\begin{bmatrix} \mathbf{0}_k \\ 1 \end{bmatrix} + \operatorname{Im}\begin{bmatrix} C_0B_{i_0} & C_0A_{i_0}B_{i_0} & \cdots & C_0A_{i_0}^{h_0-1}B_{i_0} \\ * & * & \cdots & * \end{bmatrix} + \operatorname{Im}\begin{bmatrix} C_1B_{i_1} & C_1A_{i_1}B_{i_1} & \cdots & C_1A_{i_1}^{h_1-1}B_{i_1} \\ * & * & \cdots & * \end{bmatrix}$$

$$+ \cdots + \operatorname{Im}\begin{bmatrix} C_{m_1-2}B_{i_{m_1-2}} & C_{m_1-2}A_{i_{m_1-2}}B_{i_{m_1-2}} & \cdots & C_{m_1-2}A_{i_{m_1-2}}^{h_{m_1-2}-1}B_{i_{m_1-2}} \\ * & * & \cdots & * \end{bmatrix}$$

$$+ \operatorname{Im}\begin{bmatrix} C_{m_1-1}B_{i_{m_1-1}} & C_{m_1-1}A_{i_{m_1-1}}B_{i_{m_1-1}} & \cdots & C_{m_1-1}A_{i_{m_1-1}}^{h_{m_1-1}-1}B_{i_{m_1-1}} \\ * & * & \cdots & * \end{bmatrix},$$

where $C_0 = A_{i_w}A_{i_{m_1-1}}^{h_{m_1-1}} \cdots A_{i_1}^{h_1}, C_1 = A_{i_w}A_{i_{m_1-1}}^{h_{m_1-1}} \cdots A_{i_2}^{h_2}, \ldots, C_{m_1-2} = A_{i_w}A_{i_{m_1-1}}^{h_{m_1-1}}$ and $C_{m_1-1} = A_{i_w}$. Note that $\mathcal{T}(\pi_2) = \mathcal{F}_q^k$, and we get

$$\mathcal{T}(\pi_3) = \operatorname{Im}\left[(A_{i_w})^{-1}C_0B_{i_0}, (A_{i_w})^{-1}C_0A_{i_0}B_{i_0}, \ldots, (A_{i_w})^{-1}C_0A_{i_0}^{h_0-1}B_{i_0}\right]$$
$$+ \operatorname{Im}\left[(A_{i_w})^{-1}C_1B_{i_1}, (A_{i_w})^{-1}C_1A_{i_1}B_{i_1}, \ldots, (A_{i_w})^{-1}C_1A_{i_1}^{h_1-1}B_{i_1}\right] + \cdots$$
$$+ \operatorname{Im}\left[(A_{i_w})^{-1}C_{m_1-1}B_{i_{m_1-1}}, (A_{i_w})^{-1}C_{m_1-1}A_{i_{m_1-1}}B_{i_{m_1-1}}, \ldots, (A_{i_w})^{-1}C_{m_1-1}A_{i_{m_1-1}}^{h_{m_1}-1}B_{i_{m_1-1}}\right].$$

Note that $A_{j_0}, A_{j_1}, \ldots, A_{j_{m_1}}$ are nonsingular over $\mathcal{F}_q$. Then we get $\mathcal{T}(\pi_3) = \mathcal{F}_q^{k+1}$.

(ii) We first consider the case of $n = 3$. Consider the graph (c) of Figure 6, the adjacency matrix of $\mathcal{G}^{i_s}$ is denoted as A_{i_s}. Let $\widetilde{\pi}_1 = \{(i_0, 1), (i_1, 2)\}$, $B_{i_0} = e_{3,3}$ $B_{i_1} = e_{2,3}$, and $B_{i_2} = e_{1,3}$. Then we get $\mathcal{T}(\widetilde{\pi}_1) = \operatorname{Im}[e_{1,3}, A_{i_1}e_{2,3}] + A_{i_1}^2 \operatorname{Im}[e_{3,3}] = \{\widetilde{\delta}_1, \widetilde{\delta}_2, \widetilde{\delta}_3\}$, where $\widetilde{\delta}_1 = [1, 0, 0]^{\mathrm{T}}$, $\widetilde{\delta}_2 = [0, a_{22}(i_2), 0]^{\mathrm{T}}$, $\widetilde{\delta}_3 = [0, 0, a_{33}(i_2)a_{33}(i_1)]^{\mathrm{T}}$ and $\mathcal{T}(\widetilde{\pi}_1) = \mathcal{F}_q^3$. Suppose that system (4) is controllable with $\widetilde{\pi}_2 = \{(i_0, h_0), \ldots, (i_{m_1-1}, h_{m_1-1})\}$ if $\bigcup_{s=0}^{n-1} \mathcal{G}^{i_s}$ is a directed star for $n = k$. We will prove that system (4) is controllable with $\widetilde{\pi}_3 = \{(j_0, h_0), \ldots, (j_{m_1-1}, h_{m_1-1}), (j_{m_1}, h_{m_1})\}$ if $\bigcup_{s=0}^{n-1} \mathcal{G}^{j_s}$ is a directed path for $n = k+1$. It follows that

$$\mathcal{T}(\widetilde{\pi}_2) = A_{i_{m_1-1}}^{h_{m_1-1}} \cdots A_{i_1}^{h_1} \operatorname{Im}\left[B_{i_0}, A_{i_0}B_{i_0}, \ldots, A_{i_0}^{h_0-1}B_{i_0}\right] + A_{i_{m_1-1}}^{h_{m_1-1}} \cdots A_{i_2}^{h_2} \operatorname{Im}\left[B_{i_1}, A_{i_1}B_{i_1}, \ldots, A_{i_1}^{h_1-1}B_{i_1}\right]$$
$$+ \cdots + \operatorname{Im}\left[B_{i_{m_1-1}}, A_{i_{m_1-1}}B_{i_{m_1-1}}, \ldots, A_{i_{m_1-1}}^{h_{m_1-1}-1}B_{i_{m_1-1}}\right],$$

where $\mathcal{T}(\widetilde{\pi}_2) = \mathcal{F}_q^k$. According to the structure of the directed star, we have $\mathcal{G}^{j_s} = (\mathcal{V}^{i_s} \bigcup \{k+1\}, \mathcal{E}^{i_s} \bigcup (1, k+1))$, where $\mathcal{V}^{i_s}$ is a set of vertices of $\mathcal{G}^{i_s}$, $\mathcal{E}^{i_s}$ is a set of edges of $\mathcal{G}^{i_s}$, an edge $(j, i) \in \mathcal{E}^{i_s}$ implies that the agent i can access the information of the agent j, and $\emptyset$ is an empty set. It follows that $A_{j_s} = \left[\begin{smallmatrix} A_{i_s} & \mathbf{0} \\ e_{1,k}^T & \gamma_{k+1}^{i_s} \end{smallmatrix}\right]$, where $s = 1, 2, \ldots, m_1 - 1$. Let $B_{j_s} = \left[\begin{smallmatrix} B_{i_s} \\ 0 \end{smallmatrix}\right], s = 0, 1, \ldots, m_1 - 1$, $B_{j_{m_1}} = e_{k+1,k+1}$, $A_{j_{m_1}} = \left[\begin{smallmatrix} A_{i_w} & \mathbf{0} \\ \mathbf{0} & \gamma_{k+1}^{i_w} \end{smallmatrix}\right]$ for some w and $h_{m_1} = 1$, where $w = 0, 1, \ldots, m_1 - 1$. Then we have

$$\mathcal{T}(\widetilde{\pi}_3) = \operatorname{Im}\begin{bmatrix} \mathbf{0}_k \\ 1 \end{bmatrix} + \operatorname{Im}\begin{bmatrix} C_0B_{i_0} & C_0A_{i_0}B_{i_0} & \cdots & C_0A_{i_0}^{h_0-1}B_{i_0} \\ * & * & \cdots & * \end{bmatrix} + \operatorname{Im}\begin{bmatrix} C_1B_{i_1} & C_1A_{i_1}B_{i_1} & \cdots & C_1A_{i_1}^{h_1-1}B_{i_1} \\ * & * & \cdots & * \end{bmatrix}$$
$$+ \cdots + \operatorname{Im}\begin{bmatrix} C_{m_1-1}B_{i_{m_1-1}} & C_{m_1-1}A_{i_{m_1-1}}B_{i_{m_1-1}} & \cdots & C_{m_1-1}A_{i_{m_1-1}}^{h_{m_1-1}-1}B_{i_{m_1-1}} \\ * & * & \cdots & * \end{bmatrix},$$

where $C_0 = A_{i_w}A_{i_{m_1-1}}^{h_{m_1-1}} \cdots A_{i_1}^{h_1}, C_1 = A_{i_w}A_{i_{m_1-1}}^{h_{m_1-1}} \cdots A_{i_2}^{h_2}, \ldots, C_{m_1-2} = A_{i_w}A_{i_{m_1-1}}^{h_{m_1-1}}$ and $C_{m_1-1} = A_{i_w}$. Note that $\mathcal{T}(\widetilde{\pi}_2) = \mathcal{F}_q^k$, and we get

$$\mathcal{T}(\widetilde{\pi}_3) = \operatorname{Im}\left[(A_{i_w})^{-1}C_0B_{i_0}, (A_{i_w})^{-1}C_0A_{i_0}B_{i_0}, \ldots, (A_{i_w})^{-1}C_0A_{i_0}^{h_0-1}B_{i_0}\right]$$
$$+ \operatorname{Im}\left[(A_{i_w})^{-1}C_1B_{i_1}, (A_{i_w})^{-1}C_1A_{i_1}B_{i_1}, \ldots, (A_{i_w})^{-1}C_1A_{i_1}^{h_1-1}B_{i_1}\right] + \cdots$$
$$+ \operatorname{Im}\left[(A_{i_w})^{-1}C_{m_1-1}B_{i_{m_1-1}}, (A_{i_w})^{-1}C_{m_1-1}A_{i_{m_1-1}}B_{i_{m_1-1}}, \ldots, (A_{i_w})^{-1}C_{m_1-1}A_{i_{m_1-1}}^{h_{m_1}-1}B_{i_{m_1-1}}\right].$$

Note that $A_{j_0}, A_{j_1}, \ldots, A_{j_{m_1}}$ are nonsingular over $\mathcal{F}_q$. Then we get $\mathcal{T}(\widetilde{\pi}_3) = \mathcal{F}_q^{k+1}$.

This completes the proof.

Remark 3. Ref. [29] proposes the algebraic conditions for the structural controllability with switching topology. The differences between the structural controllability problem studied in [29] and this paper are as follows. (i) In this paper, we propose graphical conditions instead of algebraic conditions for the controllability. (ii) In Theorem 3 of [29], each graph has spanning forest, i.e., leader-follower connected. However, we provide graphical conditions for the controllability even if each graph has some inaccessible nodes.

4 Conclusion

In this paper, the controllability problem was studied for multi-agent systems with switching topology over finite fields. We established several graphical conditions for controllability of multi-agent systems over finite fields. We proved that a switched multi-agent system is controllable over a finite field if each graph of the subsystem is a spanning forest. It is shown that a multi-agent system with switching topology can be controllable over a finite field even if each of its subsystems is not controllable. Finally, we showed that the switched system is controllable if the union of graphs is a path graph or a star graph.

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