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• **Supplementary File** •

Fully distributed consensus of second-order multi-agent systems using adaptive event-based control

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Appendix A The proof of Theorem 1

Proof. Let $\hat{x}(t) = L^{\frac{1}{2}}x(t)$, $\hat{v}(t) = L^{\frac{1}{2}}v(t)$, $\varepsilon(t) = (\hat{x}^T(t), \hat{v}^T(t))^T$, $e_v(t) = (e_{v_1}(t), \cdots, e_{v_N}(t))^T$. By (1) and (2), one has

$$\begin{cases} \dot{x}(t) = v(t) \\ \dot{v}(t) = -\alpha x(t) - C^2(t)Lv(t) - C^2(t)e_v(t) \end{cases} \tag{A1}$$

where $C(t) = diag\{c_1(t), \cdots, c_N(t)\}$.

Consider the following function:

$$V(t) = \varepsilon^T(t)P\varepsilon(t) + \eta\sum_{i=1}^{N} c_i^2(t),$$

where $P = diag\{\alpha, 1\}$ and $\eta > \frac{\beta^2}{2\gamma}$.

Computing the time derivative of $V(t)$, we can get

$$\dot{V}(t) = 2\varepsilon^T(t)P\dot{\varepsilon}(t) + 2\eta\sum_{i=1}^{N} c_i(t)\dot{c}_i(t).$$

By the definition of $\varepsilon(t)$ and $\dot{c}_i(t)$, one has

$$\begin{aligned} \dot{V}(t) &= 2\alpha x^T(t)L\dot{x}(t) + 2v^T(t)L\dot{v}(t) + 2\eta\sum_{i=1}^{N} c_i(t)\dot{c}_i(t) \\ &= -2v^T(t)LC^2(t)Lv(t) - 2v^T(t)LC^2(t)e_v(t) \\ &- 2\eta\sum_{i=1}^{N} c_i^2(t)q_{vi}^2(t_k^i) - \eta\gamma\sum_{i=1}^{N} c_i^2(t) \\ &\leqslant -v^T(t)LC^2(t)Lv(t) - \eta\gamma\sum_{i=1}^{N} c_i^2(t) + e_v^T(t)C^2(t)e_v(t) \\ &- v^T(t)LC^2(t)Lv(t) - 2v^T(t)LC^2(t)e_v(t) - e_v^T(t)C^2(t)e_v(t). \end{aligned} \tag{A2}$$

Using the definition of triggering time instants and triggering function (5), one has

$$\begin{aligned} 4e_v^T(t)C^2(t)e_v(t) &\leqslant 2v^T(t)LC^2(t)Lv(t) + 4v^T(t)LC^2(t)e_v(t) \\ &+ 2e_v^T(t)C^2(t)e_v(t) + 2\beta^2\sum_{i=1}^{N} c_i^2(t)e^{-\gamma(t-t_k^i)}. \end{aligned}$$

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Further, we can derive

$$\begin{aligned} e_v^T(t)C^2(t)e_v(t) \leqslant\ & \frac{1}{2}v^T(t)LC^2(t)Lv(t) + v^T(t)LC^2(t)e_v(t) \\ +\ & \frac{1}{2}e_v^T(t)C^2(t)e_v(t) + \frac{\beta^2}{2}\sum_{i=1}^{N} c_i^2(t), \end{aligned} \tag{A3}$$

Combining (A2) and (A3), one can obtain that

$$\begin{aligned} \dot{V}(t) \leqslant\ & -v^T(t)LC^2(t)Lv(t) - \eta\gamma\sum_{i=1}^{N} c_i^2(t) + \frac{\beta^2}{2}\sum_{i=1}^{N} c_i^2(t) \\ =\ & -v^T(t)LC^2(t)Lv(t) - \sum_{i=1}^{N}(\eta\gamma - \frac{\beta^2}{2})c_i^2(t). \end{aligned}$$

It follows from (3) that

$$c_i(t) = e^{-(q_{vi}^2(t_k^i)+\frac{1}{2}\gamma)(t-t_k^i)}c_i(t_k^i) \leqslant e^{-\frac{1}{2}\gamma(t-t_k^i)}c_i(t_k^i), t \in [t_k^i, t_{k+1}^i).$$

As $c_i(t_0) > 0$, one can easily derive $c_i(t) > 0$. Then, we have $\dot{V}(t) < 0$. Thus, one can derive $\lim_{t\to\infty}\varepsilon(t) = \mathbf{0}$, i.e., $\lim_{t\to\infty} L^{\frac{1}{2}}x(t) = \mathbf{0}$ and $\lim_{t\to\infty} L^{\frac{1}{2}}v(t) = \mathbf{0}$. Due to the communication graph is connected, there exists $\xi(t) \in \mathbb{R}$, $\zeta \in \mathbb{R}$ such that

$$\begin{cases} \lim\limits_{t\to\infty} |x_i(t) - \xi(t)| = 0, \\ \lim\limits_{t\to\infty} |v_i(t) - \zeta| = 0, \end{cases} \forall i = 1, 2, \cdots, N. \tag{A4}$$

Appendix B The proof of Theorem 2

Proof. For $t \in [t_k^i, t_{k+1}^i)$, computing the Dini derivative of $|e_{vi}(t)|$, one has

$$\begin{aligned} D^+|e_{vi}(t)| \leqslant\ & |\dot{e}_{vi}(t)| = |-\dot{q}_{vi}(t)| = |-\sum_{j=1}^{N} a_{ij}(\dot{v}_i(t) - \dot{v}_j(t))| \\ =\ & |\sum_{j=1}^{N} a_{ij}(\alpha x_i(t) + c_i^2(t)q_{vi}(t_k^i) - \alpha x_j(t) - c_j^2(t)q_{vj}(t_{k'}^j))| \\ =\ & |\alpha\sum_{j=1}^{N} a_{ij}(x_i(t) - x_j(t)) + \sum_{j=1}^{N} a_{ij}(c_i^2(t)q_{vi}(t_k^i) - c_j^2(t)q_{vj}(t_{k'}^j))| \\ \leqslant\ & m_1. \end{aligned} \tag{B1}$$

where $m_1 = \max\limits_i \sup\limits_{t\in[t_k^i, t_{k+1}^i)} \{|\alpha\sum_{j=1}^{N} a_{ij}(x_i(t) - x_j(t)) + \sum_{j=1}^{N}(c_i^2(t)q_{vi}(t_k^i) - c_j^2(t)q_{vj}(t_{k'}^j))|\} > 0$ and $t_{k'}^j$ is the latest update time instant of agent j.

It follows from $|e_{vi}(t_k^i)| = 0$ that

$$|e_{vi}(t)| \leqslant m_1(t - t_k^i). \tag{B2}$$

By the definition of triggering time sequence t_k^i, we always have the following inequality unless the the next event is triggered.

$$|q_{vi}(t_k^i)| + \beta e^{-\frac{1}{2}\gamma(t_{k+1}^i - t_k^i)} = 2|e_{vi}(t_{k+1}^i)| \leqslant 2m_1(t_{k+1}^i - t_k^i). \tag{B3}$$

Then, via (B3), one has

$$\beta e^{-\frac{1}{2}\gamma(t_{k+1}^i - t_k^i)} \leqslant 2m_1(t_{k+1}^i - t_k^i). \tag{B4}$$

Denote $T_k^i = t_{k+1}^i - t_k^i$, we have

$$\beta e^{-\frac{1}{2}T_k^i} \leqslant 2m_1 T_k^i. \tag{B5}$$

By (B5), one can easily derive that $\{T_k^i > 0 : 2m_1 T_k^i - \beta e^{-\frac{1}{2}T_k^i} \geqslant 0\}$ is nonempty and $T = \inf\limits_k\{T_k^i\} > 0$ for $\forall i$.

Therefore, there is no Zeno-behavior for any agent i.

Appendix C The proof of Theorem 3

Proof. For $t \in [t_k^i, t_{k+1}^i)$, computing the Dini derivative of $|e_{xi}(t)| + |e_{vi}(t)|$, one can obtain that

$$\begin{aligned} D^+(|e_{xi}(t)| + |e_{vi}(t)|) &= D^+(|e_{xi}(t)|) + D^+(|e_{vi}(t)|) \leqslant |\dot{e}_{xi}(t)| + |\dot{e}_{vi}(t)| \\ &= |-\dot{q}_{xi}(t)| + |-\dot{q}_{vi}(t)| \\ &= |-q_{vi}(t)| + |\alpha q_{xi}(t) + \sum_{j=1}^N a_{ij}(c_i^2(t)q_{vi}(t_k^i) - c_j^2(t)q_{vj}(t_{k'}^j))| \\ &\leqslant \alpha|e_{xi}(t)| + |e_{vi}(t)| + \alpha|q_{xi}(t_k^i)| + |q_{vi}(t_k^i)| \\ &+ |\sum_{j=1}^N a_{ij}(c_i^2(t)q_{vi}(t_k^i) - c_j^2(t)q_{vj}(t_{k'}^j))|, \end{aligned}$$

where $e_{xi}(t) = q_{xi}(t_k^i) - q_{xi}(t)$.

It follows from (3) that

$$c_i(t) = e^{-(q_{vi}^2(t_k^i) + \frac{1}{2}\gamma)(t - t_k^i)} c_i(t_k^i) \leqslant e^{-\frac{1}{2}\gamma(t - t_k^i)} c_i(t_k^i), t \in [t_k^i, t_{k+1}^i),$$

and then we can derive that

$$\begin{aligned} c_i(t) &\leqslant e^{-\frac{1}{2}\gamma(t - t_k^i)} \cdot e^{-\frac{1}{2}\gamma(t_k^i - t_{k-1}^i)} c_i(t_{k-1}^i) \\ &\leqslant \cdots \leqslant e^{-\frac{1}{2}\gamma(t - t_0)} c_i(t_0). \end{aligned}$$

Due to $0 < c_i(t_0) \leqslant \psi$, one can derive that

$$0 < c_i(t) \leqslant \psi, t \in [t_0, +\infty), i = 1, \cdots, N.$$

As $\alpha \geqslant 1$ and $0 < c_i(t) \leqslant \psi$, one has

$$D^+(|e_{xi}(t)| + |e_{vi}(t)|) \leqslant \alpha(|e_{xi}(t)| + |e_{vi}(t)|) + h_{kk'}^{ij}.$$

It follows from $|e_{xi}(t_k^i)| + |e_{vi}(t_k^i)| = 0$ that

$$|e_{xi}(t)| + |e_{vi}(t)| \leqslant \int_{t_k^i}^t h_{kk'}^{ij} e^{\alpha(t-s)} ds = \frac{h_{kk'}^{ij}}{\alpha}(e^{\alpha(t - t_k^i)} - 1),$$

then, we can get

$$|e_{vi}(t)| \leqslant \frac{h_{kk'}^{ij}}{\alpha}(e^{\alpha(t - t_k^i)} - 1). \tag{C1}$$

Following the proof of Theorem 1, one can derive that the formula (A4) holds. Thus, the systems (1) can reach the consensus.

In the following, the Zeno-behavior of system (1) will be excluded.

By the definition of triggering time sequence t_k^i, we always have the following inequality unless the the next event is triggered.

$$\begin{aligned} |q_{vi}(t_k^i)| + \beta e^{-\frac{1}{2}\gamma(t_{k+1}^i - t_k^i)} &= \frac{2h_{kk'}^{ij}}{\alpha}(e^{\alpha(t - t_k^i)} - 1) \\ &\leqslant \frac{2m_2}{\alpha}(e^{\alpha(t_{k+1}^i - t_k^i)} - 1), \end{aligned}$$

where $m_2 = \max_i \{h_{kk'}^{ij}\} > 0$.

Then, one can obtain that

$$\beta e^{-\frac{1}{2}\gamma T_k^i} \leqslant \frac{2m_2}{\alpha}(e^{\alpha T_k^i} - 1),$$

which implies that $T = \inf_k \{T_k^i\} > 0, \forall i$.

Thus, there is no Zeno-behavior for any agent i.

Appendix D A numerical example

In this part, an example is given to validate the availability of the proposed results.

Consider a second-order MASs with five agents, where the communication topology is described by Figure D1. Obviously, the undirected graph is connected. According to Theorem 1, 2 and 3, for any $\alpha \geqslant 1$, $\beta > 0$ and $\gamma > 0$, the system can reach the consensus and it does not exhibit Zeno-behavior.

Assume $\alpha = 1$, $\beta = 0.1$, $\gamma = 0.1$, which satisfy the conditions in Theorem 1, 2 and 3. Let the initial states $x(0) = [0, 1, 2, 3, 4]^T$, $c(0) = [5, 4, 2, 3, 5]^T$, $v(0) = [0, 0.5, 1, 1.5, 2]^T$ and $\psi = 5$. The position and velocity responses are depicted in Figure D2, Figure D3 and Figure D4, Figure D5, respectively, which illustrate that consensus has been achieved. Furthermore, Figure D6 and Figure D7 show the Zeno-behavior is excluded.

Remark 1. From the simulation results, it can be seen that Theorem 3 does more event triggering times than that in Theorem 1. In fact, the event triggering times in Theorem 1 for agents 1-5 are $26, 37, 28, 28, 49$, respectively and the event triggering times in Theorem 3 for agents 1-5 are $1181, 1353, 1453, 1536, 1197$, respectively.

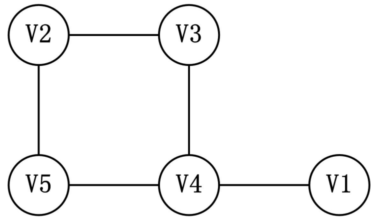

Figure D1 The communication graph.

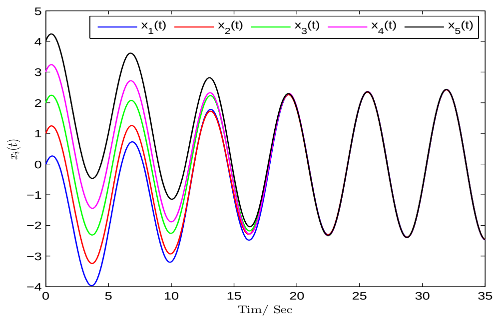

Figure D2 Positions consensus by Theorem 1.

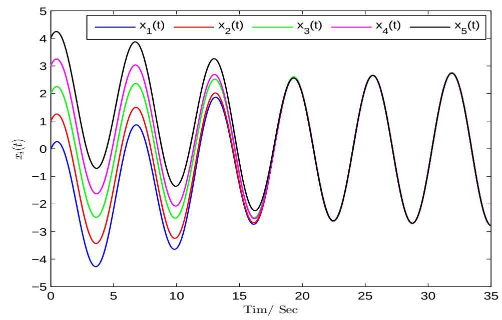

Figure D3 Positions consensus by Theorem 3.

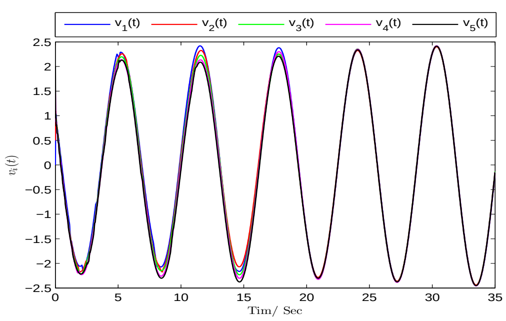

Figure D4 Velocities consensus by Theorem 1.

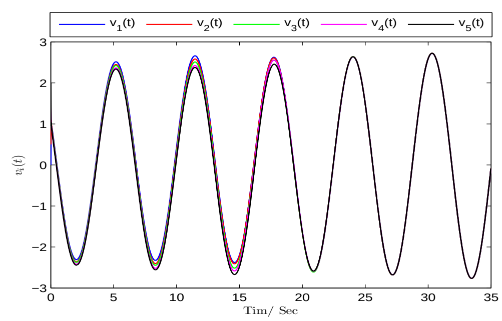

Figure D5 Velocities consensus by Theorem 3.

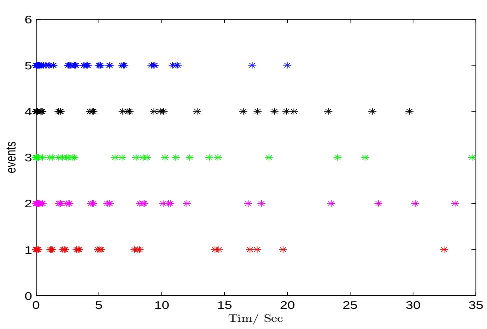

Figure D6 Events for agents by Theorem 1.

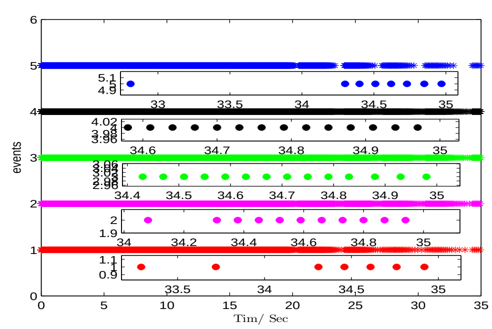

Figure D7 Events for agents by Theorem 3.