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IQC based robust stability verification for a networked system with communication delays

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Abstract In this paper, we consider robust stability analysis of a networked system with uncertain communication delays. Each of its subsystems can have different dynamics, and interconnections among its subsystems are arbitrary. It is assumed that there exists an uncertain but constant delay in each communication channel. Using the so called integral quadratic constraint (IQC) technique, a sufficient robust stability condition is derived utilizing a sparseness assumption of the interconnections, and a set of decoupled robustness conditions are further derived which depend only on parameters of each subsystem, the subsystem connection matrix (SCM) and the selected IQC multipliers. These characteristics result in an evident improvement of computational efficiency for robustness verification of the networked system with delay uncertainties, which is illustrated by some numerical results.

Keywords large scale networked system, time delay, robust stability, integral quadratic constraints, linear matrix inequalities

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1 Introduction

Over the past decades, considerable research interests have been devoted to the stability analysis and controller design of large scale networked systems (LSNS) [1–4]. In LSNS, the subsystems interact over a communication network, so that the entire system exhibits complex dynamical behaviors. Owning to the distributed nature of LSNS, the information exchanged by these subsystems may be delayed during transmission. The presence of these network-induced delays can lead to performance deterioration and even destabilization of the system. The robust stability issue of LSNS with communication delays are, therefore, of theoretical and practical importance.

One of the challenging problems for robust stability analysis is that computational burden grows rapidly with the state dimension of the system. This is especially true for LSNS that inherently have a large system scale. On the other hand, it has also been noticed that some characteristics of the interconnection structure of LSNS can be potentially utilized to reduce the computational cost [5, 6]. The objective of this paper is to develop more computationally efficient conditions for robust stability analysis of LSNS with communication delays.

As for robust stability of time-delayed systems, integral quadratic constraints (IQCs) provide a unified framework for robustness analysis by characterizing the input-output behaviors of different types of

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nonlinearities and uncertainties in a quadratic form (see [7–12]). In these literatures, a library of IQC multipliers have been presented for linear time invariant (LTI) systems with constant and time-varying delays. In contrast to the Lyapunov-Krasovskii approach [13, 14], the IQC-based approach depends only on the choice of IQCs describing the uncertainties, which can be easily extended to deal with the LSNS with combination of various uncertainties.

A major difficulty to analyze the robustness of an uncertain LSNS is the treatment of the interconnections among the subsystems in an efficient way. Spatially invariant NS were studied in [1,3], which derived some computationally attractive sufficient conditions through a spatial Laplace transformation. Ref. [15] considered an NS with cyclic interconnections, and presented an exact IQC-based delay-dependent stability condition. Identical subsystems connected over a graph with diagonalizable pattern matrix were considered in [16]. Heterogeneous subsystems connected over arbitrary undirected graphs were investigated in [2, 4], which applied dissipativity theory to LSNS and used a family of coupled IQC multipliers to model the interconnections. Sparsity in the interconnection was explicitly considered in [5, 6]. Especially in [6], a necessary and sufficient condition was derived for the stability of an NS with arbitrary interconnection structures, and some decoupled sufficient stability conditions were derived by utilizing the structural properties of the interconnections.

In this paper, we reinvestigate robust stability of the NS adopted in [6] under the IQC framework. We further assume that the interactions among the subsystems of the NS are arbitrary but with an uncertain constant delay in each communication channel. By separating the delay operators from the interconnections, we describe the uncertain NS as a linear fractional transformation (LFT) model.

In order to reduce the computational burden of performing robust stability verification, we first utilize the sparse structure reflected by the subsystem connection matrix (SCM). Inspired by [5], we employ a special IQC multiplier to represent the interconnections among the subsystems. Using this IQC in combination with a set of IQCs for the delay-difference operators, we derive a sufficient robustness condition for the uncertain NS, which allow us to apply effective sparse solvers to solve the robustness problem.

Next, we focus on exploiting some structural properties of the interconnections. Based on a general assumption on the SCM Φ and Theorem 2 in [6], we propose some decoupled LMIs as the sufficient robust stability conditions using IQC analysis, which depend only on parameters of each subsystem, the SCM Φ and the selected IQC multipliers for the delays. Hence, they can be verified for every individual subsystem independently, which is an attractive property for the robustness analysis of LSNS.

Refs. [17, 18] also introduced separate LFT channels for delay operators and the interconnections, and arrived at decoupled conditions for the robustness analysis of an NS with delay uncertainties. Unlike these studies, which considered the interconnections as uncertainties and conservatively characterized them with block-diagonal multipliers, in this paper, we concentrate on exploiting the actual structure of the interconnections.

The rest of this paper is organized as follows. The IQC stability theory is reviewed in Section 2. Section 3 gives a description of the adopted LSNS and the problem formulation. Section 4 develops some robust stability conditions of the NS with uncertain communication delays. Some numerical results are reported in Section 5. Section 6 concludes this paper. The material in this paper was partially presented in [19].

Notation. Symbol $\mathbb{Z}_+$ is used to denote the set of nonnegative integers. ℓ_2^n denotes the set of n dimensional square summable signals. Notation $\mathbf{Rl}_\infty^{m\times n}$ is used to denote the space of proper rational transfer matrix with no poles on the unit circle, while $\mathbf{Rh}_\infty^{m\times n}$ represents the subspace of $\mathbf{Rl}_\infty^{m\times n}$ consisting of functions with no poles outside the open unit disk. The transpose and conjugate transpose of a matrix X is denoted by X^{T} and X^*, respectively. The $n\times n$ identity and the $m\times n$ zero matrix are denoted by I_n and $0^{m\times n}$, respectively, or just I and 0 if dimension is clear from context. 0_m denotes the m dimensional zero column vector. $\mathcal{F}_u\left(*,\#\right)$ stands for upper LFT. $\mathbf{diag}\{X_i|_{i=1}^L\}$ denotes a block diagonal matrix, while $\mathbf{col}\{X_i|_{i=1}^L\}$ the vector/matrix stacked by $X_i|_{i=1}^L$. The Kronecker product is denoted by $\otimes$. Let $\mathcal{D}_\tau$ denote the time delay operator: $(\mathcal{D}_\tau v)(t) = v(t-\tau), \tau\in\mathbb{Z}_+$, where τ specifies the delay, and $\mathcal{S}_\tau$ be the "delay-difference" operator $(\mathcal{D}_\tau - I)$; i.e., $\mathcal{S}_\tau(v) := v(t-\tau) - v(t)$.

2 Preliminaries

IQCs is a powerful tool in robust stability analysis of uncertain systems [7, 8, 15]. They are extensively applied to characterize the uncertainty in the system. More precisely, let Δ denote a bounded and causal operator. Two signals $w \in \ell_2^m$ and $d \in \ell_2^n$ related by $d = \Delta(w)$ satisfy the IQC defined by Π if

$$\int_{-\pi}^{\pi} \begin{bmatrix} \hat{w}\left(\mathrm{e}^{\mathrm{j}\omega}\right) \\ \hat{d}\left(\mathrm{e}^{\mathrm{j}\omega}\right) \end{bmatrix}^{*} \Pi\left(\mathrm{e}^{\mathrm{j}\omega}\right) \begin{bmatrix} \hat{w}\left(\mathrm{e}^{\mathrm{j}\omega}\right) \\ \hat{d}\left(\mathrm{e}^{\mathrm{j}\omega}\right) \end{bmatrix} \mathrm{d}\omega \geqslant 0, \tag{1}$$

where $\hat{w}$ and $\hat{d}$ are Fourier transforms of w and d, respectively. Π is a bounded LTI self-adjoint operator on ℓ_2 space, while $\Pi\left(\mathrm{e}^{\mathrm{j}\omega}\right)$ is its frequency response function. The time-domain form of (1) is

$$\begin{aligned} \sigma_{\Pi}(w, d) &= \left\langle \begin{bmatrix} w \\ d \end{bmatrix}, \Pi \begin{bmatrix} w \\ d \end{bmatrix} \right\rangle \\ &= \sum_{t=0}^{\infty} \begin{bmatrix} w(t) \\ d(t) \end{bmatrix}^{\mathrm{T}} \left(\Pi \begin{bmatrix} w \\ d \end{bmatrix} \right)(t) \geqslant 0. \end{aligned} \tag{2}$$

The operator Π is referred to as the multiplier of the quadratic form $\sigma_{\Pi}(w, d)$. Next, we give the following lemma to describe the IQC framework for analyzing robust stability of discrete uncertain systems, which is the discrete-time version of the IQC stability theorem in [8].

Lemma 1. Assume that a discrete uncertain system, $w = Gd, d = \Delta(w)$, where G is a linear time-invariant transfer function matrix. Δ is a bounded and causal operator representing the uncertainty in the system. Then, this uncertain system is robustly stable, if

(i) For every $\rho \in [0, 1]$, the interconnection of G and $\rho\Delta$ is well-posed;

(ii) For every $\rho \in [0, 1]$, the IQC defined by Π is satisfied by $\rho\Delta$;

(iii) There exists $\epsilon > 0$ such that

$$\begin{bmatrix} G(\mathrm{e}^{\mathrm{j}\omega}) \\ I \end{bmatrix}^{*} \Pi(\mathrm{e}^{\mathrm{j}\omega}) \begin{bmatrix} G(\mathrm{e}^{\mathrm{j}\omega}) \\ I \end{bmatrix} \leqslant -\epsilon I, \quad \forall |\omega| \leqslant \pi. \tag{3}$$

Condition (iii) is a frequency dependent, infinite dimensional LMI. If the operator Δ satisfies a set of IQCs defined by $\Pi_k, k = 1, 2, \ldots, n$, then Δ also satisfies the IQC defined by $\Pi = \sum_{k=1}^{n} \lambda_k \Pi_k$. In this case, robust stability analysis via IQC approach becomes a search for a suitable operator Π that satisfies the LMI in (iii). When $\Pi \in \mathbf{Rl}_{\infty}^{(m+n)\times(m+n)}$, this can be handled by converting the condition (iii) to a frequency independent finite dimensional LMI using the Kalma-Yakubovich-Popov (KYP) lemma [20, 21].

3 Problem formulation

Consider a networked system $\mathbf{\Sigma}$ consisting of N linear time invariant dynamic subsystems. Each subsystem $\mathbf{\Sigma}_i$ is described by the following discrete state-space equation:

$$\begin{bmatrix} x(t+1, i) \\ z(t, i) \\ y(t, i) \end{bmatrix} = \begin{bmatrix} A_{xx}(i) & A_{xv}(i) & B_x(i) \\ A_{zx}(i) & A_{zv}(i) & B_z(i) \\ C_x(i) & C_v(i) & D_u(i) \end{bmatrix} \begin{bmatrix} x(t, i) \\ v(t, i) \\ u(t, i) \end{bmatrix}, \tag{4}$$

in which $i = 1, 2, \ldots, N$. t and i stand for the temporal variable and the index number of a subsystem, respectively. $x(t, i)$ is the state vector of the i-th subsystem $\mathbf{\Sigma}_i$ at time t. $z(t, i)/v(t, i)$ is the output/input vector of the $\mathbf{\Sigma}_i$ to/from other subsystems, which is also called internal output/input vector throughout this paper. On the contrary, $y(t, i)/u(t, i)$ is called external output/input vector of the $\mathbf{\Sigma}_i$.

Assume that the dimensions of the vectors $x(t, i)$, $v(t, i)$, $u(t, i)$, $z(t, i)$ and $y(t, i)$ to be m_{xi}, m_{vi}, m_{ui}, m_{zi} and m_{yi}, respectively. Define $z(t, j)$ and $v(t, i)$ as the partitioned vectors $z(t, j) := \mathbf{col}\left\{ z_q(t, j)|_{q=1}^{m_{zj}} \right\}$

and $v(t,i) := \mathbf{col}\left\{v_p(t,i)|_{p=1}^{m_{vi}}\right\}$, respectively. For each distinct pair of subsystems, indexed by i and j, the constraint of each interconnection can be expressed as

$$v_p\left(t,i\right) = \left(\mathcal{D}_{\tau_{i,p}} z_q\right)\left(t,j\right), \quad \forall i \neq j, \quad 1 \leqslant i,j \leqslant N, \tag{5}$$

in which $\mathcal{D}_{\tau_{i,p}}$ is the delay operator that is defined by $v_p(t,i) = z_q(t-\tau_{i,p},j)$, where the delay duration $\tau_{i,p}$ is constant but uncertain. The upper bound of $\tau_{i,p}$ is denoted by $\mathcal{T}_{i,p} \in \mathbb{Z}_+$ such that $\tau_{i,p} \in [0, \mathcal{T}_{i,p}]$. Thus, the subsystems are connected through

$$v\left(t\right) = \left(\mathcal{D}_\tau \Phi z\right)\left(t\right). \tag{6}$$

Here, $z(t) = \mathbf{col}\left\{z(t,i)|_{i=1}^{N}\right\}$ and $v(t) = \mathbf{col}\left\{v(t,i)|_{i=1}^{N}\right\}$. In addition, $\mathcal{D}_\tau$ is the delay operator generated via $\mathcal{D}_{\tau_{i,p}}$.

$$\mathcal{D}_\tau = \mathbf{diag}\left\{\mathbf{diag}\left\{\mathcal{D}_{\tau_{i,p}}|_{p=1}^{m_{vi}}\right\}|_{i=1}^{N}\right\}. \tag{7}$$

It is assumed that every row of the SCM Φ has only one non-zero element which is equal to one. That means every single internal input channel of a subsystem is only affected by one internal output channel of another subsystem, and one internal output channel of a subsystem can affect several input channels of other subsystems. As argued in [22, 23], this assumption does not introduce any restrictions on the structure of the whole system.

Introduce the vector $\overline{v}(t)$ and let $\overline{v}(t) = \Phi z(t)$. we have that $v(t) = (\mathcal{D}_\tau \overline{v})(t)$. For the subsystem $\mathbf{\Sigma}_i$, the internal input vector $v(t,i)$ can be expressed by $v(t,i) = (\mathcal{D}_{\tau_i}\overline{v})(t,i)$, where $\mathcal{D}_{\tau_i} = \mathbf{diag}\left\{\mathcal{D}_{\tau_{i,p}}|_{p=1}^{m_{vi}}\right\}$. Based on these relations and (4), the state-space description of subsystem $\mathbf{\Sigma}_i$ and the interconnections among the subsystems can be rewritten as

$$\begin{bmatrix} x\left(t+1,i\right) \\ z\left(t,i\right) \\ y\left(t,i\right) \end{bmatrix} = \begin{bmatrix} A_{xx}\left(i\right) & A_{xv}\left(i\right) & B_x\left(i\right) \\ A_{zx}\left(i\right) & A_{zv}\left(i\right) & B_z\left(i\right) \\ C_x\left(i\right) & C_v\left(i\right) & D_u\left(i\right) \end{bmatrix} \begin{bmatrix} x\left(t,i\right) \\ \left(\mathcal{D}_{\tau_i}\overline{v}\right)\left(t,i\right) \\ u\left(t,i\right) \end{bmatrix}, \tag{8}$$

$$\overline{v}\left(t\right) = \Phi z\left(t\right). \tag{9}$$

By doing this, the delay operators are removed from the interconnections into the relevant subsystems as the delay effects on the internal inputs. In next section, we present some robust stability conditions of system $\mathbf{\Sigma}$ with time delay uncertainties under the IQC framework.

4 IQC based robustness analysis with uncertain communication delays

To facilitate the robustness analysis, a model transformation is performed on the original model in order to separate the delay uncertainties from the nominal LTI subsystems. Specifically, we introduce two vectors $w(t,i)$ and $d(t,i)$. Let $w(t,i) = \overline{v}(t,i)$ and $d(t,i) = (\mathcal{D}_{\tau_i}\overline{v})(t,i) - \overline{v}(t,i)$. From (8), we have the augmented model of the subsystem $\mathbf{\Sigma}_i$ described in the following LFT form:

$$\begin{bmatrix} x\left(t+1,i\right) \\ w\left(t,i\right) \\ z\left(t,i\right) \\ y\left(t,i\right) \end{bmatrix} = \begin{bmatrix} A_{xx}\left(i\right) & A_{xv}\left(i\right) & A_{xv}\left(i\right) & B_x\left(i\right) \\ 0 & 0 & I & 0 \\ A_{zx}\left(i\right) & A_{zv}\left(i\right) & A_{zv}\left(i\right) & B_z\left(i\right) \\ C_x\left(i\right) & C_v\left(i\right) & C_v\left(i\right) & D_u\left(i\right) \end{bmatrix} \begin{bmatrix} x\left(t,i\right) \\ d\left(t,i\right) \\ \overline{v}\left(t,i\right) \\ u\left(t,i\right) \end{bmatrix}, \tag{10}$$

$$d\left(t,i\right) = \left(\mathcal{S}_{\tau_i} w\right)\left(t,i\right), \tag{11}$$

in which $\mathcal{S}_{\tau_i}$ is a diagonal delay-difference operator, and it is obvious that $\mathcal{S}_{\tau_i} = \mathbf{diag}\left\{\mathcal{S}_{\tau_{i,p}}|_{p=1}^{m_{vi}}\right\}$. This results in an LFT formulation of each subsystem with a delay-difference uncertain block, which is widely utilized in robust control theory (see [24]). Figure 1 shows the transformed LFT model structure of system $\mathbf{\Sigma}$.

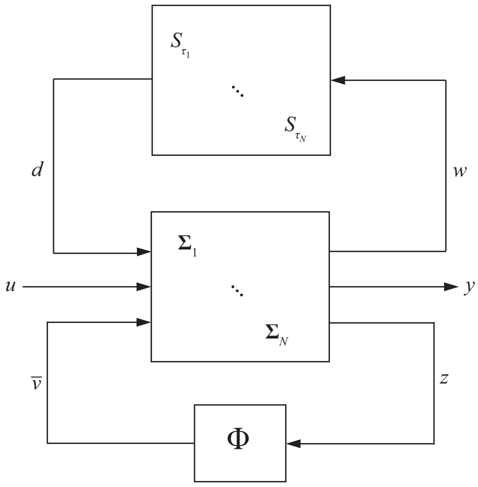

Figure 1 Transformed LFT model structure of system $\boldsymbol{\Sigma}$.

By assuming that $u(t,i)=0, i=1,2,\ldots,N$ in (10), we further describe the system $\boldsymbol{\Sigma}$ in the following form:

$$\begin{aligned} w &= G_{wd}d + G_{w\overline{v}}\overline{v}, \\ z &= G_{zd}d + G_{z\overline{v}}\overline{v}, \\ d &= \mathcal{S}_{\tau}(w), \\ \overline{v} &= \Phi z, \end{aligned} \tag{12}$$

where $\mathcal{S}_{\tau}=\mathbf{diag}\left\{\mathcal{S}_{\tau_i}|_{i=1}^{N}\right\}$, $G_{*\#}=\mathbf{diag}\{G_{*\#}(i)|_{i=1}^{N}\}$ with $*$=w, z and $\#$=d, $\overline{v}$. Define the dimensions of the vectors w, d, $\overline{v}$ and z as M_w, M_d, M_v and M_z, respectively, in which $M_w=M_d=M_v=\sum_{i=1}^{N} m_{vi}$ and $M_z=\sum_{i=1}^{N} m_{zi}$.

In this study, the input-output behavior of the delay-difference operator is characterized by multiple IQC multipliers under the LFT representation (10) and (11) of each subsystem, and the IQC stability theorem can be employed to verify the robustness of system $\boldsymbol{\Sigma}$ with delay uncertainties.

Assume the delay-difference operator $\mathcal{S}_{\tau_{i,p}}$ satisfies the IQC defined by $\Pi^{i,p}$, which can be partitioned as

$$\Pi^{i,p}=\begin{bmatrix}\Pi_{11}^{i,p} & \Pi_{12}^{i,p}\\ \Pi_{12}^{i,p*} & \Pi_{22}^{i,p}\end{bmatrix}. \tag{13}$$

Then, the diagonal delay-difference operators $\mathcal{S}_{\tau_i}$ and $\mathcal{S}_{\tau}$ satisfy the IQCs defined by

$$\Pi^{i}=\begin{bmatrix}\Pi_{11}^{i} & \Pi_{12}^{i}\\ \Pi_{12}^{i*} & \Pi_{22}^{i}\end{bmatrix} \tag{14}$$

and

$$\hat{\Pi}=\begin{bmatrix}\hat{\Pi}_{11} & \hat{\Pi}_{12}\\ \hat{\Pi}_{12}^{*} & \hat{\Pi}_{22}\end{bmatrix}, \tag{15}$$

in which the (r,c) blocks of Π^i and $\hat{\Pi}$ are denoted by $\Pi_{rc}^{i}=\mathbf{diag}\left\{\Pi_{rc}^{i,p}|_{p=1}^{m_{vi}}\right\}$ and $\hat{\Pi}_{rc}=\mathbf{diag}\left\{\Pi_{rc}^{i}|_{i=1}^{N}\right\}$, respectively.

Suppose that Π^i has a proper rational transfer matrix representation $\Psi(\overline{\zeta})^{\mathrm{T}}M\Psi(\zeta)$, in which $\Psi\in\mathbf{Rh}_{\infty}^{n_{\Psi}\times(m_{vi}+m_{vi})}$. Moreover, ζ denotes the variable of $\mathcal{Z}$-transformation, M is a self-adjoint real symmetric matrix. Let $\Psi(\zeta)$ have the following state-space realization:

$$\begin{aligned} x_{\Psi}(t+1) &= A_{\Psi}x_{\Psi}(t)+B_{\Psi,w}w(t)+B_{\Psi,d}d(t), \\ z_{\Psi}(t) &= C_{\Psi}x_{\Psi}(t)+D_{\Psi,w}w(t)+D_{\Psi,d}d(t) \end{aligned} \tag{16}$$

with zero initial conditions. Thus, w and d satisfy the IQC defined by $\Pi^i = \Psi(\overline{\zeta})^{\mathrm{T}} M \Psi(\zeta)$ if and only if the signal $z_\Psi = \Psi \left[\begin{smallmatrix} w \\ d \end{smallmatrix}\right]$ satisfies the following time domain quadratic constraint:

$$\sum_{t=0}^{\infty} z_\Psi(t)^{\mathrm{T}} M z_\Psi(t) \geqslant 0. \tag{17}$$

The classical approach to robust stability analysis of system $\mathbf{\Sigma}$ with uncertainties is to eliminate the SCM constraint in (9) so as to describe the overall networked system as a lumped one. Define $A_{*\#} = \mathbf{diag}\{A_{*\#}(i)|_{i=1}^N\}$ with $*$, $\#$=x, v, z. From (10) and (12), we obtain

$$\begin{aligned} \begin{bmatrix} x\,(t+1) \\ w\,(t) \end{bmatrix} &= \begin{bmatrix} \hat{A}_{xx} & \hat{A}_{xd} \\ \hat{A}_{wx} & \hat{A}_{wd} \end{bmatrix} \begin{bmatrix} x\,(t) \\ d\,(t) \end{bmatrix}, \\ d(t) &= \left(\mathcal{S}_\tau w\right)(t), \end{aligned} \tag{18}$$

in which $\hat{A}_{xx} = A_{xx} + A_{xv}(I - \Phi A_{zv})^{-1}\Phi A_{zx}$, $\hat{A}_{xd} = A_{xv} + A_{xv}(I - \Phi A_{zv})^{-1}\Phi A_{zv}$, $\hat{A}_{wx} = (I - \Phi A_{zv})^{-1}\Phi A_{zx}$ and $\hat{A}_{wd} = (I - \Phi A_{zv})^{-1}\Phi A_{zv}$.

We define (18) in the input-output form $w = \hat{G}d, d = \mathcal{S}_\tau w$ where $\hat{G} = G_{wd} + G_{w\overline{v}}(I - \Phi G_{z\overline{v}})^{-1}\Phi G_{zd}$ is the transfer function matrix of the nominal system. Assume that the interconnection $(\hat{G}, \mathcal{S}_\tau)$ is well-posed. Furthermore, if Theorem 1 in [6] is satisfied, $\hat{G}$ is internally stable.

According to the condition (iii) of Lemma 1, the NS with constant delay uncertainties is robustly stable if there exist $\epsilon > 0$, a set of IQC multipliers $\hat{\Pi}_k$ and scalars $\gamma_k \geqslant 0, k = 1, 2, \ldots, n$, such that

$$\sum_{k=1}^{n} \gamma_k \begin{bmatrix} \hat{G}(\mathrm{e}^{\mathrm{j}\omega}) \\ I \end{bmatrix}^* \hat{\Pi}_k(\mathrm{e}^{\mathrm{j}\omega}) \begin{bmatrix} \hat{G}(\mathrm{e}^{\mathrm{j}\omega}) \\ I \end{bmatrix} \leqslant -\epsilon I, \quad \forall |\omega| \leqslant \pi. \tag{19}$$

As argued in Section 2, the frequency dependent, infinite dimensional LMI (19) can be equivalently converted to a finite dimensional LMI using the KYP lemma.

Remark 1. Notice that the matrices $\hat{A}_{*\#}$ with $*$, $\#$=x, v, z in (18) are usually dense even if the SCM Φ is assumed to be sparse. It can be expected that the finite dimensional LMI converted from (19) is also dense. This implies that computational complexity for verifying the robustness condition based on the lumped formulation will grow prohibitively with the increment of system size.

It is obvious that it will meet considerably computational difficulties to solve the lumped formulation of the robustness problem when the NS has a large scale. However, the NS usually has a sparse structure reflected by the SCM Φ, which means each subsystem is only connected to a small number of neighbors. This structural property makes it possible to develop a sparse LMI to reduce the computational effort in some degree. According to the idea introduced by [5, 25], we define the following IQC multiplier to characterize the interconnections among the subsystems

$$\Pi_\Phi = \begin{bmatrix} -\Phi^{\mathrm{T}} X \Phi & \Phi^{\mathrm{T}} X \\ X\Phi & -X \end{bmatrix}, \tag{20}$$

where $X = xI > 0$. Using this interconnection constraint in combination with a set of IQCs for the delay uncertainties, the following sufficient condition is derived for the robustness verification of system $\mathbf{\Sigma}$.

Theorem 1. Consider the system $\mathbf{\Sigma}$. If for a given delay margin vector $\mathcal{T}_u = \mathbf{col}\{\mathcal{T}_{u_i}|_{i=1}^N\}$, in which $\mathcal{T}_{u_i} = \mathbf{col}\{\mathcal{T}_{u_{i,p}}|_{p=1}^{m_{vi}}\}$, $\mathcal{T}_{u_{i,p}} \in \mathbb{Z}_+$, there exist scalars $\lambda_k \geqslant 0, k = 1, 2, \ldots, n$, $X = xI > 0$ and $P > 0$ such that LMI (22) is satisfied with

$$\begin{aligned} A &= \mathbf{diag}\left\{ \left. \begin{bmatrix} \mathbf{diag}\{A_{\hat{\Psi}_k}|_{k=1}^n\} & 0 \\ 0 & A_{xx}(i) \end{bmatrix} \right|_{i=1}^N \right\}, \\ B &= \left[\mathbf{diag}\left\{ \left. \begin{bmatrix} \mathbf{col}\{B_{\hat{\Psi}_k,d}|_{k=1}^n\} \\ A_{xv}(i) \end{bmatrix} \right|_{i=1}^N \right\} \;\; \mathbf{diag}\left\{ \left. \begin{bmatrix} \mathbf{col}\{B_{\hat{\Psi}_k,w}|_{k=1}^n\} \\ A_{xv}(i) \end{bmatrix} \right|_{i=1}^N \right\} \right], \end{aligned}$$

$$C_k = \begin{bmatrix} \mathbf{diag}\left\{ \left[C_{\hat{\Psi}_k} \ 0^{m_{vi}\times m_{xi}} \right] \right\}\Big|_{i=1}^{N} \\ 0 \end{bmatrix}, \quad D_k = \begin{bmatrix} I_N \otimes D_{\hat{\Psi}_k,d} & I_{M_v} \\ I_{M_d} & 0 \end{bmatrix},$$

$$C_\Phi = \begin{bmatrix} \mathbf{diag}\left\{ \left[0^{m_{zi}\times M_{\hat{\Psi}}} \ A_{zx}(i) \right] \Big|_{i=1}^{N} \right\} \\ 0 \end{bmatrix}, \quad D_\Phi = \begin{bmatrix} \mathbf{diag}\{A_{zv}(i)|_{i=1}^{N}\} & \mathbf{diag}\{A_{zv}(i)|_{i=1}^{N}\} \\ 0 & I_{M_v} \end{bmatrix}. \tag{21}$$

$$\begin{bmatrix} A^{\rm T}PA - P & A^{\rm T}PB \\ B^{\rm T}PA & B^{\rm T}PB \end{bmatrix} + \sum_{k=1}^{n} \lambda_k \begin{bmatrix} C_k^{\rm T} \\ D_k^{\rm T} \end{bmatrix} \hat{M}_k(\mathcal{T}_u) \begin{bmatrix} C_k & D_k \end{bmatrix} + \begin{bmatrix} C_\Phi{}^{\rm T} \\ D_\Phi{}^{\rm T} \end{bmatrix} \Pi_\Phi(X) \begin{bmatrix} C_\Phi & D_\Phi \end{bmatrix} < 0. \tag{22}$$

Then, system $\mathbf{\Sigma}$ is robustly stable for all the communication delay $\tau_{i,p} \in [0, \mathcal{T}_{u_{i,p}}]$.

Proof. Consider the system $\mathbf{\Sigma}$ formulated by (12). Suppose that the delay-difference operator $\mathcal{S}_{\tau_{i,p}}$ satisfies a collection of IQCs defined by $\Pi^{i,p}$ for $k = 1, 2, \ldots, n$ such that $\mathcal{S}_\tau$ satisfies the IQCs defined by $\hat{\Pi}_k$ for $k = 1, 2, \ldots, n$ where $\hat{\Pi}_k$ is defined by (15). For each $\hat{\Pi}_k$, we have

$$\sigma_{\hat{\Pi}_k}(w, d) = \int_{-\pi}^{\pi} \begin{bmatrix} \hat{w}\left({\rm e}^{{\rm j}\omega}\right) \\ \hat{d}\left({\rm e}^{{\rm j}\omega}\right) \end{bmatrix}^* \hat{\Pi}_k\left({\rm e}^{{\rm j}\omega}\right) \begin{bmatrix} \hat{w}\left({\rm e}^{{\rm j}\omega}\right) \\ \hat{d}\left({\rm e}^{{\rm j}\omega}\right) \end{bmatrix} {\rm d}\omega \geqslant 0. \tag{23}$$

In addition, the interconnections are modeled by the IQC defined by (20). According to [25], there exist scalars $\lambda_k \geqslant 0, k = 1, 2, \ldots, n$ and $X = xI > 0$ such that the following LMI is feasible if and only if Eq. (19) is feasible.

$$\sum_{k=1}^{n} \lambda_k \begin{bmatrix} G_{wd} & G_{w\overline{v}} \\ G_{zd} & G_{z\overline{v}} \\ I & 0 \\ 0 & I \end{bmatrix}^* \left[\begin{array}{cc|cc} \hat{\Pi}_{11,k} & 0 & \hat{\Pi}_{12,k} & 0 \\ 0 & -\Phi^{\rm T}X\Phi & 0 & -\Phi^{\rm T}X \\ \hline \hat{\Pi}_{12,k} & 0 & \hat{\Pi}_{22,k} & 0 \\ 0 & X\Phi & 0 & -X \end{array}\right] \begin{bmatrix} G_{wd} & G_{w\overline{v}} \\ G_{zd} & G_{z\overline{v}} \\ I & 0 \\ 0 & I \end{bmatrix} \leqslant -\epsilon I, \tag{24}$$

for $\epsilon > 0$ and for all $\omega \in [0, \infty]$. Because the $\hat{G}$ is internally stable, the LMI (24) becomes a sufficient condition for the robust stability of system $\mathbf{\Sigma}$ with uncertain communication delays.

Suppose that any $\hat{\Pi}_k$ can be factorized as a proper rational TFM representation $\hat{\Pi}_k = \hat{\Psi}_k^{\rm T}\hat{M}_k\hat{\Psi}_k$. It is worth mentioning, in order to reduce the conservatism of the IQC descriptions, $\hat{M}_k$ in the selected $\hat{\Pi}_k$ is the function of the given delay margin vector $\mathcal{T}_u$. According to (16), we define a state-space realization of $\hat{\Psi}_k$ as $(A_{\hat{\Psi}_k}, [B_{\hat{\Psi}_k,w} \ B_{\hat{\Psi}_k,d}], C_{\hat{\Psi}_k}, [D_{\hat{\Psi}_k,w} \ D_{\hat{\Psi}_k,d}])$ for $k = 1, 2, \ldots, n$. Define the dimension of the vector $x_{\hat{\Psi}_k}(t)$ to be $m_{\hat{\Psi}_k}$ and $M_{\hat{\Psi}} = \sum_{k=1}^{n} m_{\hat{\Psi}_k}$. The state-space realization for the TFM $\hat{\Psi}_k[\begin{smallmatrix} G_{wd} & G_{w\overline{v}} \\ I & 0 \end{smallmatrix}]$ for $k = 1, 2, \ldots, n$ is (A, B, C_k, D_k), the parameters of which are defined in (21). Then, the KYP Lemma can be applied to demonstrate the equivalence of inequalities (22) and (24).

Remark 2. Note that in (21) the matrix A and all sub-blocks of other matrix parameters are block diagonal, and the SCM Φ is usually sparse. Meanwhile, we can normalize the scalar x such that $x = 1$, and then the left hand side of inequality (22) depends linearly on the matrix P and scalars $\lambda_k, k = 1, 2, \ldots, n$. When the IQC multipliers are selected, it is possible to use available efficient solvers for this sparse semi-definite programming, such as that developed in [26]. It is expected that the computational complexity for verifying the above condition is usually lower than that of the condition based on the lumped formulation.

Remark 3. Note that the LMI in (24) is equivalent to the LMI in (19), which means that using the IQC multiplier (20) to model the interconnections does not change the conservativeness of the robustness analysis. This is because the LMI in (24) cannot be obtained directly by applying Lemma 1 to the networked uncertain system (12) with $\mathcal{S}_\tau$ and Φ as uncertainties. In fact, the IQC of the interconnections does not satisfy the condition (ii) in Lemma 1 because $-\Phi^{\rm T}X\Phi \ngeqslant 0$. By Theorem 2 and Corollary 1 in [25], the condition (24) is derived by reformulating (19), which proves the condition (ii) in Lemma 1 is not required for this special case.

When the NS has a very large scale, numerical difficulties may still arise in verifying the condition in Theorem 1. To overcome these difficulties, we further develop a set of distributed robustness conditions for the robustness analysis of the NS with delay uncertainties by investigating the structural properties of the SCM Φ in the remaining part of this section.

Taking the $\mathcal{Z}$-transformation on both sides of (10), we have that

$$\begin{bmatrix}\zeta x(\zeta,i)\\ w(\zeta,i)\\ z(\zeta,i)\\ y(\zeta,i)\end{bmatrix}=\begin{bmatrix}A_{xx}(i) & A_{xv}(i) & A_{xv}(i) & B_x(i)\\ 0 & 0 & I & 0\\ A_{zx}(i) & A_{zv}(i) & A_{zv}(i) & B_z(i)\\ C_x(i) & C_v(i) & C_v(i) & D_u(i)\end{bmatrix}\begin{bmatrix}x(\zeta,i)\\ d(\zeta,i)\\ \overline{v}(\zeta,i)\\ u(\zeta,i)\end{bmatrix}. \tag{25}$$

From (11) and (25), straightforward algebraic manipulations show that

$$\begin{bmatrix}z(\zeta,i)\\ y(\zeta,i)\end{bmatrix}=\begin{bmatrix}G_{z\overline{v}}(\zeta,i) & G_{zu}(\zeta,i)\\ G_{y\overline{v}}(\zeta,i) & G_{yu}(\zeta,i)\end{bmatrix}\begin{bmatrix}\overline{v}(\zeta,i)\\ u(\zeta,i)\end{bmatrix}, \tag{26}$$

where

$$\begin{aligned}\begin{bmatrix}G_{z\overline{v}}(\zeta,i) & G_{zu}(\zeta,i)\\ G_{y\overline{v}}(\zeta,i) & G_{yu}(\zeta,i)\end{bmatrix}=&\begin{bmatrix}A_{zv}(i) & B_z(i)\\ C_v(i) & D_u(i)\end{bmatrix}+\begin{bmatrix}A_{zx}(i) & A_{zv}(i)\\ C_x(i) & C_v(i)\end{bmatrix}\begin{bmatrix}\zeta^{-1}I & \\ & \mathcal{S}_{\tau_i}(\zeta)\end{bmatrix}\\ &\times\left(I-\begin{bmatrix}A_{xx}(i) & A_{xv}(i)\\ 0 & 0\end{bmatrix}\begin{bmatrix}\zeta^{-1}I & \\ & \mathcal{S}_{\tau_i}(\zeta)\end{bmatrix}\right)^{-1}\begin{bmatrix}A_{xv}(i) & B_x(i)\\ I & 0\end{bmatrix}.\end{aligned} \tag{27}$$

Define $G_{*\#}=\mathbf{diag}\{G_{*\#}(\zeta,i)|_{i=1}^{N}\}$ with $*$, $\#{=}z$, y, $\overline{v}$, u. Then,

$$\begin{bmatrix}z(\zeta)\\ y(\zeta)\end{bmatrix}=\begin{bmatrix}G_{z\overline{v}}(\zeta) & G_{zu}(\zeta)\\ G_{y\overline{v}}(\zeta) & G_{yu}(\zeta)\end{bmatrix}\begin{bmatrix}\overline{v}(\zeta)\\ u(\zeta)\end{bmatrix}. \tag{28}$$

Moreover, from (9), we have that $\overline{v}(\zeta)=\Phi z(\zeta)$, which can be substituted into the above equation such that $y(\zeta)=[G_{yu}(\zeta)+G_{y\overline{v}}(\zeta)\left(I-\Phi G_{z\overline{v}}(\zeta)\right)^{-1}\Phi G_{zu}(\zeta)]u(\zeta)$. Then, the stability of system $\mathbf{\Sigma}$ can be verified by the following lemma that is presented in [6].

Lemma 2. Assume that system $\mathbf{\Sigma}$ is well-posed. Then, system $\mathbf{\Sigma}$ is stable if and only if

$$|I-\Phi G_{z\overline{v}}(\zeta)|\neq 0,\quad \forall\,|\zeta|\geqslant 1. \tag{29}$$

From Lemma 2, it is obvious that inequality (29) must be verified for each complex number ζ with $|\zeta|\geqslant 1$, which is generally hard to realize through straightforward computations. Meanwhile, the large dimension of LSNS will make the computations prohibitively costly even when the SCM Φ is usually sparse. In order to tackle these difficulties, some structural properties of the SCM Φ are investigated.

Define $M_{\star,i}$ as $M_{\star,i}=\sum_{k=1}^{i}m_{\star k}$ with $M_{\star,0}=0$, in which $\star=v,z$. Let e_k denote the M_z dimensional row vector with its k-th column element being 1 and all other elements being zero. In addition, let $j(i),i=1,2,\ldots,M_v$ denote the position of the non-zero element of the i-th row of the SCM Φ. Then, from the assumptions on this matrix, we have that $\Phi=\mathbf{col}\{e_{j(i)}|_{i=1}^{M_v}\}$. Let $m(i)$ stand for the number of subsystems that is directly affected by the i-th element of the vector $z(t)$. Let $\Sigma_j, j=1,2,\ldots,N$ denote $\mathbf{diag}\{\sqrt{m(i)}|_{i=M_{z,j-1}+1}^{M_{z,j}}\}$. Note that $e_k^{\mathrm{T}}e_k=\mathbf{diag}\left\{0_{k-1}^{\mathrm{T}},1,0_{M_z-k}^{\mathrm{T}}\right\}$. Straightforward algebraic manipulations show that

$$\begin{aligned}\Phi^{\mathrm{T}}\Phi&=\mathbf{col}^{\mathrm{T}}\left\{e_{j(i)}|_{i=1}^{M_v}\right\}\mathbf{col}\left\{e_{j(i)}|_{i=1}^{M_v}\right\}\\ &=\mathbf{diag}\left\{m(i)|_{i=1}^{M_z}\right\}\\ &=\mathbf{diag}\left\{\Sigma_j^2|_{j=1}^{N}\right\}.\end{aligned} \tag{30}$$

Based on the properties of the SCM Φ, the following sufficient stability condition of system $\mathbf{\Sigma}$ is given, which is proposed in [6].

Lemma 3. System $\mathbf{\Sigma}$ is stable if $\|\Sigma_i G_{z\overline{v}}(\zeta,i)\|_\infty < 1$ is satisfied for each subsystem.

Note that Lemma 3 is a decoupled stability condition by using the structural properties of SCM. Therefore, it can be verified independently for each subsystem and its computational complexity increases only linearly with the growth of subsystem number N. For an NS, this characteristic is quite attractive in the system analysis and synthesis. However, it is worth pointing out that there exist uncertain time delays in the TFM $G_{z\overline{v}}(\zeta,i)$. Therefore, Lemma 3 cannot be used directly to verify the robustness of system $\mathbf{\Sigma}$.

Definition 1. A quadratic functional $\sigma_P(z,v)$ is defined as

$$\sigma_P(z,v) = \sum_{t=0}^{\infty} \begin{bmatrix} z(t) \\ v(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Pi_{P,11} & \Pi_{P,12} \\ \Pi_{P,12}^{*} & \Pi_{P,22} \end{bmatrix} \begin{bmatrix} z(t) \\ v(t) \end{bmatrix}. \tag{31}$$

A system has robust performance with respect to σ_P over a set of uncertainties if the system is well-posed, internally stable and $\sigma_P(z,v) < 0$ is satisfied.

The following result is obtained for robust stability of system $\mathbf{\Sigma}$ with delay uncertainties.

Theorem 2. Consider the system $\mathbf{\Sigma}$. If for a given delay margin vector $\mathcal{T}_{u_i} = \mathbf{col}\{\mathcal{T}_{u_{i,p}}|_{p=1}^{m_{vi}}\}$, $\mathcal{T}_{u_{i,p}} \in \mathbb{Z}_+$, there exist scalars $\lambda_k^i \geqslant 0, k = 1,2,\ldots,n$ and a positive definite matrix $P > 0$, such that LMI (33) is satisfied for all $i = 1,2,\ldots,N$ with

$$\begin{gathered} A^i = \begin{bmatrix} \mathbf{diag}\{A_{\Psi_k}|_{k=1}^n\} & 0 \\ 0 & A_{xx}(i) \end{bmatrix}, \quad B_1^i = \begin{bmatrix} \mathbf{col}\{B_{\Psi_k,d}|_{k=1}^n\} \\ A_{xv}(i) \end{bmatrix}, \quad B_2^i = \begin{bmatrix} \mathbf{col}\{B_{\Psi_k,w}|_{k=1}^n\} \\ A_{xv}(i) \end{bmatrix}, \\ C_k^i = \begin{bmatrix} C_{\Psi_k} & 0^{m_{vi}\times m_{xi}} \\ 0 & 0 \end{bmatrix}, \quad D_{k_1}^i = \begin{bmatrix} D_{\Psi_k,d} \\ I_{m_{vi}} \end{bmatrix}, \quad D_{k_2}^i = \begin{bmatrix} I_{m_{vi}} \\ 0 \end{bmatrix}, \\ C_G^i = \begin{bmatrix} 0^{m_{zi}\times M_\Psi} & \Sigma_i A_{zx}(i) \\ 0 & 0 \end{bmatrix}, \quad D_{G_1}^i = \begin{bmatrix} \Sigma_i A_{zv}(i) \\ 0 \end{bmatrix}, \quad D_{G_2}^i = \begin{bmatrix} \Sigma_i A_{zv}(i) \\ I_{m_{vi}} \end{bmatrix}, \end{gathered} \tag{32}$$

$$\begin{aligned} &\begin{bmatrix} A^{i\mathrm{T}}PA^i - P & A^{i\mathrm{T}}PB_1^i & A^{i\mathrm{T}}PB_2^i \\ B_1^{i\mathrm{T}}PA^i & B_1^{i\mathrm{T}}PB_1 & B_1^{i\mathrm{T}}PB_2 \\ B_2^{i\mathrm{T}}PA^i & B_2^{i\mathrm{T}}PB_1 & B_2^{i\mathrm{T}}PB_2 \end{bmatrix} + \sum_{k=1}^{n} \lambda_k \begin{bmatrix} C_k^{i\,\mathrm{T}} \\ D_{k_1}^{i\,\mathrm{T}} \\ D_{k_2}^{i\,\mathrm{T}} \end{bmatrix} M_k\left(\mathcal{T}_{u_i}\right) \begin{bmatrix} C_k^i & D_{k_1}^i & D_{k_2}^i \end{bmatrix} \\ &+ \begin{bmatrix} C_G^{i\,\mathrm{T}} \\ D_{G_1}^{i\,\mathrm{T}} \\ D_{G_2}^{i\,\mathrm{T}} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} C_G^i & D_{G_1}^i & D_{G_2}^i \end{bmatrix} < 0. \end{aligned} \tag{33}$$

Then, system $\mathbf{\Sigma}$ is robustly stable for all the communication delay $\tau_{i,p} \in [0, \mathcal{T}_{u_{i,p}}]$.

Proof. Define a TFM $\widetilde{G}(i)$. Its state-space realization is described as

$$\begin{bmatrix} x(t+1,i) \\ w(t,i) \\ z(t,i) \end{bmatrix} = \begin{bmatrix} A_{xx}(i) & A_{xv}(i) & A_{xv}(i) \\ 0 & 0 & I \\ \Sigma_i A_{zx}(i) & \Sigma_i A_{zv}(i) & \Sigma_i A_{zv}(i) \end{bmatrix} \begin{bmatrix} x(t,i) \\ d(t,i) \\ \overline{v}(t,i) \end{bmatrix}, \tag{34}$$

$$d\left(t,i\right) = \left(\mathcal{S}_{\tau_i} w\right)\left(t,i\right). \tag{35}$$

The input-output form of this interconnected system is represented by

$$\begin{gathered} \begin{bmatrix} w(\zeta,i) \\ z(\zeta,i) \end{bmatrix} = \widetilde{G}(i) \begin{bmatrix} d(\zeta,i) \\ \overline{v}(\zeta,i) \end{bmatrix}, \\ d\left(\zeta,i\right) = \mathcal{S}_{\tau_i} w\left(\zeta,i\right). \end{gathered} \tag{36}$$

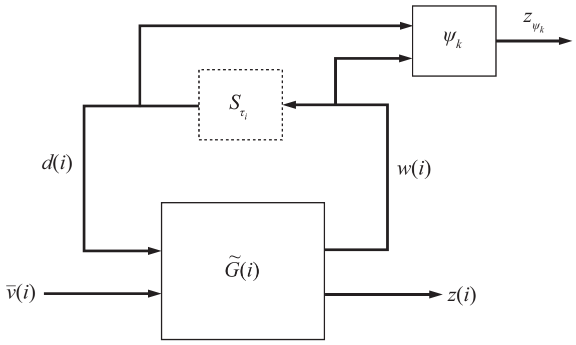

Figure 2 Stability analysis structure under the IQC framework.

Define an upper LFT with respect to $\mathcal{S}_{\tau_i}$ as $\mathcal{F}_u(\widetilde{G}(i), \mathcal{S}_{\tau_i})$ with

$$\mathcal{F}_u\left(\widetilde{G}(i), \mathcal{S}_{\tau_i}\right) = \Sigma_i A_{zv}(i) + \Sigma_i \begin{bmatrix} A_{zx}(i) & A_{zv}(i) \end{bmatrix} \begin{bmatrix} \zeta^{-1} I & \\ & \mathcal{S}_{\tau_i} \end{bmatrix} \times \left(I - \begin{bmatrix} A_{xx}(i) & A_{xv}(i) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \zeta^{-1} I & \\ & \mathcal{S}_{\tau_i} \end{bmatrix} \right)^{-1} \begin{bmatrix} A_{xv}(i) \\ I \end{bmatrix}. \tag{37}$$

From (26) and (37), we have that $\Sigma_i G_{z\overline{v}}(\zeta, i) = \mathcal{F}_u(\widetilde{G}(i), \mathcal{S}_{\tau_i})$. Then, the inequality $\|\Sigma_i G_{z\overline{v}}(\zeta, i)\|_\infty < 1$ in Lemma 3 is satisfied if and only if the inequality $\|\mathcal{F}_u(\widetilde{G}(i), \mathcal{S}_{\tau_i})\|_\infty < 1$ is satisfied.

Because $z = \mathcal{F}_u(\widetilde{G}(i), \mathcal{S}_{\tau_i})\overline{v}$, $\|\mathcal{F}_u(\widetilde{G}(i), \mathcal{S}_{\tau_i})\|_\infty < 1$ means the induced ℓ_2 gain from $\overline{v}$ to z is less than 1 for a given delay vector τ_i, which corresponds to the following IQC description:

$$\sigma_P(z, \overline{v}) = \sum_{t=0}^{\infty} \left(|z(t,i)|^2 - |\overline{v}(t,i)|^2 \right) < 0. \tag{38}$$

Assume that $\mathcal{S}_{\tau_i}$ satisfies a collection of IQCs defined by Π_k^i for $k = 1, 2, \ldots, n$. For each $\Pi_k^i = \Psi_k(\overline{\zeta})^{\mathrm{T}} M_k \Psi_k(\zeta)$, we have

$$\sigma_{\Pi_k^i}(w, d) = \sum_{t=0}^{\infty} \begin{bmatrix} w(t) \\ d(t) \end{bmatrix}^{\mathrm{T}} \left(\Pi_k^i \begin{bmatrix} w \\ d \end{bmatrix} \right)(t) = \sum_{t=0}^{\infty} z_{\Psi_k}(t)^{\mathrm{T}} M_k z_{\Psi_k}(t) \geqslant 0, \tag{39}$$

where M_k in the selected Π_k is the function of the given delay margin vector $\mathcal{T}_{u_{i,p}}$. This IQC description essentially replaces the original relation $d(i) = \mathcal{S}_{\tau_i}(w(i))$, which is shown in Figure 2. Let

$$\mathcal{H} = \left\{ (z, w, \overline{v}, d) \in \ell_2^{m_{zi} + 3m_{vi}} : \begin{bmatrix} w \\ z \end{bmatrix} = \widetilde{G}(i) \begin{bmatrix} d \\ \overline{v} \end{bmatrix} \right\}.$$

Then, we need $\sigma_P(z, \overline{v}) < 0$ satisfies for all $(z, w, \overline{v}, d) \in \mathcal{H}$ such that $\sigma_{\Pi_k^i}(w, d) \geqslant 0, k = 1, 2, \ldots, n$. This is clearly the case if there exist scalars $\lambda_k \geqslant 0, k = 1, 2, \ldots, n$ such that $\sigma(z, w, \overline{v}, d) = \sigma_P(z, \overline{v}) + \sum_{k=1}^{n} \lambda_k \sigma_{\Pi_k^i}(w, d) < 0$. Then, the condition (iii) of Lemma 1 is satisfied if for $k = 1, 2, \ldots, n$ such that

$$\begin{bmatrix} \widetilde{G}_{wd}(i) & \widetilde{G}_{w\overline{v}}(i) \\ \widetilde{G}_{zd}(i) & \widetilde{G}_{z\overline{v}}(i) \\ I & 0 \\ 0 & I \end{bmatrix}^* \left[\begin{array}{cc|cc} \Pi_{11,k}^i & 0 & \Pi_{12,k}^i & 0 \\ 0 & I & 0 & 0 \\ \hline \Pi_{12,k}^{i*} & 0 & \Pi_{22,k}^i & 0 \\ 0 & 0 & 0 & -I \end{array}\right] \begin{bmatrix} \widetilde{G}_{wd}(i) & \widetilde{G}_{w\overline{v}}(i) \\ \widetilde{G}_{zd}(i) & \widetilde{G}_{z\overline{v}}(i) \\ I & 0 \\ 0 & I \end{bmatrix} \leqslant -\epsilon I. \tag{40}$$

By realizing the TFM $\Psi_k \left[\begin{smallmatrix} \widetilde{G}_{wd}(i) & \widetilde{G}_{w\overline{v}}(i) \\ I & 0 \end{smallmatrix}\right]$ for $k = 1, 2, \ldots, n$, we derive an extended system

$$\begin{bmatrix} x_e(t+1,i) \\ z_{\Psi_k}(t,i) \\ z(t,i) \end{bmatrix} = \begin{bmatrix} A^i & B_1^i & B_2^i \\ C_k^i & D_{k_1}^i & D_{k_2}^i \\ C_G^i & D_{G_1}^i & D_{G_2}^i \end{bmatrix} \begin{bmatrix} x_e(t,i) \\ d(t,i) \\ \overline{v}(t,i) \end{bmatrix}, \tag{41}$$

where $x_e := [x \quad x_\Psi]^{\mathrm{T}}$ with x and x_Ψ denoting the state vectors of subsystem $\mathbf{\Sigma}_i$ and $\mathbf{diag}\{\Psi_k|_{k=1}^n\}$, respectively. Define the dimension of the vector $x_\Psi(t)$ as M_Ψ. The parameters of the extended system above are defined in (32).

An application of the KYP lemma now shows that the inequality (40) is satisfied if and only if Eq. (33) is satisfied. Thus, $\|\Sigma_i G_{z\overline{v}}(\zeta,i)\|_\infty < 1$ is satisfied for each subsystem of $\mathbf{\Sigma}$. This proves Theorem 2.

Remark 4. Note that the robust stability condition in Theorem 2 is completely determined by the SCM Φ, the parameters of the i-th subsystem and $\Pi_k^i, 0 \leqslant k \leqslant n$. Therefore, it can be verified independently for each subsystem.

From the result in Theorem 2, we derive a decouped LMI-based condition for the robustness verification of the NS with delay uncertainties, which has lower computational complexity than Theorem 1. However, there is a trade-off between the computational complexity and the degree of conservatism. To be more specific, the condition in Theorem 2 is conservative in part because it is based on Lemma 3. As discussed in [6], it is obvious that using the maximum singular value $\bar{\sigma}(\Sigma_i G_{z\overline{v}}(\zeta,i))$ to bound the spectral radius $\rho(\Phi G_{z\overline{v}}(\zeta))$ is only sufficient, but its conservatism is still not clear. Another reason of conservatism in Theorem 2 is the selected IQC multipliers that model the delay uncertainties. To reduce this conservatism, we need to develop more compact IQCs to bound the delay operators.

5 Numerical simulations

To illustrate properties of the obtained robustness verification conditions, two numerical examples are reported in this section. The first is to demonstrate the computation efficiency of Theorems 1 and 2, while the second is to compare Theorems 1 and 2 with the existing method in terms of the degree of conservatism. All the simulations are performed with a personal computer with an Intel(R) Core(TM) i5-2400 CPU.

5.1 Numerical Example 1

In this subsection, we perform multiple numerical simulations to evaluate computational efficiency of Theorems 1 and 2. Some typical results are provided here. In these simulations, let $m_{xi} = m_{vi} = m_{zi} = 2$ and every parameter of the subsystems is independently and randomly generated according to a continuous uniform distribution over the interval $[-0.5, 0.5]$. Additionally, each row of the SCM Φ is generated randomly and independently, in which the non-zero element is selected according to a discrete uniform distribution over all the possible locations.

Three methods are utilized in checking the robust stability of the generated system with constant delay uncertainties. One is based on existing lumped formulations, while the others are based on Theorems 1 and 2.

For all the three IQC-based robustness conditions, We choose two IQC multipliers from [10] to characterize the associated delay-difference operators $\mathcal{S}_\tau$, which are

$$\Pi_1 = \begin{bmatrix} I_n & 0 \\ I_n & I_n \end{bmatrix}^* \begin{bmatrix} (\mathcal{T}_u+1)\,X_1 & 0 \\ 0 & -X_1 \end{bmatrix} \begin{bmatrix} I_n & 0 \\ I_n & I_n \end{bmatrix}, \tag{42}$$

$$\Pi_2 = \begin{bmatrix} \left(1-\zeta^{-1}\right) I_n & 0 \\ 0 & I_n \end{bmatrix}^* \begin{bmatrix} \mathcal{T}_u^2 X_2 & 0 \\ 0 & -X_2 \end{bmatrix} \begin{bmatrix} \left(1-\zeta^{-1}\right) I_n & 0 \\ 0 & I_n \end{bmatrix}, \tag{43}$$

where $X_1 = X_1^{\mathrm{T}} \geqslant 0$ and $X_2 = X_2^{\mathrm{T}} \geqslant 0$. $\mathcal{T}_u$ is the upper bound of delay duration.

Table 1 Computation time for robust stability verification

Subsystem	Average CPU time (s)			Standard deviation (s)		
number N	LMI (lumped)	Theorem 1	Theorem 2	LMI (lumped)	Theorem 1	Theorem 2
2	0.0184	0.0343	0.0330	0.0316	0.0076	0.0389
4	0.1539	0.1834	0.0607	0.0472	0.0293	0.0395
6	0.6426	0.7331	0.0911	0.0982	0.0921	0.0413
8	1.9633	1.7141	0.1207	0.3007	0.2309	0.0430
10	5.1660	4.0375	0.1692	0.4497	0.5571	0.0532
12	12.5412	8.6380	0.1964	1.1057	1.0417	0.0551
14	25.8146	16.3217	0.2293	1.8817	1.2748	0.0579
16	53.4153	26.5453	0.2654	1.8952	1.3492	0.0598
18	90.0755	41.5946	0.2952	0.8409	1.5985	0.0623
20	148.5057	60.6832	0.3255	1.5072	1.5344	0.0610
30	1164.2107	247.0348	0.4883	15.8555	6.5307	0.0669

In order to utilize the sparsity in inequality (22), we use the sparse solvers DSDP [26] to verify the condition in Theorem 1, while the other two conditions are checked using Matlab LMI toolbox. For each subsystem number, one hundred computations are conducted for each condition. According to the computations, both the average and standard deviation of computation time are calculated for their robust stability verification. Table 1 reports some representative results when N is increased from 2 to 30, and Figure 3 illustrates the averaged computation time required to verify the aforementioned conditions for each value of N.

As is shown in Figure 3, the condition in Theorem 2 is more efficient than the other two conditions when the subsystem number is large. It appears that the computation time of Theorem 2 increases linearly with the subsystem number, while the existing condition based on the lumped formulation increases almost exponentially. With the increment of the subsystem number N, the condition in Theorem 1 is also more efficient than the existing condition based on the lumped formulation, which demonstrates the advantage of the sparse formulation. On the other hand, when N is small to a certain degree, the condition based on the lumped formulation is more computationally efficient than the conditions in Theorems 1 and 2. This is not a surprise, noting that the dimension of inequality (22) is generally larger than that of the condition based on the lumped formulation, and inequality (33) is checked for every subsystem in the verification of Theorem 2. In the simulation process, we chose N to be larger than 30 to conduct the verifications of the three methods. However, it was impossible to compute the solutions for the condition based on the lumped formulation or that in Theorem 1 because of the memory limit, while the verification of Theorem 2 worked normally. Owning to the low computational complexity of the condition in Theorem 2, it allows us to compute the solution even for larger N.

5.2 Numerical Example 2

In this numerical example, we compare the degree of conservatism associated with the proposed conditions in Theorems 1 and 2, with the existing condition based on the lumped formulation of the IQC analysis. The NS consists of two subsystems whose state space model-like representations are given below.

$$\begin{bmatrix} x(t+1,1) \\ z(t,1) \\ y(t,1) \end{bmatrix} = \begin{bmatrix} 0.7 & -0.2 & 1 \\ 0.5 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t,1) \\ v(t,1) \\ u(t,1) \end{bmatrix}, \tag{44}$$

$$\begin{bmatrix} x(t+1,2) \\ z(t,2) \\ y(t,2) \end{bmatrix} = \begin{bmatrix} 0.8 & -0.3 & 1 \\ 0.3 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t,2) \\ v(t,2) \\ u(t,2) \end{bmatrix}. \tag{45}$$

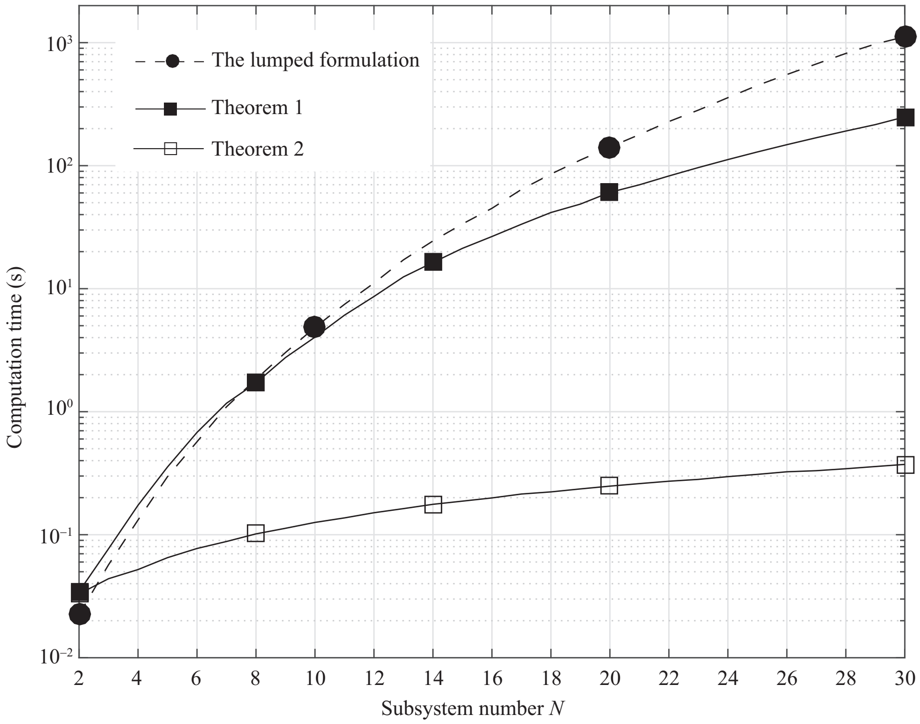

Figure 3 Averaged computation time of the robustness verifications.

These subsystems are connected through

$$v(t)=\left(\begin{bmatrix}\mathcal{D}_{\tau_1} & 0\\ 0 & \mathcal{D}_{\tau_2}\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}z\right)(t). \tag{46}$$

There are two delay operators in the interconnections between the two subsystems. Assume that the delay durations τ_1 and τ_2 vary independently, and we still employ the IQC defined by $\Pi_1+\Pi_2$ to characterize the relevant delay-difference operators $\mathcal{S}_{\tau_1}$ and $\mathcal{S}_{\tau_2}$. Three robustness verification methods, the condition based on the lumped formulation, Theorems 1, and Theorem 2, are applied to this system. The objective of these verifications is to estimate the stability region with τ_1 and τ_2 as parameters such that the NS given in (44)–(46) is robustly stable for all the delays τ_1 and τ_2 that belong to this region. The estimated stability boundary for each method is plotted as a function of delays τ_1 and τ_2 in Figure 4.

In Figure 4, the curve of Theorem 1 is almost identical to that of the lumped formulation, and the difference may be caused by unavoidable errors due to numerical calculations. Thus, for this particular example the condition in Theorem 1 is no more conservative than that based on the lumped formulation. Evidently, utilizing the decoupled condition in Theorem 2 renders the most conservative results. However, the robustness verification of Theorem 2 requires less computational time than those of the other two methods as shown in Figure 3 when the NS has a very large scale.

6 Conclusion

This paper investigates robust stability of a networked system with uncertain communication delays. A distinct feature of the problem we consider is that the dynamic properties of each subsystem are without any constraint, and interconnections among the subsystems are arbitrary but with an uncertain constant delay in each communication channel. By employing a set of IQCs to bound the delay uncertainties and the interconnections, a sparse LMI is derived as a sufficient condition for the robust stability of the NS. Furthermore, some sufficient robustness conditions are derived which depend only on parameters of each subsystem, the SCM and the selected IQC multipliers. Numerical experiments demonstrate the

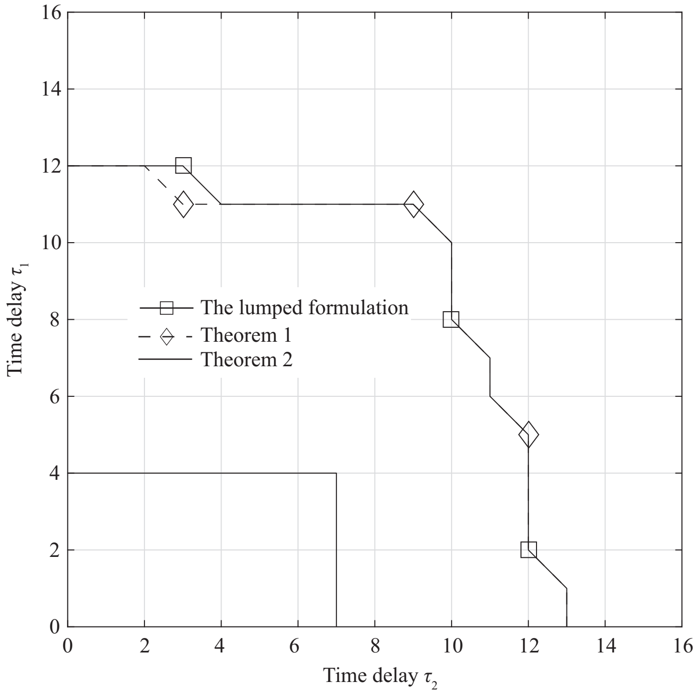

Figure 4 Estimated boundary for the stability region in τ_1-τ_2 plain.

conditions we propose can improve the computational efficiency for the robustness verification in different degrees.

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