Supporting Materials (Appendices A and B)

Appendix A Preliminaries

1. Petri Nets

A *Petri net* is a four-tuple $N = (P, T, F, W)$ where P and T are called the sets of *places* and *transitions*, respectively and they are finite, non-empty, and disjoint sets. $F \subseteq (P \times T) \cup (T \times P)$ is the set of *flow relation* that are graphically denoted by directed arcs connecting places to transitions. The function W: $(P \times T) \cup (T \times P) \rightarrow \mathbf{N}$ assigns each arc a weight. Given $x, y \in P \cup T$, $W(x, y)>0$ if $(x, y) \in F$, and $W(x, y)=0$ otherwise. The *preset* of a node $x \in P \cup T$ is ${}^{\bullet}x = \{y \in P \cup T \mid (y, x) \in F\}$ and the *post-set* of a node $x \in P \cup T$ is $x^{\bullet} = \{y \in P \cup T \mid (x, y) \in F\}$. The *incidence matrix* of N is a matrix $[N]$: $P \times T \rightarrow \mathbf{Z}$ such that $[N](p, t)=W(t, p)-W(p, t)$. $[N](p, \bullet)$ ($[N](\bullet, t)$) denotes the incidence vector with respect to a place p (transition t), i.e., a row (column) in $[N]$.

An *ordinary marking* μ of N is a mapping from P to $\mathbf{N}$. $t \in T$ is *enabled* at an ordinary marking μ if $\forall p \in {}^{\bullet}t$, $\mu(p) \geq W(p, t)$. An enabled transition t at an ordinary marking μ can fire. $\mu[t>\mu'$ denotes that the firing of t at μ leads to a new marking μ', where $\mu'(p)=\mu(p) + [N](p, t)$, $\forall p \in P$. A transition sequence $\alpha = t_1 t_2 \ldots t_k$, $t_i \in T$ ($i \in \mathbf{N}_k=\{1, 2, \ldots, k\}$), is feasible from a marking μ_1 if there exists μ_{i+1}, $i \in \mathbf{N}_k$, such that $\mu_i[t_i>\mu_{i+1}$, $i \in \mathbf{N}_k$. We use $\mu_1[\alpha>\mu_{k+1}$ to denote the case that μ_{k+1} is reachable from μ_1 by firing α. The set of all reachable markings of N from initial marking μ_0 is denoted by $R(N, \mu_0)$.

A Petri net (N, μ_0) is *bounded* if the token count of each place p does not exceed a finite number $B \in \mathbf{N}^+$ for any marking μ reachable from μ_0, i.e., $\mu(p) \leq B$. Otherwise, the net is *unbounded*.

2. ω–numbers

We review the related notations of ω–numbers defined in [5-7] .

A subset of integers S is called an *ω-number* if $\exists k \in \mathbf{N}^+$, $n, q \in \mathbf{Z}$ such that $S=\{ik + q \mid i \geq n\}$. S can be uniquely expressed as $S=\omega(k, n, q) \equiv k\omega_n + q \equiv \{ik + q \mid k \in \mathbf{N}^+, n \in \mathbf{Z}, 0 \leq q < k, i \geq n\}$, where $\omega(k, n, q)$ or $k\omega_n + q$ is called a *canonical* ω-number with k as its *base*, n as the *least bound*, and q as the *remainder*.

A vector $x \in \mathbf{Z}_\omega^n$ is called an *ω-vector* if at least one of its components is an ω-number, where $\mathbf{Z}_\omega$ is the set of integers and ω-numbers. Clearly, an ω-vector can be viewed as a set of ordinary integer vectors. A marking μ is called an *ω-marking* if it can be represented by an ω-vector. An ω-marking can be viewed as a set of ordinary markings.

At an ω-marking μ, $t \in T$ is *enabled* if t is enabled at all ordinary markings of μ; t is *not enabled* at μ if t is not enabled at any ordinary marking of μ; t is *conditionally enabled* at μ if it is not enabled at some ordinary markings of μ but enabled at any other ordinary markings of μ. Note that if t is enabled at μ and $\mu' \geq \mu$, it holds that t is enabled at μ'.

Appendix B Proofs

***Property* 1:** Let $S_1=\boldsymbol{\omega}(k_1^{(1)}, k_2^{(2)}, \ldots, k_m^{(m)}; q_1)$ and $S_2=\boldsymbol{\omega}(k_1^{(1)}, k_2^{(2)}, \ldots, k_m^{(m)}; q_2)$ be two ω-numbers with the same form. $S_1 \subseteq S_2$ *iff* $q_1-q_2= c_1k_1+c_2k_2+\ldots+ c_mk_m$, $c_1, c_2, \ldots, c_m \in \mathbf{N}$.

Proof: (Sufficiency) It is clear that $S_1=\boldsymbol{\omega}(k_1^{(1)}, k_2^{(2)}, \ldots, k_m^{(m)}; q_1)\equiv\{i^{(1)}k_1+i^{(2)}k_2+ \ldots + i^{(m)}k_m+ q_1 \mid i^{(1)}, i^{(2)}, \ldots, i^{(m)}\in \mathbf{N}\}$ and $S_2=\boldsymbol{\omega}(k_1^{(1)}, k_2^{(2)}, \ldots, k_m^{(m)}; q_2)\equiv\{i^{(1)}k_1+i^{(2)}k_2+ \ldots + i^{(m)}k_m+ q_2 \mid i^{(1)}, i^{(2)}, \ldots, i^{(m)}\in \mathbf{N}\}$. Since $q_1-q_2=c_1k_1+c_2k_2+\ldots+c_mk_m$, we have $S_1=\{(i^{(1)}+c_1)k_1+ (i^{(2)}+c_2)k_2 + \ldots + (i^{(m)}+c_m)k_m+q_2 \mid i^{(1)}, i^{(2)}, \ldots, i^{(m)}\in \mathbf{N}\}= \{i^{(1)}k_1+i^{(2)}k_2+ \ldots + i^{(m)}k_m+q_2 \mid i^{(1)}\geq c_1, i^{(2)}\geq c_2, \ldots, i^{(m)}\geq c_m\}$. Obviously, $S_1\subseteq S_2$ holds.

(Necessity) Since $S_1 \subseteq S_2$ and $q_1\in S_1$, we have $q_1\in S_2= \{i^{(1)}k_1+i^{(2)}k_2+ \ldots + i^{(m)}k_m+ q_2 \mid i^{(1)}, i^{(2)}, \ldots, i^{(m)}\in \mathbf{N}\}$. Clearly, $\exists\, c_1, c_2, \ldots, c_m\in \mathbf{N}$ such that $q_1-q_2= c_1k_1+c_2k_2+\ldots+ c_mk_m$. ■

***Property* 2:** Let $\mu= (S_1, S_2, \ldots, S_n)$ be an ω-vector. We have $\mu=\Delta$ *iff* μ is an independent ω-vector, where $\Delta=\{(a_1, a_2, \ldots, a_n) \mid a_g\in S_g$ (or $a_g=S_g$ if S_g is an integer), $\forall g\in \{1, 2, \ldots, n\}\}$.

Proof: (Sufficiency) The proof is trivial.

(Necessity) By contradiction, suppose that μ is not an independent ω-vector. Hence, $\exists x, y\in \{1, 2, \ldots, n\}$ and $x\neq y$ such that S_x is not independent of S_y. Let $S_x=\{i^{(1)}k_{x1}+i^{(2)}k_{x2}+ \ldots + i^{(m)}k_{xm}+q_x \mid i^{(1)}, i^{(2)}, \ldots, i^{(m)}\in \mathbf{N}\}$ and $S_y=\{i^{(1)}k_{y1}+i^{(2)}k_{y2}+ \ldots + i^{(m)}k_{ym}+q_y \mid i^{(1)}, i^{(2)}, \ldots, i^{(m)}\in \mathbf{N}\}$. According to Definition 7, $\exists j\in \{1, 2, \ldots, m\}$ such that $k_{xj} \bullet k_{yj}\neq 0$. Now, let $i^{(j)}=c \geq 1$ and $\forall h\neq j$, $i^{(h)}=0$. In this case, $a_x= ck_{xj}+ q_x$ and $a_y=ck_{yj}+q_y$. This means when $a_x=ck_{xj}+q_x\in S_x$, a_y cannot be equal to any number in S_y except $ck_{yj}+q_y$. Hence, $\mu\neq\Delta$, which obviously contradicts the fact $\mu=\Delta$. Therefore, μ is an independent ω-vector. ■

Note that Δ is a set consisting of ordinary vectors, which results from an arbitrary combination of n components that are either an element in a set represented by an ω-number or an integer.

***Property* 3**: Let $\mu_1=(S_{11}, S_{12}, \ldots, S_{1n})$ and $\mu_2=(S_{21}, S_{22}, \ldots, S_{2n})$ be two independent ω-vectors. $\mu_1 \subseteq \mu_2$ *iff* $S_{1i}\subseteq S_{2i}$ or $S_{1i} = S_{2i}$ or $S_{1i} \in S_{2i}$, $\forall i\in \{1, 2, \ldots, n\}$.

Proof: Straightforward from Property 2. ■

***Property* 4:** Let $\mu_1=(S_{11}, S_{12}, \ldots, S_{1x}, \ldots, S_{1n})$ and $\mu_2=(S_{21}, S_{22}, \ldots, S_{2x}, \ldots, S_{2n})$ be two ω-vectors with the same form. $\mu_1 \subseteq \mu_2$ *iff* there exists $C_{1\times m}=(c_1, c_2, \ldots, c_m) \in \mathbf{N}^m$ such that $\forall\, x\in \{1, 2, \ldots, n\}$,

1) $q_x - q_x'=C_{1\times m} \bullet (k_{x1}, k_{x2}, \ldots, k_{xm})^T$ if $S_{1x}=\boldsymbol{\omega}(k_{x1}^{(1)}, k_{x2}^{(2)}, \ldots, k_{xm}^{(m)}; q_x)$ and $S_{2x}=\boldsymbol{\omega}(k_{x1}^{(1)}, k_{x2}^{(2)}, \ldots, k_{xm}^{(m)}; q_x')$ are ω-numbers with the same form, and

2) $S_{1x}= S_{2x}$ if S_{1x} and S_{2x} are both integers.

Proof: Clearly, the sets represented by μ_1 and μ_2 are accordingly as follows.

$\mu_1=(S_{11}, S_{12}, \ldots, S_{1x}, \ldots, S_{1n})$

$\equiv\{I_{1\times m}\bullet K_{m\times n}+Q_{1\times n} \mid I_{1\times m}\in \mathbf{N}^m\}$, and

$\mu_2=(S_{21}, S_{22}, \ldots, S_{2x}, \ldots, S_{2n})$

$\equiv\{I_{1\times m}\bullet K_{m\times n}+Q_{1\times n}' \mid I_{1\times m}\in \mathbf{N}^m\}$,

where

$I_{1\times m} =(i^{(1)}, i^{(2)}, \ldots, i^{(m)})$,

$$K_{m\times n} = \begin{bmatrix} k_{11} & k_{21} & \cdots & k_{n1} \\ k_{12} & k_{22} & \cdots & k_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ k_{1m} & k_{2m} & \cdots & k_{nm} \end{bmatrix} \neq 0,$$

$Q_{1\times n}=(q_1, q_2, \ldots, q_n)$,

$Q_{1\times n}'=(q_1', q_2', \ldots, q_n')$, and

$k_{xy}\in\mathbf{N}$, $q_x, q_x'\in\mathbf{Z}$, $\forall x\in\{1, 2, \ldots, n\}$, $\forall y\in\{1, 2, \ldots, m\}$.

(Sufficiency) We can know that there exists $C_{1\times m}=(c_1, c_2, \ldots, c_m)\in\mathbf{N}^m$, such that $\forall x\in\{1, 2, \ldots, n\}$, $q_x- q_x'= C_{1\times m}\bullet(k_{x1}, k_{x2}, \ldots, k_{xm})^T$. Hence, we have

$\mu_1=(S_{11}, S_{12}, \ldots, S_{1x}, \ldots, S_{1n})$

$\equiv\{I_{1\times m}\bullet K_{m\times n}+Q_{1\times n}| I_{1\times m}\in\mathbf{N}^m\}$

$=\{(I_{1\times m}+C_{1\times m})\bullet K_{m\times n}+Q_{1\times n}' \mid I_{1\times m}\in\mathbf{N}^m\}$.

Obviously, $\mu_1\subseteq\mu_2$ holds.

(Necessity) It is clear that $Q_{1\times n}\in\mu_1$. Since $\mu_1\subseteq\mu_2$, we have $Q_{1\times n}\in\mu_2$. Hence, $\exists\ I_{1\times m}\in\mathbf{N}^m$, such that $Q_{1\times n}= I_{1\times m}\bullet K_{m\times n}+Q_{1\times n}'$. In more detail, there exists $I_{1\times m}\in\mathbf{N}^m$ such that $\forall x\in\{1, 2, \ldots, n\}$, $q_x- q_x'= I_{1\times m}\bullet(k_{x1}, k_{x2}, \ldots, k_{xm})^T$. That is to say, there exists $C_{1\times m}=(c_1, c_2, \ldots, c_m)\in\mathbf{N}^m$ such that $\forall x\in\{1, 2, \ldots, n\}$, $q_x- q_x'=C_{1\times m}\bullet(k_{x1}, k_{x2}, \ldots, k_{xm})^T$ if S_{1x} and S_{2x} are two ω-numbers with the same form and $S_{1x}= S_{2x}$ if they are both integers. ■

In the following, we prove Theorems 1-3. Before that, we present some necessary results.

***Property* 5:** Let $\mu_0, \mu_1, \mu_2, \ldots, \mu_n$ be a sequence of markings corresponding to a path starting from the root node of NRT. ω-numbers with the same dimension in each coordinate of $\mu_0, \mu_1, \mu_2, \ldots$, and μ_n have the same form.

Proof: By Algorithm 1, once a new ω-element $\omega^{(j)}$ is introduced into a component of a marking, the base related to $\omega^{(j)}$ can never be changed in the corresponding component of any marking that is generated subsequently in the same path. Hence, the conclusion holds. ■

***Property* 6**: Let $\mu_0, \mu_1, \mu_2, \ldots, \mu_n$ be a sequence of markings corresponding to a path starting from the root node of NRT. $u_j\subseteq u_i$ if $u_j>u_i$, where $i, j\in\{0, 1, 2, \ldots, n\}$ and $j>i$.

Proof: Since $u_j>u_i$, we know that u_j and u_i are ω-markings with the same form or both ordinary markings. According to the construction algorithm of NRT, it can be concluded that $u_j\subseteq u_i$ since otherwise a new superscript has to be introduced into u_j. ■

***Lemma* 1** [r1]: In any infinite directed tree where each node has only a finite number of direct successors, there exists an infinite path leading from the root.

***Theorem* 1** (*Finiteness*): The NRT of an unbounded PN is finite.

Proof: By contradiction, suppose that there exists an infinite NRT. Due to Lemma 1, there exists an infinite path $x_0, x_1, x_2, \ldots$ from the root node x_0. Accordingly, we have an infinite sequence of makings, denoted as $u[x_0], u[x_1], u[x_2], \ldots$.

Consider the first coordinate of $u[x_0], u[x_1], u[x_2], \ldots$, denoted as $u[x_0](p_1), u[x_1](p_1), u[x_2](p_1), \ldots$. Clearly, it is impossible to introduce infinite superscripts during constructing NRT. Hence, there exists $u[x_a](p_1)$ in $u[x_0](p_1), u[x_1](p_1), u[x_2](p_1), \ldots$, which is an ω-number with the maximal dimension. Besides, it is easy to know that $u[x_a](p_1), u[x_{a+1}](p_1), u[x_{a+2}](p_1), \ldots$ is an infinite sequence of ω-numbers with the same dimension. According to Property 5, $u[x_a](p_1), u[x_{a+1}](p_1), u[x_{a+2}](p_1), \ldots$ is an infinite sequence of ω-numbers with the same form. Hence, an infinite nondecreasing subsequence can be definitely found in $u[x_a](p_1), u[x_{a+1}](p_1), u[x_{a+2}](p_1), \ldots$. In other words, we can find an infinite node subsequence of $x_0, x_1, x_2, \ldots$, denoted as $x_0', x_1', x_2', \ldots$ such that $u[x_0'](p_1)\le u[x_1'](p_1)\le u[x_2'](p_1)\le \ldots$. Now, consider the second coordinate of $u[x_0']$, $u[x_1']$, $u[x_2']$, …, denoted as $u[x_0'](p_2)$, $u[x_1'](p_2)$, $u[x_2'](p_2)$, …. Similarly, we can also find an infinite node subsequence of x_0', x_1', x_2', …, denoted as x_0'', x_1'', x_2'', …, such that $u[x_0''](p_2)\leq u[x_1''](p_2)\leq u[x_2''](p_2)\leq$ …. Finally, we can definitely find an infinite node subsequence of x_0, x_1, x_2, …, denoted as x_0^*, x_1^*, x_2^*, … such that $u[x_0^*]\leq u[x_1^*]\leq u[x_2^*]\leq$ …. But by construction, it must be an infinite strictly increasing subsequence, i.e., $u[x_0^*]<u[x_1^*]<u[x_2^*]<$ …, since otherwise a node would be a duplicate one with no successors. Besides, $\forall u[x_j^*]>u[x_i^*]$, $u[x_j^*]\not\subset u[x_i^*]$, since otherwise a node would be a ω-duplicate one with no successors. However, this contradicts Property 6. Therefore, the NRT of an unbounded net is finite. ■

***Theorem* 2** (*Reachability*): The NRT of an unbounded PN consists of only but all reachable markings from its initial marking.

Proof: According to the construction algorithm for NRT, it is easy to conclude that the reachable marking set is contained in NRT. In what follows, we have to prove that any marking in NRT belongs to the reachable marking set.

First, consider the direct successors of the root node. Obviously, the marking sets that correspond to these direct successors are contained in the reachable marking set. Next, consider the director successors of a node with the marking set being contained in the reachable marking set. Similarly, it is easy to see that the marking sets of these director successors are also contained in the reachable marking set. As a result, we can conclude that any marking in NRT belongs to the reachable marking set. ■

***Theorem* 3** (*Deadlock-checking*): An unbounded PN has deadlocks *iff* its NRT contains terminal nodes or full conditional nodes.

Proof: (Sufficiency) It is clear that a terminal node or a full conditional node definitely contains a dead marking. Hence, we can see that there is a dead marking in the NRT. By Theorem 2, the dead marking is reachable in the unbounded PN. As a result, the unbounded PN has deadlocks.

(Necessity) Since the unbounded PN has deadlocks, we can see that there is a dead marking in the NRT according to Theorem 2. Clearly, dead markings are only contained in terminal nodes or full conditional nodes in the NRT. As a result, the NRT contains terminal nodes or full conditional nodes. ■

Reference

[r1] König D. Über eine Schlussweise aus dem Endlichen ins Unendliche. Acta Sci. Math. (Szeged) (in German), 1927, 3(2-3): 121-130