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Evasion strategies of a three-player lifeline game

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Abstract This study examines a multi-player pursuit-evasion game, more specifically, a three-player lifeline game in a planar environment, where a single evader is tasked with reaching a lifeline prior to capture. A decomposition method based on an explicit policy is proposed to address the game qualitatively from two main aspects: (1) the evader's position distribution to guarantee winning the game (i.e., the escape zone), which is based on the premise of knowing the pursuers' positions initially, and (2) evasion strategies in the escape zone. First, this study decomposes the three-player lifeline game into two two-player sub-games and obtains an analytic expression of the escape zone by constructing a barrier, which is an integration of the solutions of two sub-games. This study then explicitly partitions the escape zone into several regions and derives an evasion strategy for each region. In particular, this study provides a resultant force method for the evader to balance the active goal of reaching the lifeline and the passive goal of avoiding capture. Finally, some examples from a lifeline game involving more than one pursuer are used to verify the effectiveness and scalability of the evasion strategies.

Keywords lifeline game, three-player pursuit-evasion game, barrier, evasion strategy, game of kind

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1 Introduction

In recent years, pursuit-evasion games have been applied in numbers of areas [1–7] such as missile interception, collision avoidance of satellites, and aerial combat. Generally, the traditional two-player pursuit-evasion game focuses on finding the winning region, the capture zone, or escape zone, within the game's state space to guarantee capture or escape, as well as finding the optimal control strategies for the players [8]. However, with the increasing number of players, these objectives of finding winning regions and optimal strategies become difficult to achieve. Because some connotative features exist in a multi-player pursuit-evasion game [9,10], such as the cooperation of identic players, complicated terminal conditions of the game, and some uncertain information (e.g., knowledge about the opponent, unexpected events).

The problem of pursuit and evasion can be examined from either a quantitative or qualitative perspective, referred to by Isaacs [11] as a game of degree or game of kind, respectively. In a quantitative pursuit-evasion game, the main objective is to solve equilibrium strategies for the pursuers and evaders with respect to the quantitative payoff. In a qualitative pursuit-evasion game, the main objective for

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either pursuers or evaders is to win the game based on the premise that all players play optimally. Here, the payoff is qualitative: win or lose. Several classic qualitative two-player games, such as the Homicidal Chauffeur game, the game of two cars, the lifeline game, and the deadline game, can be found in [11].

This study concerns with the solutions of a qualitative multi-player pursuit-evasion game, involving the initial conditions under which capture or escape is possible and the corresponding optimal state feedback strategies for the pursuers and evaders. In previous studies, the multi-player pursuit-evasion game was considered appropriate to start with a three-player pursuit-evasion game, namely that of two pursuers and one evader or one pursuer and two evaders. For example, the authors in [12] address a class of multi-player pursuit-evasion games involving one superior evader, which is an extension of their underlying work concerning the solutions for a fishing game with two pursuers and one superior evader [13]. Similar studies are also conducted in [14–18].

Extending from the pioneering work of Isaacs [11], this study considers a three-player lifeline game in which one evader and two pursuers play in a planar environment. At the beginning of the game, the three players all move on the same side of but far removed from a line that extends indefinitely. The goal of the pursuers is to capture the evader as soon as possible and before the evader reaches the line. For the evader, the line is known as the "lifeline". This game model can be applied to many situations, for example, patrolling a borderline using multiple unmanned aerial vehicles (UAVs). In this example, a reconnaissance UAV secretly flies over a borderline and sneaks into enemy territory to search for and obtain valuable intelligence. When enemy UAVs attempt to block or intercept it, the reconnaissance UAV must escape from its current position. In this case, the borderline becomes the reconnaissance UAV's lifeline. Similar civilian-based scenarios can be identified, such as the problem of protecting oil pipelines using a sensor network [8, 19].

It is well known that when the number of players increases from two to three or more, the problem of lifeline game becomes intractable with many challenges [20–23] such as the variety of terminal conditions, the cooperation of the pursuers, and the tradeoff of the evader's multiple goals (e.g., reaching the lifeline and avoiding capture). These challenges also exist in other traditional pursuit-evasion games such as the homicidal chauffeur game [24] and its variant, the cooperative homicidal chauffeur game [25], the game of two cars [26] and its variant, the game of three cars [27], and the suicidal pedestrian game [28] and its variant, the three-player suicidal pedestrian game [29].

Two main approaches for coping with the aforementioned challenges are given in the literature. The first is called a decomposition method, which involves dividing the game into several sub-games and then analyzing the optimal behaviors of the players in each sub-game [13]. The decomposition is usually implemented on the terminal manifold of the whole game based on the characteristics (time-sequence, variety) of terminal conditions. For example, Shankaran et al. [27] divided their game of three cars into two sub-games and obtained the solution of the whole game by analyzing qualitatively each sub-game. Exarchos et al. [29] also decomposed the three-player suicidal pedestrian game into two sub-games and provided the necessary conditions for evasion. The second approach is using the method of explicit policy that demonstrates the possibility of capture or escape by giving particular strategies or policies of players. As an example of this method, Bopardikar et al. [25] proposed a multi-phase cooperative strategy for multiple pursuers to move in specific formations and confine a single evader to a bounded region. This strategy is inspired by the hunting and foraging behaviors of fish. The method of explicit policy generally involves geometric analysis which is computationally efficient for generating control strategies but is limited to relatively simple game environments and simple motion models of players.

Because of lifeline and different game rules, the aforementioned methods cannot be used in a three-player lifeline game directly. The main objective of this study is to present a decomposition method based on explicit policy by properly integrating the advantages of the two methods. First, we divide the three-player lifeline game into two two-player problems with respect to the terminal conditions of winning the game. For each problem, we analyze qualitatively the winning regions for the players and obtain the barrier, which is a semipermeable surface separating the state space into disjoint parts associated with each player's winning region [13]. We then integrate the barriers in the solution to the entire three-player game. Based on these qualitative results, we explicitly partition the game space into several regions and derive the corresponding evasion strategy for each region. It is worth mentioning that the method of this study is more scalable than that of [25, 27, 29] for lifeline games with more than two pursuers and can be applied to similar pursuit-evasion games with accessible boundaries.

This paper is organized as follows. In Section 2, the problem formulation of the three-player lifeline game is presented. In Section 3, the escape zone of the game is obtained by solving the barrier analytically. In Section 4, evasion strategies when the evader resides in the escape zone are derived and the effectiveness and scalability of the proposed method are verified in the lifeline game when more than one pursuer and only one evader are involved. Section 5 contains our conclusion and discusses future work.

2 Problem formulation

Before presenting the problem formulation, we should explain some important terms. More detailed descriptions of terminology can be found in [11]. The pursuit-evasion game is generally considered in Euclidean n-space, where the state $\boldsymbol{x} = \{x_1, x_2, \ldots, x_n\}$ is represented by the following kinematic equations:

$$\dot{x_i} = f_i(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_p, v_1, v_2, \ldots, v_q), \quad i = 1, 2, \ldots, n, \tag{1}$$

$\boldsymbol{u} = (u_1, u_2, \ldots, u_p)^{\mathrm{T}} \in \mathbb{R}^p$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_q)^{\mathrm{T}} \in \mathbb{R}^q$ are the control strategies of the pursuer and evader, respectively.

Definition 1 (Terminal manifold)**.** Supposing the pursuit-evasion game proceeds from an initial time t_0 to a terminal time t_f, the terminal manifold is a close set $\mathscr{D} \in \mathbb{R}^n \times [t_0, t_f]$,

$$\mathscr{D} = \{\boldsymbol{x}, t \mid \boldsymbol{G}(\boldsymbol{x}(t), t) = 0\}, \tag{2}$$

where $\boldsymbol{G}(\boldsymbol{x}(t), t) = 0$ is the terminal condition of the game. For example, in a general two-player pursuit-evasion game, the terminal condition is that the distance between the pursuer and evader is less than or equal to a given non-negative value.

Definition 2 (Game of kind)**.** Game of kind, which was introduced by Isaacs [11], is a pursuit-evasion game in which the focus is on the conditions that make capture possible for the pursuer or escape for the evader rather than seeking the best procedures in terms of optimizing a continuous payoff.

In other words, the payoff in a game of kind is qualitative and discrete, and corresponds to the outcome: pursuer wins, evader wins, or both of them fail.

Definition 3 (Barrier)**.** In a game of kind, barrier is a semipermeable surface that splits the state space into two disjoint zones: capture zone and escape zone. If the state lies in the capture zone, there will exist a suitable strategy for the pursuer to capture the evader regardless of the evader's strategy. If the state is in the escape zone, the evader will also have a suitable strategy to escape no matter what strategy the pursuer employs. Consequently, if the state is on the barrier, the pursuer must adopt an optimal strategy to prevent the state from entering the escape zone. In the meantime, the evader should also exert himself to keep the state out of the capture zone.

The classic lifeline game is also described in Example 9.5.1 of [11]. We can now present the problem formulation of the three-player lifeline game. Consider two pursuers P_1 and P_2 and one evader E being at positions $(x_1^p, y_1^p), (x_2^p, y_2^p)$, and (x_e, y_e), respectively. Each of them performs a simple motion in a half-plane $\mathscr{R}$ bounded by a line $\mathscr{L}$ known as the lifeline. Without loss of generality, suppose that the half-plane $\mathscr{R} = \{(x, y) \in \mathbb{R}^2 : y \geqslant 0\}$ and the lifeline $\mathscr{L} = \{(x, y) \in \mathbb{R}^2 : y = 0\}$. Then, the game in the realistic space can be depicted as in Figure 1.

The kinematic equations of the players in the realistic game space are given by

$$\dot{x}_i^p = v_p \sin\phi_i, \quad \dot{y}_i^p = -v_p \cos\phi_i, \quad i = 1, 2, \tag{3}$$

$$\dot{x}_e = v_e \sin\varphi, \quad \dot{y}_e = -v_e \cos\varphi, \tag{4}$$

where v_p and v_e are the speeds of P_i $(i = 1, 2)$ and E, respectively, and ϕ_i and φ are their corresponding control inputs (i.e., strategies).

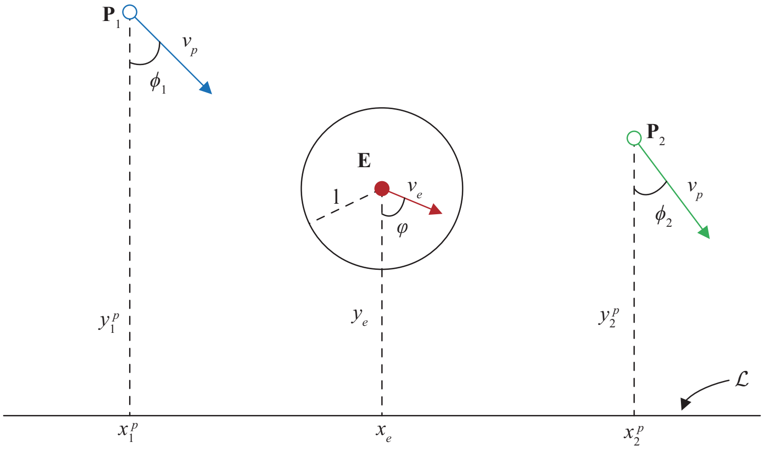

Figure 1 (Color online) Three-player lifeline game in the plane, where φ, ϕ_1, and ϕ_2 are the moving directions of players E, P_1, and P_2 in the realistic game space, respectively.

We assume that the instantaneous positions and velocities of the players are available to their opponents. Then, the purpose of the pursuers P_1 and P_2 is to capture the evader as soon as possible, which occurs when the distance $\min\{|\mathrm{EP}_1|, |\mathrm{EP}_2|\} < l$, where $l \geqslant 0$ is known as a capture radius. The evader's goal is to reach the lifeline $\mathscr{L}$ prior to capture. This study focuses on the following two problems: (a) What is the initial positional distribution of players that guarantees that the evader will win the game? (b) What is the corresponding evasion strategy necessary for the evader to win the game? With respect to the game of kind, the states that meet the definition of problem (a) constitute an escape zone, while the remaining states constitute a capture zone. It is trivial if $v_p < v_e$. The evader is free to put as great distances as he hopes to the two pursuers and then to streak unhindered to $\mathscr{L}$. Here then, the entire game space is the escape zone. Thus, we only consider the case $w := v_p/v_e \geqslant 1$ in the three-player lifeline game, and expect to obtain the game's deterministic solution.

3 Escape zone in a three-player lifeline game

From the problem formulation of a three-player lifeline game, we know that the game will terminate when the following situations occur:

(1) If $y_e = 0$ and $\min\{d_1, d_2\} \geqslant l$, then the evader E wins, where $d_1 = |\mathrm{EP}_1|$ and $d_2 = |\mathrm{EP}_2|$;

(2) If $y_e > 0$ and $\min\{d_1, d_2\} < l$, then the pursuers P_1 and P_2 win.

Clearly, if $y_e > 0$ and $\min\{d_1, d_2\} \geqslant l$, the game will proceed. In general, to obtain the escape zone, the barrier of the game should be constructed. The classic approach [11] to construct the barrier involves integrating the retrogressive path equations (RPEs), starting from a point on the boundary of the usable part (BUP) of the terminal manifold. But this approach cannot be applied directly in a three-player lifeline game because a component of the terminal manifold, $\min\{d_1, d_2\} = l$, is indeterminate. This is a major problem in the game of kind with the deterministic analytical solution as the primary feature. Therefore, we use the decomposition method to divide the three-player game into two two-player sub-games (one sub-game of E and P_1 and another sub-game of E and P_2) and obtain an analytical solution for each sub-game. Then we integrate the solutions of the two sub-games into the solution of the whole game.

3.1 Solution to the sub-game of E and P_1

First, we can reduce the dimension of the state space by using $x_1 = x_1^p - x_e$, which is the distance between the x-coordinates of the pursuer P_1 and the evader E. Then, the kinematic equations of the sub-game in the reduced state space are given by

$$\dot{x}_1 = v_p \sin\phi_1 - v_e \sin\varphi, \quad \dot{y}_1 = -v_p \cos\phi_1, \quad \dot{y}_e = -v_e \cos\varphi, \tag{5}$$

where $y_1 = y_1^p$, ϕ_1 is the strategy of pursuer P_1 and φ is the strategy of evader E.

Theorem 1. The barrier of the sub-game of E and P_1, denoted as $\mathscr{B}_1$, is given by

$$\begin{cases} x_1(\tau) = \left[l + \left(v_p - \dfrac{v_e}{w}\right)\tau\right]\sin s, \\ y_1(\tau) = (l + v_p\tau)\cos s, \\ y_e(\tau) = \dfrac{v_e\tau}{w}\sqrt{w^2 - \sin^2 s}, \end{cases} \tag{6}$$

where $\tau = t_f - t \in [0, \infty]$ is the retrogressive time from the terminal manifold, and the parameter $s \in [-\pi/2, \pi/2]$.

Proof. We use the classic approach to solve the lifeline game with E and P_1. The main equation [11] for this sub-game of kind is

$$\min_{\phi_1}\max_{\varphi}\ [\lambda_1\dot{x}_1 + \lambda_2\dot{y}_1 + \lambda_3\dot{y}_e] = 0, \tag{7}$$

which can be rewritten as

$$\min_{\phi_1}\max_{\varphi}\ [v_p(\lambda_1\sin\phi_1 - \lambda_2\cos\phi_1) - v_e(\lambda_1\sin\varphi + \lambda_3\cos\varphi)] = 0. \tag{8}$$

Solving the main equation (8), we can obtain the optimal strategies of P_1 and E, namely, ϕ_1^* and φ^*, respectively, by

$$\sin\phi_1^* = -\frac{\lambda_1}{\rho_1}, \quad \cos\phi_1^* = \frac{\lambda_2}{\rho_1}, \quad \rho_1 = \sqrt{\lambda_1^2 + \lambda_2^2}, \tag{9}$$

$$\sin\varphi^* = -\frac{\lambda_1}{\rho_2}, \quad \cos\varphi^* = -\frac{\lambda_3}{\rho_2}, \quad \rho_2 = \sqrt{\lambda_1^2 + \lambda_3^2}. \tag{10}$$

Substitute (9) and (10) into (8), and we have

$$-w\rho_1 + \rho_2 = 0, \quad w = v_p/v_e. \tag{11}$$

According to the optimal strategies ϕ_1^* and φ^*, the RPEs for the sub-game are

$$\mathring{x}_1 = -\dot{x}_1 = v_e\lambda_1\left(\frac{w}{\rho_1} - \frac{1}{\rho_2}\right), \quad \mathring{y}_1 = -\dot{y}_1 = \frac{v_p\lambda_2}{\rho_1}, \quad \mathring{y}_e = -\dot{y}_e = -\frac{v_e\lambda_3}{\rho_2}, \quad \mathring{\lambda}_i = 0, \tag{12}$$

where "$\circ$" denotes the temporal derivative in retrogressive time. Next, we analyze the boundary of terminal manifold of the lifeline game. From the problem formulation, the evader's terminal manifold is $\mathscr{D}_e = \{x_1, y_1, y_e \,|\, y_e = 0 \text{ and } d_1 \geqslant l\}$, whereas that of the pursuer P_1 is $\mathscr{D}_p = \{x_1, y_1, y_e \,|\, y_e > 0$ and $d_1 < l\}$. Then, obviously, there is a boundary $\mathscr{G}_1 = \{x_1, y_1, y_e \,|\, y_e = 0 \text{ and } d_1 = l\}$ separating the regions $\mathscr{D}_e$ and $\mathscr{D}_p$, which can be parameterized by

$$x_1 = l\sin s, \quad y_1 = l\cos s, \quad y_e = 0, \quad -\pi/2 \leqslant s \leqslant \pi/2. \tag{13}$$

A normal vector $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ to and on the boundary must satisfy

$$\lambda_1\dot{x}_1 + \lambda_2\dot{y}_1 + \lambda_3\dot{y}_e = 0. \tag{14}$$

Therefore, we substitute (13) into (14) to obtain

$$\lambda_1 = \sin s, \quad \lambda_2 = \cos s. \tag{15}$$

Then, from (9)–(11), we have $\rho_1 = 1$ and

$$\rho_2 = w = \sqrt{\sin^2 s + \lambda_3^2} \quad \text{or} \quad \lambda_3 = \pm\sqrt{w^2 - \sin^2 s}. \tag{16}$$

Because $w \geqslant 1$, λ_3 is meaningful in the real domain and the whole boundary $\mathscr{G}_1$ is usable (i.e., is BUP [11]). We then integrate RPEs (12) and notice that the positions of P_1 and E are all in the upper half-plane of x-coordinate, $\mathscr{R}$, from which we obtain the expression of the barrier (6).

Considering $y_1 \geqslant 0$ and $y_e \geqslant 0$, we can delineate the barrier in the reduced state space, as shown in Figure 2. The region enclosed by the barrier and the plane $0 - x_1y_e$ is the capture zone, whereas the remaining region is the escape zone.

3.2 Integration of the solutions of the two sub-games

Similarly, the solution to the sub-game with E and P_2 is available. The barrier, denoted as $\mathscr{B}_2$, can be described by

$$\begin{cases} x_2(\tau) = \left[l + \left(v_p - \dfrac{v_e}{w}\right)\tau\right]\sin\bar{s}, \\ y_2(\tau) = (l + v_p\tau)\cos\bar{s}, \\ y_e(\tau) = \dfrac{v_e\tau}{w}\sqrt{w^2 - \sin^2\bar{s}}, \end{cases} \tag{17}$$

where $x_2 = x_2^p - x_e$, $y_2 = y_2^p$ and $\bar{s} \in [-\pi/2, \pi/2]$.

To integrate the barriers, we can fix the positions of the pursuers P_1 and P_2 with respect to the lifeline $\mathscr{L}$ by setting $P_1 = (-u, h_1)$ and $P_2 = (u, h_2)$. The evader's position is set as $E = (X, Y)$. Then from (6) and (17), the trajectories of E on the barriers $\mathscr{B}_1$ and $\mathscr{B}_2$ are, respectively, given by

$$\begin{cases} X = \left[l + \dfrac{(w^2-1)(h_1 - l\cos s)}{w^2\cos s}\right]\sin s - u, \\ Y = \dfrac{(h_1 - l\cos s)\sqrt{w^2 - \sin^2 s}}{w^2\cos s}, \end{cases} \quad \begin{cases} X = \left[l + \dfrac{(w^2-1)(h_2 - l\cos\bar{s})}{w^2\cos\bar{s}}\right]\sin\bar{s} + u, \\ Y = \dfrac{(h_2 - l\cos\bar{s})\sqrt{w^2 - \sin^2\bar{s}}}{w^2\cos\bar{s}}, \end{cases} \tag{18}$$

where $s, \bar{s} \in (-\pi/2, \pi/2)$. In addition, when $s, \bar{s} = \pm\pi/2$, curves from (18) are degenerated to the following respective lines

$$X = \pm\big(Y\sqrt{w^2-1} + l\big) - u, \quad X = \pm\big(Y\sqrt{w^2-1} + l\big) + u. \tag{19}$$

Suppose the intersection of the trajectories of E on the barriers $\mathscr{B}_1$ and $\mathscr{B}_2$ is $(\tilde{x}, \tilde{y})$. We then have Theorem 2.

Theorem 2. The barrier of the three-player lifeline game, $\mathscr{B}$, is the bond of $\mathscr{B}_1$ and $\mathscr{B}_2$. In other words, if we fix the positions of P_1 and P_2 on $(-u, h_1)$ and (u, h_2), respectively, then the trajectory of E $= (X, Y)$ on the barrier $\mathscr{B}$ is

$$(X,Y) = \begin{cases} \left(\left[l + \dfrac{(w^2-1)(h_1 - l\cos s)}{w^2\cos s}\right]\sin s - u, \ \dfrac{(h_1 - l\cos s)\sqrt{w^2 - \sin^2 s}}{w^2\cos s}\right), & X \leqslant \tilde{x}, \\ \left(\left[l + \dfrac{(w^2-1)(h_2 - l\cos\bar{s})}{w^2\cos\bar{s}}\right]\sin\bar{s} + u, \ \dfrac{(h_2 - l\cos\bar{s})\sqrt{w^2 - \sin^2\bar{s}}}{w^2\cos\bar{s}}\right), & X > \tilde{x}. \end{cases} \tag{20}$$

Proof. The curves of (18) are depicted in Figure 3, where the blue and green curves correspond to the barriers $\mathscr{B}_1$ and $\mathscr{B}_2$, respectively. Because both ends of the blue curve can extend infinitely, in the upper half plane of x-axis, the region above the blue curve corresponds to the capture zone of sub-game of E and P_1, and the region blew it corresponds to the escape zone. Similarly, the region above the green curve corresponds to the capture zone of sub-game of E and P_2. In addition, because the pursuers play individually in the three-player lifeline game, the whole game can be regarded as a superposition of two sub-games. In other words, if the evader starts from any point above the two curves (i.e., in the capture zone of Figure 3), the pursuers can play optimally to catch the evader. However, if the evader starts from any point blew the two curves (i.e., the escape zone), he can play optimally to reach the lifeline $\mathscr{L}$. Thus, the barrier of the three-player lifeline game, $\mathscr{B}$, is the bond of $\mathscr{B}_1$ and $\mathscr{B}_2$. Furthermore, from Figure 3, we can see that the trajectory of E on the barrier $\mathscr{B}$ satisfies (20).

4 Evasion strategies of the three-player lifeline game

There are two irrelevant goals in the three-player lifeline game, $y_e = 0$ and $\min\{d_1, d_2\} < l$, which correspond to victory by the evader and pursuers, respectively. In the process of playing the game, the evader tries to realize his own goal while preventing the pursuers from realizing their own goals, and vice

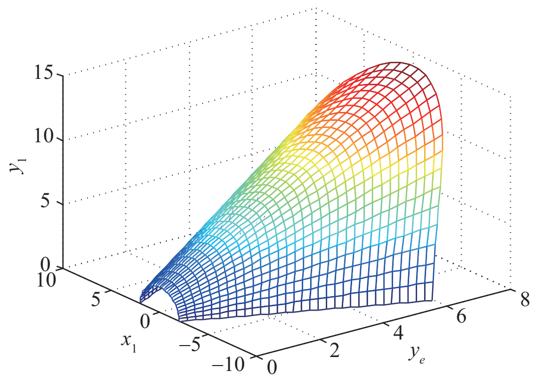

Figure 2 (Color online) Barrier of the two-player lifeline game with $v_p = 1.5$, $v_e = 1$, and $l = 2$.

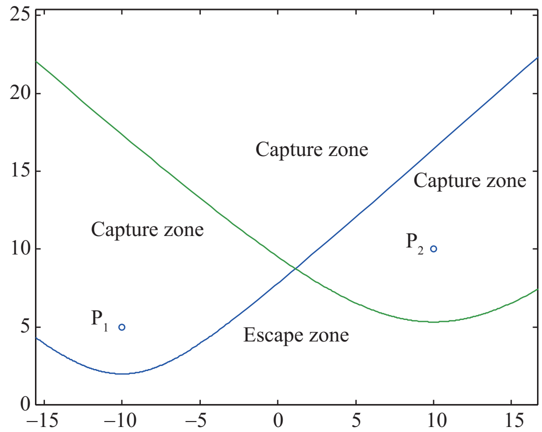

Figure 3 (Color online) Barrier of the three-player lifeline game with $u = 10$, $h_1 = 5$, $h_2 = 10$, $l = 2$, and $w = 1.5$.

versa. Therefore, the game can be regarded as a two-goal game. For such a two-goal game, analyzing the optimal strategies of the players is difficult, particularly in terms of the tradeoff for one player of advancing his or her own goal while preventing that of the other player.

However, the optimal strategies of the pursuers are easy to obtain. Because their goals are to make the distances d_1 and d_2 as short as possible and to make y_e as long as possible, we can construct the players' payoff functions as

$$J_i = k_1 d_i^2 - k_2 y_e^2 = k_1[x_i^2 + (y_i - y_e)^2] - k_2 y_e^2, \quad i = 1, 2, \tag{21}$$

where $k_1, k_2 > 0$ are the weights of d_i^2 and $(-y_e^2)$, respectively. Let us take the derivative of the payoff function J_i:

$$\begin{aligned} \dot{J}_i &= 2k_1 \dot{x}_i x_i + 2k_1 \dot{y}_i y_i - 2k_1 \dot{y}_i y_e - 2k_1 y_i \dot{y}_e + 2(k_1 - k_2) y_e \dot{y}_e \\ &= 2k_1 v_p [x_i \sin\phi_i + (y_e - y_i)\cos\phi_i] - 2v_e[k_1 x_i \sin\varphi + (k_1 y_e - k_2 y_e - k_1 y_i)\cos\varphi]. \end{aligned} \tag{22}$$

It can be seen that no matter what strategy the evader uses, the pursuer P_i can adopt the following strategy to minimize the payoff function J_i, $i = 1, 2$:

$$\sin\phi_i^* = -\frac{x_i}{x_i^2 + (y_i - y_e)^2} = -\frac{x_i}{d_i}, \quad \cos\phi_i^* = \frac{y_i - y_e}{d_i}. \tag{23}$$

Obviously, Eq. (23) defines an optimal open-loop strategy, meaning that pursuer P_i's decision is only relay on the current game state rather than the evader's strategy. From a geometric perspective, Eq. (23) represents a strategy of P_i to move toward the evader to maintain a line-of-sight.

We next analyze the evader's strategy. Clearly, the evader cannot maximize the last term of (22), $2v_e[k_1 x_i \sin\varphi + (k_1 y_e - k_2 y_e - k_1 y_i)\cos\varphi]$, simultaneously for $i = 1, 2$ to win the game. The evader must consider the active goal (reaching the lifeline as soon as possible) and the passive goal (escaping from the pursuers as far away as possible). Here, we use the method of explicit policy to obtain the evasion strategy.

First, it is clear that if the evader starts from any point in the escape zone left of pursuer P_1 or right of pursuer P_2 (as shown in Figure 3), the evader only needs to adopt strategy in (10) to reach the lifeline $\mathscr{L}$, regardless of the pursuers' strategies, because in this case only one sub-game has an effect on the outcome of the entire game.

We next concentrate on the escape zone between with the pursuers P_1 and P_2. Suppose that the initial positions of P_1 and P_2 are $(-u, h_1)$ and (u, h_2), respectively, and the initial position of the evader is (X, Y). We know that, in terms of strategy, going vertically down the lifeline will be the shortest time for the evader. In other words, the evader only seeks to achieve the active goal while ignoring the passive

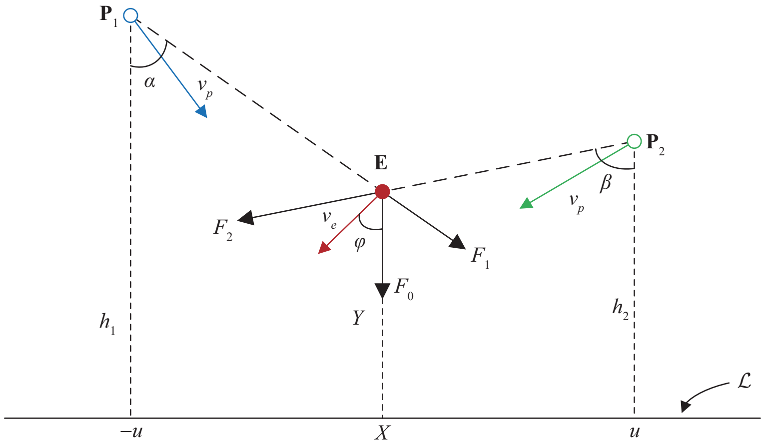

Figure 4 (Color online) Force analysis of the evader, where α and β are the directions of the forces F_1 and F_2, respectively, and φ is the direction of the resultant force of the evader E.

one. If this evasion strategy is feasible, the pursuers P_1 and P_2 can adopt close-loop strategies by moving straight towards the point $(X,0)$ rather than choosing the open-loop strategies given in (23). Thus

$$\triangle t=\frac{Y}{v_e}\leqslant\min\left\{\frac{D_1-l}{v_p},\frac{D_2-l}{v_p}\right\},\quad D_1=\sqrt{h_1^2+(X+u)^2},\quad D_2=\sqrt{h_2^2+(X-u)^2}. \tag{24}$$

That is, in the escape zone, if the position of the evader satisfies

$$(Yw+l)^2\leqslant\min\left\{h_1^2+(X+u)^2,h_2^2+(X-u)^2\right\}, \tag{25}$$

the evader can move vertically down to reach the lifeline before the pursuers.

However, if the position of the evader in the escape zone does not satisfy (25), we must consider the evader's tradeoff between the active and passive goals. From the perspective of mechanics, we can regard the players as the mass points. Suppose that a mass point E is effected by three forces, repulsive force F_1 from a mass point P_1, repulsive force F_2 from a mass point P_2, and gravitational force F_0 from the line $\mathscr{L}$, as shown in Figure 4. We then can calculate the resultant force for the evader to obtain his motion direction.

The components of the resultant force in the horizontal and vertical directions are F_x and F_y, respectively:

$$F_x=-F_1\sin\alpha-F_2\sin\beta,\quad F_y=F_1\cos\alpha+F_2\cos\beta+F_0, \tag{26}$$

where

$$\sin\alpha=\frac{-u-X}{d_1},\quad \cos\alpha=\frac{h_1-Y}{d_1},\quad \sin\beta=\frac{u-X}{d_2},\quad \cos\beta=\frac{h_2-Y}{d_2}. \tag{27}$$

Then, the direction of the resultant force can be described by

$$\sin\varphi^*=\frac{F_x}{\sqrt{F_x^2+F_y^2}},\quad \cos\varphi^*=\frac{F_y}{\sqrt{F_x^2+F_y^2}}. \tag{28}$$

Obviously, the magnitude of repulsive force or gravitational force is inversely proportional to the distance. Thus, we can assign $F_0=2/Y$, $F_1=1/d_1^2$ and $F_2=1/d_2^2$ in (26) and (28) to obtain the value of φ^*.

In addition, we must explain the derivation of F_0. The gravitational force F_0 is a resultant force of the gravitational forces of all mass points on the lifeline $\mathscr{L}$ to the mass point E. According to symmetry, these gravitational forces will cancel out each other in the horizontal direction. Thus, we have

$$F_0=\int_{-\infty}^{+\infty}Y(s^2+Y^2)^{-3/2}\mathrm{d}s=\frac{2}{Y}, \tag{29}$$

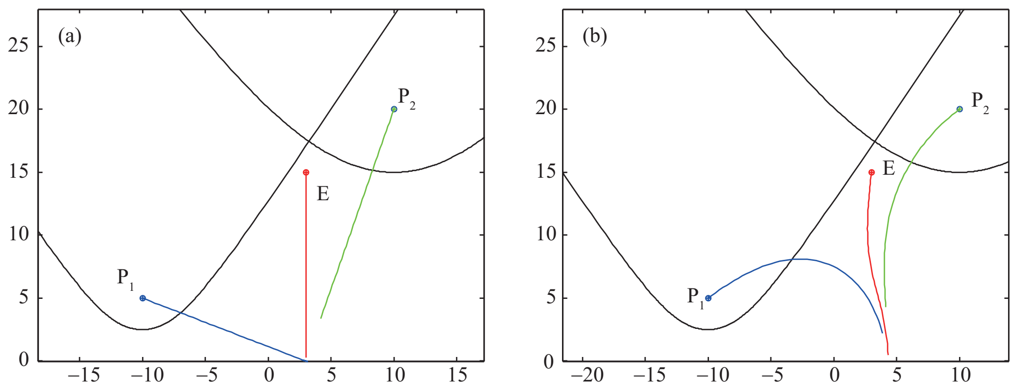

Figure 5 (Color online) Three-player lifeline game with two different evasion strategies. (a) The evader moves vertically down the lifeline; (b) the evader uses the resultant force strategy (28). The initial positions of the players are E $= (3, 15)$, P$_1 = (-10, 5)$, P$_2 = (10, 20)$. $w = 1.2$ and $l = 2$. The black curves are the barriers of the game, the blue, green and red curves are the trajectories of the pursuers P$_1$, P$_2$, and the evader E, respectively.

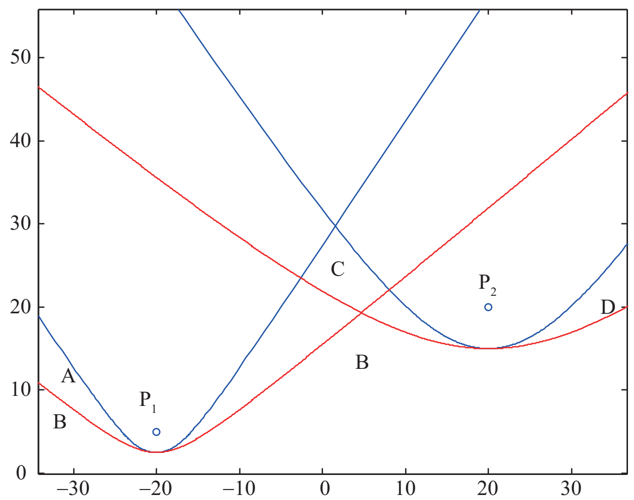

Figure 6 (Color online) Partition of the escape zone for the three-player lifeline game, where the blue curves are the barriers, and the red curves are described by (25). The initial positions of P$_1$ and P$_2$ are $(-20, 5)$ and $(20, 20)$, respectively. $w = 1.2$ and $l = 2$.

where $Y(s^2+Y^2)^{-3/2}$ is the vertical component force of the gravitational force of any mass point $(X+s, 0)$ on the lifeline to E.

Figure 5 shows the advantage of the resultant force strategy in (28). From the same initial positions, if the evader goes vertically down the lifeline (i.e., towards the point $(X, 0)$), the pursuers P$_1$ and P$_2$ will also go directly toward the point $(X, 0)$. Obviously then, the evader will be captured. However, if the evader adopts the resultant force strategy, the pursuers P$_1$ and P$_2$ will use the optimal open-loop strategies (moving toward the evader as close as possible). The result is that the evader reaches the lifeline safely.

For the different evasion strategies, we can divide the escape zone into several parts, as shown in Figure 6. If the evader's initial position is in region A or D, it can use the strategy in (10) to escape. If its initial position is in region B, the evader can use the shortest time strategy, moving vertically down the lifeline; the shortest escape time is $y_e(0)/v_e$. If its initial position is in region C, it can choose the resultant force strategy given by (28) to win the game. Certainly, if the evader resides in other regions (i.e., the capture zone), it has to accept failure. Because we know the optimal open-loop strategies of the pursuers, the maximum capture time will be $\min\{d_1(0), d_2(0)\}/(v_p - v_e)$.

Furthermore, the evasion strategies described previously are scalable for the lifeline game with more than two pursuers. The examples are shown in Figure 7. From Figure 7(a), we can see that the evader

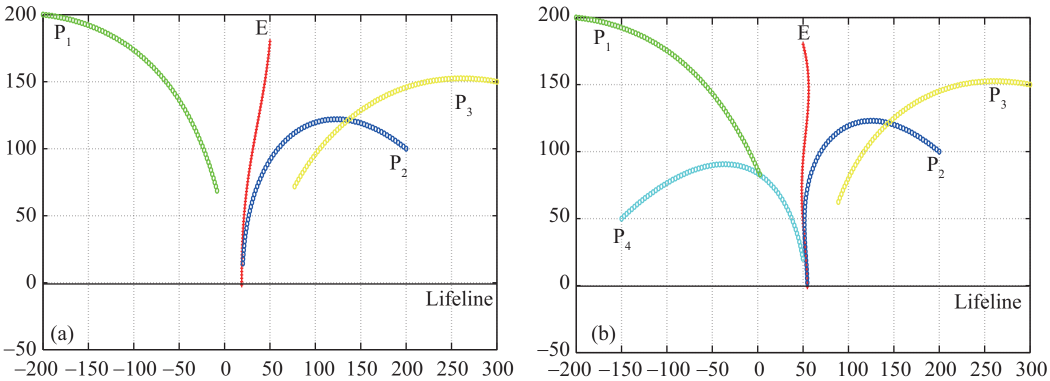

Figure 7 (Color online) Lifeline game with more than two pursuers. (a) The evader escapes from three pursuers and reaches the lifeline; (b) the evader escapes from four pursuers and reaches the lifeline. The initial positions of the players are $\mathrm{E} = (50, 180)$, $\mathrm{P}_1 = (-200, 200)$, $\mathrm{P}_2 = (200, 100)$, $\mathrm{P}_3 = (300, 150)$, and $\mathrm{P}_4 = (-150, 50)$. $w = 4/3$ and $l = 2$.

escapes from three pursuers and lightly reaches the lifeline. Moreover, even if adding a pursuer P_4 to the game, as shown in Figure 7(b), the evader can still win the game by following the evasion strategies. This analysis also reveals that the evader can nicely balance the active and passive goals. We can predict that if we replace the lifeline with a circle or polygon in the game environment, the evasion strategies derived from the decomposition method based on explicit policy are still effectual.

5 Conclusion

In this study, a three-player lifeline game was analyzed qualitatively. The escape and capture zones were derived by integrating and solving two two-player sub-games. Then, evasion strategies were determined that correspond to different regions of the escape zone. Because the three-player lifeline game has two terminal manifolds, and one of them, $\min\{d_1, d_2\} = l$ is indeterminate, using the classic approach in the game of kind to construct the barrier is difficult. This study proposed a decomposition method based on explicit policy that divides the game into two sub-games to reduce the difficulty of computation and analysis, and then determined the evasion strategies by explicitly partitioning the game space into several regions. In addition, to balance the two goals of reaching the lifeline and escaping from the pursuers, for the evader, a resultant force method was provided. Finally, the effectiveness and scalability of the evasion strategies were verified using some examples involving a lifeline game with more than two pursuers.

In a future study, our focus on both qualitative and quantitative solutions of multi-player pursuit-evasion games will involve more complex rules and environments, including incomplete knowledge about opponents, resource constraints, complicated dynamics, and obstacle environments. In addition, we will continue to examine general methodologies of solving multi-player pursuit-evasion games.

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References

1 Yuksek B, Ure N K. Optimization of allocation and launch conditions of multiple missiles for three-dimensional collaborative interception of ballistic target. Int J Aerospace Eng, 2016, 2016: 9582816
2 Liu Y F, Li R F, Wang S Q. Orbital three-player differential game using semi-direct collocation with nonlinear programming. In: Proceedings of the 2nd International Conference on Control Science and Systems Engineering, Singapore, 2016. 217–222
3 Shen D, Pham K, Blasch E, et al. Pursuit-evasion orbital game for satellite interception and collision avoidance. Proc SPIE, 2011, 8044: 284–287
4 Casbeer D W, Garcia E, Pachter M. The target differential game with two defenders. In: Proceedings of 2016 International Conference on Unmanned Aircraft Systems, Arlington, 2016. 202–210

5 Li Y, Mu Y F, Yuan S, et al. The game theoretical approach for multi-phase complex systems in chemical engineering. J Syst Sci Complex, 2017, 30: 4–19
6 Wei N, Zhang Z P. Competitive access in multi-RAT systems with regulated interference constraints. Sci China Inf Sci, 2017, 60: 022306
7 Zhu B, Xie L H, Han D, et al. A survey on recent progress in control of swarm systems. Sci China Inf Sci, 2017, 60: 070201
8 Chen P, Sastry S. Pursuit controller performance guarantees for a lifeline pursuit-evasion game over a wireless sensor network. In: Proceedings of the 45th IEEE Conference on Decision and Control, San Diego, 2016. 691–696
9 Mu Y F, Guo L. How cooperation arises from rational players? Sci China Inf Sci, 2013, 56: 112201
10 Gao H W, Petrosyan L, Qiao H, et al. Cooperation in two-stage games on undirected networks. J Syst Sci Complex, 2017, 30: 680–693
11 Isaacs R. Differential Games: A Mathematical Theory with Applications to Warfare and Pursuit, Control and Optimization. Hoboken: John Wiley and Sons, 1965
12 Chen J, Zha W Z, Peng Z H, et al. Multi-player pursuit-evasion games with one superior evader. Automatica, 2016, 71: 24–32
13 Zha W Z, Chen J, Peng Z H, et al. Construction of barrier in a fishing game with point capture. IEEE Trans Cybern, 2017, 47: 1409–1422
14 Pan S, Huang H M, Ding J, et al. Pursuit, evasion and defense in the plane. In: Proceedings of American Control Conference, Montreal, 2012. 4167–4173
15 Chen M, Zhou Z Y, Tomlin C J. Multiplayer reach-avoid games via low dimensional solutions and maximum matching. In: Proceedings of American Control Conference, Portland, 2014. 1444–1449
16 Ibragimov G, Karapanan P, Alias I A. Pursuit differential game of two pursuers and one evader in R2 with coordinate-wise integral constraints. In: Proceedings of 2015 International Conference on Research and Education in Mathematics, Kuala Lumpur, 2015. 223–226
17 Alias I A, Ibragimov G, Rakhmanov A. Evasion differential game of infinitely many evaders from infinitely many pursuers in hilbert space. Dyn Games Appl, 2017, 7: 347–359
18 Kuchkarov A, Ibragimov G, Ferrara M. Simple motion pursuit and evasion differential games with many pursuers on manifolds with euclidean metric. Discrete Dyn Nat Soc, 2016, 2016: 1386242
19 Zhou Y, Li J X, Wang D L. Target tracking in wireless sensor networks using adaptive measurement quantization. Sci China Inf Sci, 2012, 55: 827–838
20 Selvakumar J, Bakolas E. Evasion from a group of pursuers with a prescribed target set for the evader. In: Proceedings of 2016 American Control Conference, Boston, 2016. 155–160
21 Li W. Formulation of a cooperative-confinement-escape problem of multiple cooperative defenders against an evader escaping from a circular region. Commun Nonlinear Sci Num Simul, 2016, 39: 442–457
22 Yao Y, Zhang P, Liu H, et al. Optimal switching target-assignment based on the integral performance in cooperative tracking. Sci China Inf Sci, 2013, 56: 012203
23 Garcia E, Casbeer D W, Pachter M. Active target defense using first order missile models. Automatica, 2017, 78: 139–143
24 Merz A. Homicidal chauffeur: a differential game. Dissertation for Ph.D. Degree. Palo Alto: Stanford University, 1971
25 Bopardikar S D, Bullo F, Hespanha J P. A cooperative homicidal chauffeur game. Automatica, 2009, 45: 1771–1777
26 Merz A W. The game of two identical cars. J Opt Theory Appl, 1972, 9: 324–343
27 Shankaran S, Stipanovic D M, Tomlin C J. Collision avoidance strategies for a three-player game. In: Proceedings of International Society of Dynamic Games, Wroclaw, 2008. 253–271
28 Averboukh Y, Baklanov A. Stackelberg solutions of differential games in the class of nonanticipative strategies. Dyn Games Appl, 2014, 4: 1–9
29 Exarchos I, Tsiotras P, Pachter M. UAV collision avoidance based on the solution of the suicidal pedestrian differential game. In: Proceedings of AIAA Guidance, Navigation, and Control Conference, San Diego, 2016