SCIENCE CHINA
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• **Supplementary File** •

Exponential tracking of adaptive control systems

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Appendix A Proof of Lemma 2

Proof. Let a gain K be chosen such that (2) has a solution $P > 0$. Defining $S = P^{-1} > 0$ and multiplying the inequality (2) on both sides by S yield $S(A + BK)^T + (A + BK)S + 2\sigma S < 0$. Hence, for $z \in R^n$ and $z \neq 0$, we have $z^T[S(A + BK)^T + (A + BK)S + 2\sigma S]z < 0$, that is,

$$z^T(SA^T + AS + 2\sigma S)z + 2z^T BKSz < 0. \tag{A1}$$

Let $X = BB^T, Y = -(SA^T + AS + 2\sigma S)$, which implies that $X = X^T \geqslant 0, Y = Y^T$. If $z \neq 0, z^T Xz = 0$, i.e. $z^T BB^T z = 0, z^T B = 0$, then it follows from (A1) that $z^T(SA^T + AS + 2\sigma S)z < 0$, i.e. $z^T Y z > 0$. According to Lemma 1, there exists a constant $\tau > 0$ such that $Y + \tau X > 0$, i.e. $SA^T + AS + 2\sigma S - \tau BB^T < 0$. By multiplying P on both sides, it is obtained that (3) holds.

On the other hand, we suppose that there exists a constant $\tau > 0$ such that (3) has a symmetric positive definite solution P. We rewrite (3) as $P\left(A - \frac{\tau}{2}BB^T P\right) + \left(A^T - \frac{\tau}{2}PBB^T\right)P + 2\sigma P < 0$. Letting $K = -\frac{\tau}{2}B^T P$, we have $P(A + BK) + (A^T + K^T B^T)P + 2\sigma P < 0$, which immediately leads to (2). This completes the proof.

Appendix B Proof of Lemma 3

Proof. Multiplying (4) on both sides by P, we have $PA + A^T P + 2\sigma P - \tau PBB^T P < 0$, i.e. (3) holds. According to the proof of the second part in Lemma 2, the choice $K = -\frac{\tau}{2}B^T P$ can guarantee that (2) has a symmetric positive definite solution P.

Appendix C Proof of Lemma 5

Proof. From Lemma 4, we know that $(A + \sigma I, B)$ is a controllable pair. Thus, there exists a constant matrix K satisfying that $(A + \sigma I) + BK$ is stable. Hence, $P = P^T > 0$ can be found such that the following Lyapunov equation holds:

$$[(A + \sigma I) + BK]^T P + P[(A + \sigma I) + BK] = -Q, \tag{C1}$$

for any given $Q = Q^T > 0$, that is, $(A + BK)^T P + P(A + BK) = -Q - 2\sigma P < -2\sigma P$.

Appendix D Proof of Lemma 7

Proof. From Lemma 6, we have

$$\begin{aligned} V(t) &\leqslant \exp(-\sigma t)V(0) + \int_0^t \exp[-\sigma(t-\tau)]l\exp(-\lambda\tau)d\tau \\ &= \exp(-\sigma t)V(0) + \frac{l\exp(-\sigma t)}{\lambda - \sigma}\left[1 - \exp(-(\lambda - \sigma)t)\right], \quad \forall t \geqslant 0. \end{aligned} \tag{D1}$$

By noting $\lambda > \sigma$, (D1) can be rewritten as

$$V(t) \leqslant \exp(-\sigma t)V(0) + \frac{l\exp(-\sigma t)}{\lambda - \sigma} = \left(V(0) + \frac{l}{\lambda - \sigma}\right)\exp(-\sigma t), \quad \forall t \geqslant 0.$$

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Appendix E Proof of Theorem 1

Proof. We first rewrite (5) as

$$\dot{e} = (A + BK)e + B\left[-\sum_{i=1}^{r} a_i Y_i(X) + bu(t) - x_d^{(n)} - Ke\right]. \tag{E1}$$

Define a positive Lyapunov function

$$V = e^T Pe + \frac{b}{\gamma}\exp(-2\sigma t)\tilde{\theta}^2(t), \quad \tilde{\theta}(t) = \theta^* - \hat{\theta}(t), \tag{E2}$$

whose derivative is

$$\begin{aligned}\dot{V} \ &= \ e^T\left[P(A+BK) + (A+BK)^T P\right]e - \frac{2\sigma b}{\gamma}\exp(-2\sigma t)\tilde{\theta}^2 \\ &\quad -\frac{2b}{\gamma}\exp(-2\sigma t)\tilde{\theta}\dot{\hat{\theta}} + 2e^T PB\left[-\sum_{i=1}^{r} a_i Y_i(X) + bu(t) - x_d^{(n)} - Ke\right].\end{aligned} \tag{E3}$$

Substituting (2) and (10) into (E3) and noting (E2), we have

$$\begin{aligned}\dot{V} \ &\leqslant \ -2\sigma e^T Pe - \frac{2\sigma b}{\gamma}\exp(-2\sigma t)\tilde{\theta}^2 - 2b\tilde{\theta}|e^T PB|f(X,t) + 2e^T PB\left[-\sum_{i=1}^{r} a_i Y_i(X) - x_d^{(n)} - Ke\right] + 2be^T PBu(t) \\ &= \ -2\sigma V - 2b\tilde{\theta}|e^T PB|f(X,t) + 2be^T PBu(t) + 2e^T PB\left[-\sum_{i=1}^{r} a_i Y_i(X) - x_d^{(n)} - Ke\right].\end{aligned} \tag{E4}$$

Noting the definitions in (7) and (8), we have

$$\begin{aligned}e^T PB\left[-\sum_{i=1}^{r} a_i Y_i(X) - x_d^{(n)} - Ke\right] \ &\leqslant \ |e^T PB|\left[\sum_{i=1}^{r} |a_i|\cdot|Y_i(X)| + \|K\|\cdot\|e\| + \sup_{t\geqslant 0}|x_d^{(n)}|\right] \\ &\leqslant \ |e^T PB|\theta\left(\sum_{i=1}^{r}|Y_i(X)| + \|e\| + 1\right) \\ &\leqslant \ |e^T PB|\theta f(X,t) = b|e^T PB|\theta^* f(X,t).\end{aligned} \tag{E5}$$

Combining (E4) and (E5) implies that

$$\begin{aligned}\dot{V} \ &\leqslant \ -2\sigma V - 2b\tilde{\theta}|e^T PB|f(X,t) + 2be^T PBu(t) + 2b|e^T PB|\theta^* f(X,t) \\ &= \ -2\sigma V + 2be^T PBu(t) + 2b|e^T PB|\hat{\theta}f(X,t).\end{aligned} \tag{E6}$$

Then, substituting (9) into (E6) results in

$$\dot{V} \leqslant -2\sigma V + 2b|e^T PB|\hat{\theta}f(X,t) - \frac{2b(e^T PB)^2\hat{\theta}^2 f^2(X,t)}{e^T PB\tanh\left[l^{-1}e^T PB\exp(2\lambda t)\right]\hat{\theta}f(X,t) + l\exp(-2\lambda t)}. \tag{E7}$$

Using the inequality $0 \leqslant x\tanh\left(\frac{x}{a}\right) \leqslant |x|, \forall x \in R, a > 0$, and noting the nonnegativeness of $b, \hat{\theta}(t)$ and $f(X,t)$, we have

$$\begin{aligned}\dot{V} \ &\leqslant \ -2\sigma V + 2b|e^T PB|\hat{\theta}f(X,t) - \frac{2b(e^T PB)^2\hat{\theta}^2 f^2(X,t)}{|e^T PB|\hat{\theta}f(X,t) + l\exp(-2\lambda t)} \\ &= \ -2\sigma V + 2bl\exp(-2\lambda t)\frac{|e^T PB|\hat{\theta}f(X,t)}{|e^T PB|\hat{\theta}f(X,t) + l\exp(-2\lambda t)}.\end{aligned} \tag{E8}$$

Applying the inequality $\frac{a}{a+b} \leqslant 1, \forall a \geqslant 0, b > 0$ or $\forall a > 0, b \geqslant 0$, to (E8), we get $\dot{V} \leqslant -2\sigma V + 2bl\exp(-2\lambda t)$. Thus, using Lemma 7, we obtain $V(t) \leqslant \left(V(0) + \frac{bl}{\lambda-\sigma}\right)\exp(-2\sigma t)$. Owing to (E2), we conclude that

$$e^T Pe \leqslant \left(V(0) + \frac{bl}{\lambda-\sigma}\right)\exp(-2\sigma t), \quad \frac{b}{\gamma}\exp(-2\sigma t)\tilde{\theta}^2 \leqslant \left(V(0) + \frac{bl}{\lambda-\sigma}\right)\exp(-2\sigma t), \tag{E9}$$

which further implies that $\|e\| \leqslant \sqrt{\frac{V(0)+\frac{bl}{\lambda-\sigma}}{\lambda_{\min}(P)}}\exp(-\sigma t), |\tilde{\theta}| \leqslant \sqrt{\frac{\gamma\left(V(0)+\frac{bl}{\lambda-\sigma}\right)}{b}}$. Clearly, it can be seen that the tracking error converges to zero exponentially, and the convergence rate is not less than σ. Moreover, it follows that the parameter estimate $\hat{\theta}(t)$ is bounded. By Assumption 1, it is shown that X is bounded. Examining (7), we obtain the boundedness of $f(X,t)$. Next, we will prove $u(t)$ is bounded. Using (9) and then applying Lemma 8, we get

$$\begin{aligned}|u(t)| \ &\leqslant \ \hat{\theta}^2 f^2(X,t)\frac{|e^T PB|}{e^T PB\tanh\left[l^{-1}e^T PB\exp(2\lambda t)\right]\hat{\theta}f(X,t) + l\exp(-2\lambda t)} \\ &\leqslant \ \hat{\theta}^2 f^2(X,t)\frac{e^T PB\tanh\left[l^{-1}e^T PB\exp(2\lambda t)\right] + \kappa l\exp(-2\lambda t)}{e^T PB\tanh\left[l^{-1}e^T PB\exp(2\lambda t)\right]\hat{\theta}f(X,t) + l\exp(-2\lambda t)} \\ &= \ \hat{\theta}f(X,t)\frac{e^T PB\tanh\left[l^{-1}e^T PB\exp(2\lambda t)\right]\hat{\theta}f(X,t)}{e^T PB\tanh\left[l^{-1}e^T PB\exp(2\lambda t)\right]\hat{\theta}f(X,t) + l\exp(-2\lambda t)} \\ &\quad +\kappa\hat{\theta}^2 f^2(X,t)\frac{l\exp(-2\lambda t)}{e^T PB\tanh\left[l^{-1}e^T PB\exp(2\lambda t)\right]\hat{\theta}f(X,t) + l\exp(-2\lambda t)},\end{aligned} \tag{E10}$$

which leads to $|u(t)| \leqslant \hat{\theta}f(X,t) + \kappa\hat{\theta}^2 f^2(X,t)$. Noting the boundedness of $\hat{\theta}$ and $f(X,t)$, we can obtain the boundedness of $u(t)$. Therefore, all the closed-loop signals are bounded. This completes the proof.

Appendix F Simulation results

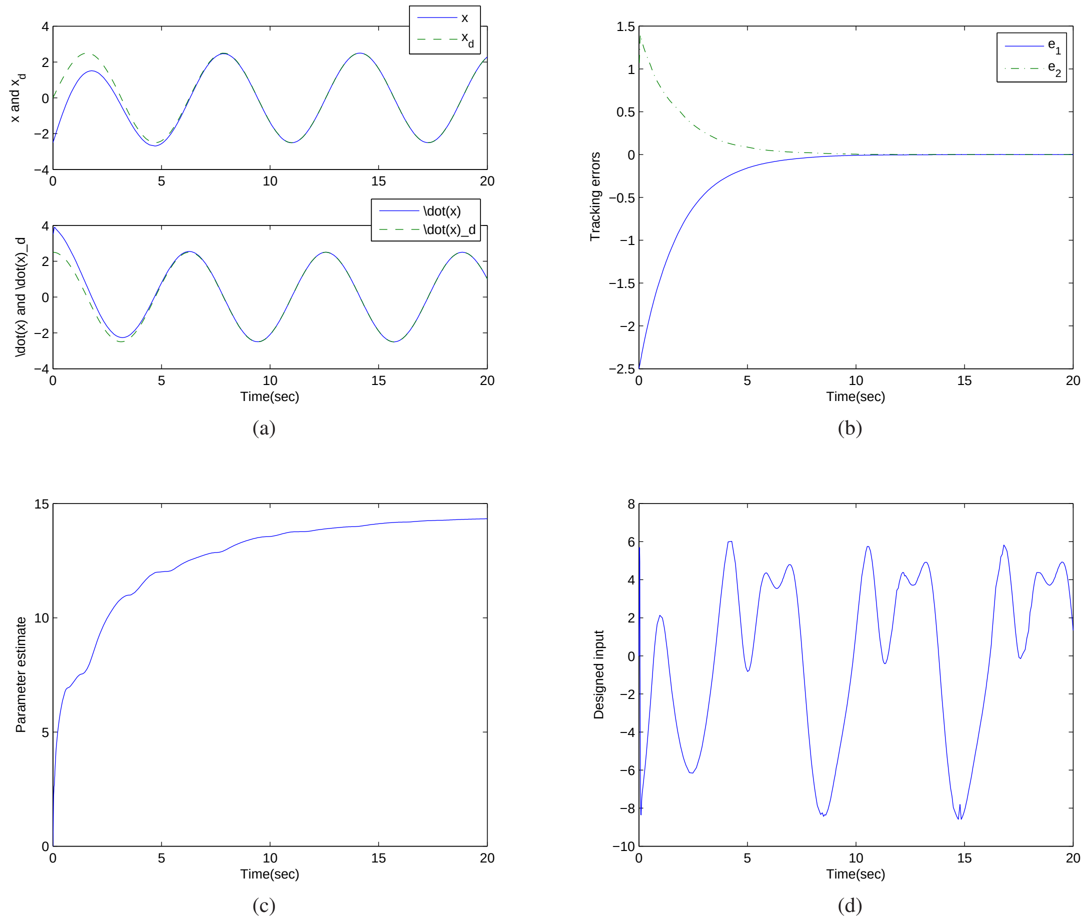

Figure F1 (a) plant states and reference signals x, x_d (top), $\dot{x}, \dot{x}_d$ (bottom); (b) tracking errors e_1, e_2; (c) parameter estimate $\hat{\theta}$; (d) designed input u