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A leader-follower stochastic linear quadratic differential game with time delay

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Abstract In this paper, we are concerned with the leader-follower stochastic differential game of Itô type with time delay appearing in the leader's control. The open-loop solution is explicitly given in the form of the conditional expectation with respect to several symmetric Riccati equations. The key technique is to establish the nonhomogeneous relationship between the forward variables and the backward ones obtained in the optimization problems of both the follower and the leader.

Keywords time delay, leader-follower differential game, open-loop strategy, Riccati equation

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1 Introduction

The leader-follower games have received much attention due to the wide applications in economic, social science, political science and biology [1–8]. This kind of game is hierarchical due to the fact that the leader knows the rational reaction of the follower and reveals first his strategy, while the follower does not know the rational reaction of the leader and has to optimize his criterion for a given control of the leader. In this paper, we are concerned with the stochastic leader-follower game of Itô's type. The solution with an open-loop information structure is applied to the game, where the players are committed to follow a predetermined strategy or no state measurements are available. The leader is seeking a strategy, a function of time only, that it expresses before the game starts knowing the follower's rational reaction; the follower will then minimize its cost function with its strategy, also a function of time only.

In the framework of the deterministic dynamics, the optimal strategy is studied in [9–11] and references therein. It is shown that the open-loop strategy is in terms of three coupled and nonsymmetric Riccati equations within the context of the linear quadratic game. Stochastic differential game is considered in [12–16] where the controls do not enter into the diffusion term of the state dynamic. Ref. [17] considers the leader-follower game with both state-dependent and control-dependent noise by solving the forward-backward stochastic differential equation (FBSDE) and sufficient conditions to ensure the existence of the Stackelberg strategy are given.

Considering the nature of past dependence for many practical problems, it is of great significance to include time delay in the dynamic of the game into consideration [18–20]. Nash games for a class of linear stochastic delay system are discussed in [21] where a strategy set is established in terms of matrix

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inequality by using Lyapunov-Krasovskii method and a non-convex optimization approach. To the best of our knowledge, there is little literature on the leader-follower stochastic game with time delay. In fact, the problem with delay becomes much more complex and difficult because it lacks of efficient tools of stochastic analysis such as Itô's formula to deal with the delay term. Moreover, since the problem is infinite dimensional, no explicit solution exists and numerical solutions are very difficult to obtain [22–26].

In this paper, we study the open-loop solution of the leader-follower stochastic game governed by Itô's differential equation with delay appearing in the leader's control. The open-loop strategy is explicitly given in the form of the conditional expectation with respect to several symmetric Riccati equations, by establishing the nonhomogeneous relationship between the forward variables and the backward ones obtained in the optimization problems of both the follower and the leader. We point out that the approach to find the open-loop solution in this paper is different from that in [17] but similar to [27] where the explicit strategy has been obtained for the leader-follower deterministic game. Moreover, we overcome the difficulty from both the stochastic Brownian motion and the time-delayed term in the leader's control.

The rest of this paper is organized as follows. Section 2 formulates the stochastic leader-follower game with time delay. The main results are shown and proved in Section 3. Section 4 gives some concluding remarks.

2 Problem formulation

Consider the leader-follower stochastic game with the system dynamic

$$\begin{aligned} &\mathrm{d}x(t) = \big[Ax(t) + Bu(t) + Cw(t-h)\big]\mathrm{d}t + \big[\bar{A}x(t) + \bar{B}u(t) + \bar{C}w(t-h)\big]\mathrm{d}W(t),\\ &x(0) = x_0, \quad w(s) = \omega(s), \quad s \in [-h, 0), \end{aligned} \tag{1}$$

and the corresponding cost functionals

$$J_1(u, w) = \mathrm{E}\left[\int_0^T \big[x'(t)Q_1x(t) + u'(t)R_1u(t)\big]\mathrm{d}t + x(T)'H_1x(T)\right], \tag{2}$$

$$J_2(u, w) = \mathrm{E}\left[\int_0^T x'(t)Q_2x(t)\mathrm{d}t + \int_0^{T-h} w'(t)R_2w(t)\mathrm{d}t + x(T)'H_2x(T)\right], \tag{3}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^k$ and $w(t) \in \mathbb{R}^s$ are the controls of the follower and the leader, respectively. $x_0 \in \mathbb{R}^n$ as well as $\omega(s)$ is the prescribed initial data, and $h > 0$ represents the constant time delay. $W(t)$ is a one-dimensional standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$. The information structure is given by a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$, which is generated by $W(\cdot)$ and augmented by all the P-null sets. $\mathrm{E}[\cdot]$ means the mathematical expectation. A, B, C, $\bar{A}$, $\bar{B}$, $\bar{C}$ are constant matrices with compatible dimension. The matrices Q_1, Q_2, R_1, R_2, H_1, H_2 are positive semi-definite.

In this paper, we mainly consider the unique strategy. The detailed definition is given below.

Definition 1. The pair $(u^*, w^*) = (Tw^*, w^*)$ provides a unique Stackelberg solution for the two-player game if it satisfies the following conditions. First, for each w, there exists a unique u minimizing $J_1(u, w)$. This implies that there exists a unique map T such that $J_1(Tw, w) \leqslant J_1(u, w)$ for all u and for every w. Second, there exists a unique w such that $J_2(Tw, w) \leqslant J_2(Tw, w)$.

Our aim is to study the unique open-loop strategy which is a function of time only. This is different from the closed-loop strategy which is a function of the time and the state of the game as well.

3 Main results

In this section, we state the unique open-loop strategy. Before giving the main results, the derivations are shown in the first three subsections.

3.1 Optimization problem of the follower

In view of the nature of the leader-follower game, the derivations begin with the optimization problem of the follower which is a standard stochastic linear quadratic (LQ) optimal control problem for any choice $w(t)$ of the leader, i.e.,

$$\min_u \ J_1(u,w)$$
$$\text{s.t.} \ \begin{cases} \mathrm{d}x(t) = \big[Ax(t)+Bu(t)+Cw(t-h)\big]\mathrm{d}t + \big[\bar{A}x(t)+\bar{B}u(t)+\bar{C}w(t-h)\big]\mathrm{d}W(t), \\ x(0)=x_0, \quad w(s)=\omega(s), \quad s\in[-h,0). \end{cases}$$

Following the maximum principle obtained in [17, 28] for delay-free case, the optimal solution satisfies that

$$0 \ = \ R_1u(t) + B'p(t) + \bar{B}'q(t), \tag{4}$$

where $(p(t), q(t))$ is the solution to the following backward stochastic differential equation:

$$\mathrm{d}p(t) \ = \ -\big[A'p(t)+\bar{A}'q(t)+Q_1x(t)\big]\mathrm{d}t + q(t)\mathrm{d}W(t), \quad p(T)=H_1x(T). \tag{5}$$

Since our aim is to discuss the existence and uniqueness of the solution to the leader-follower game, the follower's optimization problem $\min_u J_1(u,w)$ admits a unique solution for any w. In the case of $w(t-h)=0$ for $t\in[0,T]$, the problem is reduced to the standard LQ problem which also has a unique solution. Define

$$\begin{cases} -\dot{P}_1(t) = P_1(t)A + A'P_1(t) + \bar{A}'P_1(t)\bar{A} - \big[P_1(t)B+\bar{A}'P_1(t)\bar{B}\big] \\ \qquad\qquad \times\big[R_1+\bar{B}'P_1(t)\bar{B}\big]^{-1}\big[B'P_1(t)+\bar{B}'P_1(t)\bar{A}\big] + Q_1, \quad P_1(T)=H_1, \\ R_1+\bar{B}'P_1(t)\bar{B}>0. \end{cases} \tag{6}$$

It has been shown that the equivalent condition is $R_1+\bar{B}'P_1(t)\bar{B}>0$ for the unique solvability of the LQ problem. Specifically, the following result can be found in Theorem 2.3 of [29].

Lemma 1. The LQ problem $\min_u J_1(u,w)$ subject to (1) with $w(t-h)=0$ for $t\geqslant 0$ has a unique optimal control for any initial condition x_0 if and only if the Riccati differential equation (6) admits a solution.

We now consider the leader-follower game, i.e., $w(t-h)\neq 0$ for $t\in[0,T]$. Denote

$$\begin{aligned} &\gamma_1(t) \triangleq R_1+\bar{B}'P_1(t)\bar{B}, \\ &D(t) \triangleq A - B\gamma_1^{-1}(t)\big[B'P_1(t)+\bar{B}'P_1(t)\bar{A}\big], \\ &\bar{D}(t) \triangleq \bar{A} - \bar{B}\gamma_1^{-1}(t)\big[B'P_1(t)+\bar{B}'P_1(t)\bar{A}\big], \\ &\bar{\bar{D}}(t) \triangleq \bar{C}'P_1(t)\big\{\bar{A}-\bar{B}\gamma_1^{-1}(t)\big[B'P_1(t)+\bar{B}'P_1(t)\bar{A}\big]\big\} + C'P_1(t). \end{aligned}$$

Lemma 2. Assume that the open-loop strategy exists and is unique, then $u(t)$ admits a feedback form as follows:

$$u(t) \ = \ -\gamma_1^{-1}(t)\big\{\big[B'P_1(t)+\bar{B}'P_1(t)\bar{A}\big]x(t) + B'\zeta_1(t) + \bar{B}'P_1(t)\bar{C}w(t-h) + \bar{B}'\bar{\zeta}_1(t)\big\}, \tag{7}$$

where $P_1(t)$ is the solution to the Riccati equation (6) and $\zeta_1(t)$ satisfies

$$\mathrm{d}\zeta_1(t) \ = \ -\big[D'(t)\zeta_1(t)+\bar{D}'(t)\bar{\zeta}_1(t)+\bar{\bar{D}}'(t)w(t-h)\big]\mathrm{d}t + \bar{\zeta}_1(t)\mathrm{d}W(t) \tag{8}$$

with $\zeta_1(T)=0$.

Proof. Motivated by [30], there exists a nonhomogeneous relationship between $p(t)$ and $x(t)$ due to the involvement of w which is denoted by

$$\zeta_1(t) \triangleq p(t) - P_1(t)x(t), \tag{9}$$

where $P_1(t)$ is the solution to (6) and $\mathrm{d}\zeta_1(t) = \Upsilon(t)\mathrm{d}t + \bar{\zeta}_1(t)\mathrm{d}W(t)$ while $\Upsilon(t)$, $\bar{\zeta}_1(t)$ are to be determined in the sequel. Applying Itô's formula to (9) and comparing with (5) yield that

$$\begin{aligned}\mathrm{d}p(t) &= \dot{P}_1(t)x(t)\mathrm{d}t + P_1(t)\big[Ax(t) + Bu(t) + Cw(t-h)\big]\mathrm{d}t \\ &\quad + P_1(t)\big[\bar{A}x(t) + \bar{B}u(t) + \bar{C}w(t-h)\big]\mathrm{d}W(t) + \Upsilon(t)\mathrm{d}t + \bar{\zeta}_1(t)\mathrm{d}W(t) \\ &= -\big[A'P_1(t)x(t) + A'\zeta_1(t) + \bar{A}'q(t) + Q_1x(t)\big]\mathrm{d}t + q(t)\mathrm{d}W(t),\end{aligned} \tag{10}$$

then

$$q(t) = P_1(t)\big[\bar{A}x(t) + \bar{B}u(t) + \bar{C}w(t-h)\big] + \bar{\zeta}_1(t). \tag{11}$$

Substituting (9) and (11) into the equilibrium condition (4) yields that

$$\begin{aligned}0 &= R_1u(t) + B'P_1(t)x(t) + B'\zeta_1(t) + \bar{B}'P_1(t)\big[\bar{A}x(t) + \bar{B}u(t) + \bar{C}w(t-h)\big] + \bar{B}'\bar{\zeta}_1(t) \\ &= \big[R_1 + \bar{B}'P_1(t)\bar{B}\big]u(t) + \big[B'P_1(t) + \bar{B}'P_1(t)\bar{A}\big]x(t) + B'\zeta_1(t) + \bar{B}'P_1(t)\bar{C}w(t-h) + \bar{B}'\bar{\zeta}_1(t).\end{aligned} \tag{12}$$

It is known that $\gamma_1(t) > 0$ provided that the leader-follower game admits a unique solution by using Lemma 1. Accordingly, the controller u can be formulated as (7).

Substituting (7) into (10) yields that

$$\begin{aligned}0 &= \dot{P}_1(t)x(t) + P_1(t)Ax(t) - \big[P_1(t)B + \bar{A}'P_1(t)\bar{B}\big]\gamma_1(t)^{-1}[B'P_1(t) + \bar{B}'P_1(t)\bar{A}]x(t) \\ &\quad - \big[P_1(t)B + \bar{A}'P_1(t)\bar{B}\big]\gamma_1^{-1}(t)B'\zeta_1(t) - \big[P_1(t)B + \bar{A}'P_1(t)\bar{B}\big]\gamma_1^{-1}(t)\bar{B}'P_1(t)\bar{C}w(t) \\ &\quad - \big[P_1(t)B + \bar{A}'P_1(t)\bar{B}\big]\gamma_1^{-1}(t)\bar{B}'\bar{\zeta}_1(t) + P_1(t)Cw(t-h) + \Upsilon(t) + A'P_1(t)x(t) + Q_1x(t) + A'\zeta_1(t) \\ &\quad + \bar{A}'P_1(t)\bar{A}x(t) + \bar{A}'P_1(t)\bar{C}w(t-h) + \bar{A}'\bar{\zeta}_1(t).\end{aligned} \tag{13}$$

Combining with (6), we have

$$\begin{aligned}-\Upsilon(t) &= \big\{A' - \big[P_1(t)B + \bar{A}'P_1(t)\bar{B}\big]\gamma_1^{-1}(t)B'\big\}\zeta_1(t) + \big\{\bar{A}' - \big[P_1(t)B + \bar{A}'P_1(t)\bar{B}\big]\gamma_1^{-1}(t)\bar{B}'\big\}\bar{\zeta}_1(t) \\ &\quad + \Big[\big\{\bar{A}' - \big[P_1(t)B + \bar{A}'P_1(t)\bar{B}\big]\gamma_1^{-1}(t)\bar{B}'\big\}P_1(t)\bar{C} + P_1(t)C\Big]w(t-h) \\ &\triangleq D(t)'\zeta_1(t) + \bar{D}'(t)\bar{\zeta}_1(t) + \bar{\bar{D}}'(t)w(t-h).\end{aligned}$$

That is, Eq. (8) follows. This completes the proof.

As a consequence of substituting $u(t)$ in (7) into the dynamic of the state (1), we have

$$\begin{aligned}\mathrm{d}x(t) &= \Big[\big\{A - B\gamma_1^{-1}(t)\big[B'P_1(t) + \bar{B}'P_1(t)\bar{A}\big]\big\}x(t) + \big\{C - B\gamma_1^{-1}(t)\bar{B}'P_1(t)\bar{C}\big\}w(t-h) \\ &\quad - B\gamma_1(t)^{-1}B'\zeta_1(t) - B\gamma_1^{-1}(t)\bar{B}'\bar{\zeta}_1(t)\Big]\mathrm{d}t + \Big[\big\{\bar{A} - \bar{B}\gamma_1^{-1}(t)\big[B'P_1(t) + \bar{B}'P_1(t)\bar{A}\big]\big\}x(t) \\ &\quad + \big\{\bar{C} - \bar{B}\gamma_1^{-1}(t)\bar{B}'P_1(t)\bar{C}\big\}w(t-h) - \bar{B}\gamma_1(t)^{-1}B'\zeta_1(t) - \bar{B}\gamma_1^{-1}(t)\bar{B}'\bar{\zeta}_1(t)\Big]\mathrm{d}W(t) \\ &= \big[D(t)x(t) + E(t)w(t-h) + F_1(t)\zeta_1(t) + F_2(t)\bar{\zeta}_1(t)\big]\mathrm{d}t \\ &\quad + \big[\bar{D}(t)x(t) + \bar{E}(t)w(t-h) + F_2'(t)\zeta_1(t) + \bar{F}_2(t)\bar{\zeta}_1(t)\big]\mathrm{d}W(t),\end{aligned} \tag{14}$$

where

$$\begin{aligned}&E(t) \triangleq C - B\gamma_1^{-1}(t)\bar{B}'P_1(t)\bar{C}, \quad F_1(t) \triangleq -B\gamma_1(t)^{-1}B', \quad F_2(t) \triangleq -B\gamma_1^{-1}(t)\bar{B}', \\ &\bar{E}(t) \triangleq \bar{C} - \bar{B}\gamma_1^{-1}(t)\bar{B}'P_1(t)\bar{C}, \quad \bar{F}_2(t) \triangleq -\bar{B}\gamma_1^{-1}(t)\bar{B}'.\end{aligned}$$

3.2 Optimization problem of the leader

Now we are in the position to discuss the optimization problem of the leader which is in fact an optimization control problem with the forward and backward constraints, that is,

$$\min_{w} \ J_2(u,w) \quad \text{s.t.} \quad \begin{cases}(8), \\ (14).\end{cases}$$

The maximum principle for the above optimization problem in the delay-free case has been obtained in [17]. Combining with the techniques dealing with input delay in [25], it is immediately derived that

$$\mathrm{d}p_2(t) = -\big[D'(t)p_2(t) + \bar{D}'(t)q_2(t) + Q_2x(t)\big]\mathrm{d}t + q_2(t)\mathrm{d}W(t), \tag{15}$$

$$\mathrm{d}\xi(t) = \big[D(t)\xi(t) + F_1(t)p_2(t) + F_2(t)q_2(t)\big]\mathrm{d}t + \big[\bar{D}(t)\xi(t) + F_2'(t)p_2(t) + \bar{F}_2(t)q_2(t)\big]\mathrm{d}W(t), \tag{16}$$

$$p_2(T) = H_2x(T), \quad \xi(0) = 0,$$

$$0 = R_2w(t-h) + \mathrm{E}\big[E'(t)p_2(t) + \bar{E}'(t)q_2(t) + \bar{\bar{D}}(t)\xi(t)|\mathcal{F}_{t-h}\big]. \tag{17}$$

We now introduce a new costate ζ_2 satisfying

$$p_2(t) = P_2(t)x(t) + \zeta_2(t), \tag{18}$$

where $P_2(t)$ is the solution to

$$-\dot{P}_2(t) = P_2(t)D(t) + D'(t)P_2(t) + \bar{D}'(t)P_2(t)\bar{D}(t) + Q_2 \tag{19}$$

with $P_2(T) = H_2$. The dynamic of ζ_2 can be given in Lemma 3.

Lemma 3. The new costate ζ_2 satisfies

$$\begin{aligned}\mathrm{d}\zeta_2(t) = &-\big\{D'(t)\zeta_2(t) + \bar{D}'(t)\bar{\zeta}_2(t) + \big[P_2(t)E(t) + \bar{D}'(t)P_2(t)\bar{E}(t)\big]w(t-h) + \big[P_2(t)F_1(t)\\ &+\bar{D}'(t)P_2(t)F_2'(t)\big]\zeta_1(t) + \big[P_2(t)F_2(t) + \bar{D}'(t)P_2(t)\bar{F}_2(t)\big]\bar{\zeta}_1(t)\big\}\mathrm{d}t + \bar{\zeta}_2(t)\mathrm{d}W(t)\end{aligned} \tag{20}$$

with terminal value $\zeta_2(T) = 0$.

Proof. Denote $\mathrm{d}\zeta_2(t) = \Gamma(t)\mathrm{d}t + \bar{\zeta}_2(t)\mathrm{d}W(t)$. Applying Itô's formula to $p_2(t)$ in (18) and making a comparison with (15) yield that

$$\begin{aligned}\mathrm{d}p_2(t) = &\ \dot{P}_2(t)x(t)\mathrm{d}t + P_2(t)\big[D(t)x(t) + E(t)w(t-h) + F_1(t)\zeta_1(t) + F_2(t)\bar{\zeta}_1(t)\big]\mathrm{d}t + P_2(t)\big[\bar{D}(t)x(t)\\ &+\bar{E}(t)w(t-h) + F_2'(t)\zeta_1(t) + \bar{F}_2(t)\bar{\zeta}_1(t)\big]\mathrm{d}W(t) + \Gamma(t)\mathrm{d}t + \bar{\zeta}_2(t)\mathrm{d}W(t)\\ = &-\big[D'(t)P_2(t)x(t) + D'(t)\zeta_2(t) + \bar{D}'(t)q_2(t) + Q_2x(t)\big]\mathrm{d}t + q_2(t)\mathrm{d}W(t).\end{aligned} \tag{21}$$

Thus,

$$q_2(t) = P_2(t)\big[\bar{D}(t)x(t) + \bar{E}(t)w(t-h) + F_2'(t)\zeta_1(t) + \bar{F}_2(t)\bar{\zeta}_1(t)\big] + \bar{\zeta}_2(t),$$

and

$$\begin{aligned}0 = &\ \dot{P}_2(t)x(t) + P_2(t)\big[D(t)x(t) + E(t)w(t-h) + F_1(t)\zeta_1(t) + F_2(t)\bar{\zeta}_1(t)\big] + \Gamma(t)\\ &+\big[D'(t)P_2(t)x(t) + D'(t)\zeta_2(t) + \bar{D}'(t)q_2(t) + Q_2x(t)\big]\\ = &\ \big[\dot{P}_2(t) + P_2(t)D(t) + D'(t)P_2(t) + \bar{D}'(t)P_2(t)\bar{D}(t) + Q_2\big]x(t) + \big[P_2(t)E(t)\\ &+\bar{D}'(t)P_2(t)\bar{E}(t)\big]w(t-h) + \big[P_2(t)F_1(t) + \bar{D}'(t)P_2(t)F_2'(t)\big]\zeta_1(t)\\ &+\big[P_2(t)F_2(t) + \bar{D}'(t)P_2(t)\bar{F}_2(t)\big]\bar{\zeta}_1(t) + \Gamma(t) + D'(t)\zeta_2(t) + \bar{D}'(t)\bar{\zeta}_2(t).\end{aligned}$$

With the using of (19), it is obtained that the dynamic of $\zeta_2(t)$ is exactly (20).

To further obtain the causal and adapted controller from (7) and (17), we stack the forward variables and backward variables obtained in the optimization of the follower and leader, that is, denote

$$\phi(t) \triangleq \begin{bmatrix}\xi(t)\\ x(t)\end{bmatrix}, \quad \psi(t) \triangleq \begin{bmatrix}\zeta_1(t)\\ \zeta_2(t)\end{bmatrix}, \quad \bar{\psi}(t) \triangleq \begin{bmatrix}\bar{\zeta}_1(t)\\ \bar{\zeta}_2(t)\end{bmatrix},$$

and

$$M(t) \triangleq \begin{bmatrix}D(t) & F_1(t)P_2(t) + F_2(t)P_2(t)\bar{D}(t)\\ 0 & D(t)\end{bmatrix}, \quad S_1(t) \triangleq \begin{bmatrix}F_2(t)P_2F_2'(t) & F_1(t)\\ F_1(t) & 0\end{bmatrix},$$

$$S_2(t) \triangleq \begin{bmatrix} F_2(t)P_2(t)\bar{F}_2(t) & F_2(t) \\ F_2(t) & 0 \end{bmatrix}, \quad N_1(t) \triangleq \begin{bmatrix} F_2(t)P_2(t)\bar{E}(t) \\ E(t) \end{bmatrix},$$
$$\bar{N}_1(t) \triangleq \begin{bmatrix} \bar{F}_2(t)P_2(t)\bar{E}(t) \\ \bar{E}(t) \end{bmatrix}, \quad \bar{M}(t) \triangleq \begin{bmatrix} \bar{D}(t) & F_2'(t)P_2(t) + \bar{F}_2(t)P_2(t)\bar{D}(t) \\ 0 & \bar{D}(t) \end{bmatrix},$$
$$\bar{S}_2(t) \triangleq \begin{bmatrix} \bar{F}_2(t)P_2(t)\bar{F}_2(t) & \bar{F}_2(t) \\ \bar{F}_2(t) & 0 \end{bmatrix}, \quad N_2(t) \triangleq \begin{bmatrix} \bar{\bar{D}}'(t) \\ P_2(t)E(t) + \bar{D}'(t)P_2(t)\bar{E}(t) \end{bmatrix}.$$

From (8), (14), (16), (20) and using (18), it is straightforward to obtain the following system:

$$\begin{aligned} \mathrm{d}\phi(t) &= \big[M(t)\phi(t) + S_1(t)\psi(t) + S_2(t)\bar{\psi}(t) + N_1(t)w(t-h)\big]\mathrm{d}t \\ &\quad + \big[\bar{M}(t)\phi(t) + S_2'(t)\psi(t) + \bar{S}_2(t)\bar{\psi}(t) + \bar{N}_1(t)w(t-h)\big]\mathrm{d}W(t), \end{aligned} \tag{22}$$
$$\mathrm{d}\psi(t) = -\big[M'(t)\psi(t) + \bar{M}'(t)\bar{\psi}(t) + N_2(t)w(t-h)\big]\mathrm{d}t + \bar{\psi}(t)\mathrm{d}W(t), \tag{23}$$

with boundary values $\phi(0) = [0 \ x_0']'$ and $\psi(T) = 0$. Denote $\gamma_2(t) \triangleq R_2 + \bar{E}'(t)P_2(t)\bar{E}(t)$, then the equilibrium condition (17) becomes

$$0 = \gamma_2(t)w(t-h) + \mathrm{E}\big[N_2'(t)\phi(t) + N_1'(t)\psi(t) + \bar{N}_1'(t)\bar{\psi}(t)|\mathcal{F}_{t-h}\big]. \tag{24}$$

3.3 Establishment of a nonhomogeneous relationship between $\psi(t)$ and $\phi(t)$

In the sequel, the relationship between the variables $\psi(t)$ and $\phi(t)$ will be established. To this end, denote

$$\begin{aligned} \Omega(t) &\triangleq \gamma_2(t) + \bar{N}_1'(t)\big[I - L(t)\bar{S}_2(t)\big]^{-1}L(t)\bar{N}_1(t), \\ \Lambda(t) &\triangleq N_2'(t) + \bar{N}_1'(t)\big[I - L(t)\bar{S}_2(t)\big]^{-1}L(t)\bar{M}(t) + \big\{N_1'(t) + \bar{N}_1'(t)\big[I - L(t)\bar{S}_2(t)\big]^{-1} \\ &\quad \times L(t)S_2'(t)\big\}\left[L(t) - \int_t^{\min(T,t+h)} \Pi(t,\theta)\mathrm{d}\theta\right]. \end{aligned}$$

In the above, the matrix $L(t)$ is the solution to the following equation:

$$-\dot{L}(t) = \begin{cases} L(t)M(t) + \big[L(t)S_2(t) + \bar{M}'(t)\big]\big[I - L(t)\bar{S}_2(t)\big]^{-1}L(t)\bar{M}(t) + L(t)S_1(t)L(t) \\ +M'(t)L(t) + \big[L(t)S_2(t) + \bar{M}'(t)\big]\big[I - L(t)\bar{S}_2(t)\big]^{-1}L(t)S_2'(t)L(t), & t \geqslant T-h, \\ L(t)M(t) + \big[L(t)S_2(t) + \bar{M}'(t)\big]\big[I - L(t)\bar{S}_2(t)\big]^{-1}L(t)\bar{M}(t) + L(t)S_1(t)L(t) \\ +M'(t)L(t) + \big[L(t)S_2(t) + \bar{M}'(t)\big]\big[I - L(t)\bar{S}_2(t)\big]^{-1}L(t)S_2'(t)L(t) \\ -\Pi(t,t+h), & t \leqslant T-h, \end{cases} \tag{25}$$

with $\det\big[I - L(t)\bar{S}_2\big] \neq 0$, $L(T) = 0$ and $\Pi(t,\theta)$ satisfies

$$\begin{aligned} -\frac{\partial}{\partial t}\Pi(t,\theta) &= \Pi(t,\theta)M(t) + \big\{L(t)S_1(t) + M'(t) + \big[L(t)S_2(t) + \bar{M}'(t)\big]\big[I - L(t)\bar{S}_2(t)\big]^{-1} \\ &\quad \times L(t)S_2'(t)\big\}\Pi(t,\theta) + \Pi(t,\theta)S_2(t)\big[I - L(t)\bar{S}_2(t)\big]^{-1}L(t)\bar{M}(t) + \Pi(t,\theta)\big\{S_1(t) \\ &\quad + S_2(t)\big[I - L(t)\bar{S}_2(t)\big]^{-1}L(t)S_2'(t)\big\}L(t) - \int_\theta^{\min(T,t+h)} \Pi(t,\theta)\big\{S_1(t) \\ &\quad + S_2(t)\big[I - L(t)\bar{S}_2(t)\big]^{-1}L(t)S_2'(t)\big\}\Pi(t,\tau)\mathrm{d}\tau - \int_\theta^{\min(T,t+h)} \Pi(t,\tau)\big\{S_1(t) \\ &\quad + S_2(t)\big[I - L(t)\bar{S}_2(t)\big]^{-1}L(t)S_2'(t)\big\}\Pi(t,\theta)\mathrm{d}\tau, \ t \in [0,T], \ \theta \in (t, \min(T,t+h)], \end{aligned} \tag{26}$$

with

$$\Pi(t,t) = \Lambda'(t)\Omega(t)^{-1}\Lambda(t), \ t \in [0,T], \tag{27}$$
$$\det\big(\Omega(t)\big) \neq 0, \ t \in [0,T]. \tag{28}$$

Remark 1. It is unavailable to give the analytical solutions to the partial differential Riccati equations (25)–(28). However, there exists a numerical algorithm to solve these equations. Specifically, given a partition: $0 = t_0 < \cdots < t_{N+1} = T$, let $\delta = t_{k+1} - t_k$, $\delta d = h$, then the iteration algorithms can be given as follows: for $k = N - d + 1, \ldots, N$, $L(k+1) = 0$ and

$$\begin{aligned}
L(k) &= L(k+1) + \delta L(k+1)M(k) + \big(L(k+1)S_2(k) + \bar{M}'(k)\big)\,(I - L(k+1)\\
&\quad \times S_2(k))^{-1}\, L(k+1)\bar{M}(k) + L(k+1)S_1(k)L(k+1) + M'(k)L(k+1)\\
&\quad + \big(L(k+1)S_2(k) + \bar{M}'(k)\big)\,(I - L(k+1)S_2(k))^{-1}\, L(k+1)S_2'(k)L(k+1)\\
&\quad - \Pi(k+1, k+d), \quad k \leqslant N - d + 1,\\
\Pi(k, k+i) &= \Pi(k+1, k+i) + \delta\, Pi(k+1, k+i)\Bigg(\hat{M}(k+1) - \sum_{j=k+i+1}^{\min(N+1,k+d)} \delta\hat{S}(k+1)\\
&\quad \times \Pi(k+1, j)\Bigg) + \delta\Bigg(\hat{M}(k+1) - \sum_{j=k+i+1}^{\min(N+1,k+d)} \delta\hat{S}(k+1)\Pi(k+1, j)\Bigg)'\\
&\quad \times \Pi(k+1, k+i), \quad i = 1, 2, \ldots, \min(N+1-k, d),\\
\Pi(k+1, k+1) &= \Lambda'(k+1)\Omega^{-1}(k+1)\Lambda(k+1), \quad k = 1, \ldots, N,\\
\Omega^{-1}(k+1) &= \gamma_2(k+1) + \bar{N}_1'(k)(I - L(k+1)S_2(k))^{-1}L(k+1)\bar{N}_1(k), \quad k = 1, \ldots, N,\\
\Lambda(k+1) &= N_2'(k) + \bar{N}_1'(k)\,(I - L(k+1)S_2(k))^{-1}\, L(k+1)\bar{M}(k)\\
&\quad + \Big(N_1'(k) + \bar{N}_1'(k)\,(I - L(k+1)S_2(k))^{-1}\, L(k+1)S_2'(k)\Big)\\
&\quad \times \Bigg(L(k+1) - \sum_{j=k+2}^{\min(N+1,k+d)} \delta\Pi(k+1, j)\Bigg), \quad k = 1, \ldots, N,
\end{aligned}$$

where $\hat{M}(k+1) = M(k) + L(k+1)S_1(k) + S_2(k)[I - L(k+1)\bar{S}_2(k)]^{-1}L(k+1)(\bar{M}(k) + S_2'(k)L(k+1))$, $\hat{S}(k+1) = S_1(k+1) + S_2(k+1)[I - L(k+1)\bar{S}_2(k+1)]^{-1}L(k+1)S_2'(k+1)$. By selecting sufficiently small δ, it is obtained that $L(k)$, $\Pi(k, k+i)$ approximate $L(t)$ and $\Pi(t, \theta)$, respectively.

Based on these equations, we have the nonhomogeneous relationship between $\psi(t)$ and $\phi(t)$ which is shown below.

Lemma 4. Provided that Eqs. (25)–(28) admit solutions, it holds that

$$\psi(t) = L(t)\phi(t) - \int_t^{\min(T,t+h)} \Pi(t,\theta)\hat{\phi}(t|\theta - h)\mathrm{d}\theta, \tag{29}$$

where $\hat{\phi}(t|\theta - h) \triangleq \mathrm{E}\big[\phi(t)\big|\mathcal{F}_{\theta-h}\big]$, $t \leqslant \theta \leqslant \min(T, t+h)$.

Proof. Denote

$$\Theta(t) \triangleq \psi(t) - L(t)\phi(t), \tag{30}$$

where $\Theta(t)$ satisfying $\mathrm{d}\Theta(t) = \Theta_1(t)\mathrm{d}t + \bar{\Theta}(t)\mathrm{d}W(t)$, and $L(t)$ is the solution to (25). Applying Itô's formula to (30) yields that

$$\begin{aligned}
\mathrm{d}\Theta(t) &= \mathrm{d}\psi(t) - \dot{L}(t)\phi(t)\mathrm{d}t - L(t)\mathrm{d}\phi(t)\\
&= -\big[M'(t)\psi(t) + \bar{M}'(t)\bar{\psi}(t) + N_2(t)w(t-h)\big]\mathrm{d}t + \bar{\psi}(t)\mathrm{d}W(t) - \dot{L}(t)\phi(t)\mathrm{d}t\\
&\quad - L(t)\big[M(t)\phi(t) + S_1(t)\psi(t) + S_2(t)\bar{\psi}(t) + N_1(t)w(t-h)\big]\mathrm{d}t\\
&\quad - L(t)\big[\bar{M}(t)\phi(t) + S_2'(t)\psi(t) + \bar{S}_2(t)\bar{\psi}(t) + \bar{N}_1(t)w(t-h)\big]\mathrm{d}W(t).
\end{aligned} \tag{31}$$

This implies that

$$\bar{\Theta}(t) = \bar{\psi}(t) - L(t)\big[\bar{M}(t)\phi(t) + S_2'(t)\psi(t) + \bar{S}_2(t)\bar{\psi}(t) + \bar{N}_1(t)w(t-h)\big].$$

By using the invertibility of $I-L(t)\bar{S}_2$, it yields that

$$\bar{\psi}(t) = \big[I-L(t)\bar{S}_2(t)\big]^{-1}\big\{\bar{\Theta}(t)+L(t)\big[\bar{M}(t)\phi(t)+S_2'(t)\psi(t)+\bar{N}_1(t)w(t-h)\big]\big\}. \tag{32}$$

Substituting (32) into (24), we have

$$\begin{aligned}
0 &= \gamma_2(t)w(t-h)+\mathrm{E}\big[N_2'(t)\phi(t)+N_1'(t)\psi(t)+\bar{N}_1'(t)\bar{\psi}(t)|\mathcal{F}_{t-h}\big]\\
&= \gamma_2(t)w(t-h)+\mathrm{E}\big[N_2'(t)\phi(t)+N_1'(t)\psi(t)+\bar{N}_1'(t)\big[I-L(t)\bar{S}_2(t)\big]^{-1}\bar{\Theta}(t)\\
&\quad +\bar{N}_1'(t)\big[I-L(t)\bar{S}_2(t)\big]^{-1}L(t)[\bar{M}(t)\phi(t)+S_2'(t)\psi(t)+\bar{N}_1(t)w(t-h)]\Big|\mathcal{F}_{t-h}\big]\\
&\triangleq \Omega(t)w(t-h)+\mathrm{E}\Big[\big\{N_2'(t)+\bar{N}_1'(t)\big[I-L(t)\bar{S}_2(t)\big]^{-1}L(t)\bar{M}(t)\big\}\phi(t)+\bar{N}_1'(t)\big[I-L(t)\bar{S}_2(t)\big]^{-1}\bar{\Theta}(t)\\
&\quad +\big\{N_1'(t)+\bar{N}_1'(t)\big[I-L(t)\bar{S}_2(t)\big]^{-1}L(t)S_2'(t)\big\}\psi(t)\Big|\mathcal{F}_{t-h}\Big].
\end{aligned}$$

Provided that the matrix $\Omega(t)$ is invertible, one has

$$\begin{aligned}
w(t-h) &= -\Omega(t)^{-1}\mathrm{E}\Big[\big\{N_2'(t)+\bar{N}_1'(t)\big[I-L(t)\bar{S}_2(t)\big]^{-1}L(t)\bar{M}(t)\big\}\phi(t)+\bar{N}_1'(t)\big[I-L(t)\bar{S}_2(t)\big]^{-1}\bar{\Theta}(t)\\
&\quad +\big\{N_1'(t)+\bar{N}_1'(t)\big[I-L(t)\bar{S}_2(t)\big]^{-1}L(t)S_2'(t)\big\}\psi(t)\Big|\mathcal{F}_{t-h}\Big]\\
&= -\Omega(t)^{-1}\mathrm{E}\Big[\big\{N_2'(t)+\bar{N}_1'(t)\big[I-L(t)\bar{S}_2(t)\big]^{-1}L(t)\bar{M}(t)\big\}\phi(t)+\bar{N}_1'(t)\big[I-L(t)\bar{S}_2(t)\big]^{-1}\bar{\Theta}(t)\\
&\quad +\big\{N_1'(t)+\bar{N}_1'(t)\big[I-L(t)\bar{S}_2(t)\big]^{-1}L(t)S_2'(t)\big\}\big[L(t)\phi(t)+\Theta(t)\big]\Big|\mathcal{F}_{t-h}\Big].
\end{aligned} \tag{33}$$

Plugging (33) into (31) and using (25), (32), one has

$$\begin{aligned}
\Theta_1(t) &= -\big\{L(t)S_1(t)L(t)+\dot{L}(t)+\big[\bar{M}'(t)+L(t)S_2(t)\big]\big[I-L(t)\bar{S}_2(t)\big]^{-1}L(t)\big[\bar{M}(t)+S_2'(t)L(t)\big]\\
&\quad +M'(t)L(t)+L(t)M(t)\big\}\phi(t)-M'(t)\Theta(t)-\big[\bar{M}'(t)+L(t)S_2(t)\big]\big[I-L(t)\bar{S}_2(t)\big]^{-1}\bar{\Theta}(t)\\
&\quad -L(t)S_1(t)\Theta(t)-\big[\bar{M}'(t)+L(t)S_2(t)\big]\big[I-L(t)\bar{S}_2(t)\big]^{-1}L(t)S_2'(t)\Theta(t)\\
&\quad -\big\{N_2(t)+\big[\bar{M}'(t)+L(t)S_2(t)\big]\big[I-L(t)\bar{S}_2(t)\big]^{-1}L(t)\bar{N}_1(t)+L(t)N_1(t)\big\}w(t-h).
\end{aligned}$$

That is, for $t\in(T-h,T]$, $\Theta(t)$ satisfies

$$\begin{aligned}
\mathrm{d}\Theta(t) &= \big[-M'(t)\Theta(t)-L(t)S_1(t)\Theta(t)-\big[\bar{M}'(t)+L(t)S_2(t)\big]\\
&\quad \times\big[I-L(t)\bar{S}_2(t)\big]^{-1}\bar{\Theta}(t)-\big[\bar{M}'(t)+L(t)S_2(t)\big]\big[I-L(t)\bar{S}_2(t)\big]^{-1}L(t)S_2'(t)\Theta(t)\\
&\quad -\big\{N_2(t)+\big[\bar{M}'(t)+L(t)S_2(t)\big]\big[I-L(t)\bar{S}_2(t)\big]^{-1}L(t)\bar{N}_1(t)\\
&\quad +L(t)N_1(t)\big\}w(t-h)\big]\,\mathrm{d}t+\bar{\Theta}(t)\mathrm{d}W(t),
\end{aligned} \tag{34}$$

and for $t\in[0,T-h]$, $\Theta(t)$ satisfies

$$\begin{aligned}
\mathrm{d}\Theta(t) &= \big[-\Pi(t,t+h)\phi(t)-M'(t)\Theta(t)-L(t)S_1(t)\Theta(t)-\big[\bar{M}'(t)+L(t)S_2(t)\big]\\
&\quad \times\big[I-L(t)\bar{S}_2(t)\big]^{-1}\bar{\Theta}(t)-\big[\bar{M}'(t)+L(t)S_2(t)\big]\big[I-L(t)\bar{S}_2(t)\big]^{-1}L(t)S_2'(t)\Theta(t)\\
&\quad -\big\{N_2(t)+\big[\bar{M}'(t)+L(t)S_2(t)\big]\big[I-L(t)\bar{S}_2(t)\big]^{-1}L(t)\bar{N}_1(t)\\
&\quad +L(t)N_1(t)\big\}w(t-h)\big]\,\mathrm{d}t+\bar{\Theta}(t)\mathrm{d}W(t).
\end{aligned} \tag{35}$$

On the other hand, for $t\in[0,T-h]$, differentiating on $\int_t^{t+h}\Pi(t,\theta)\hat{\phi}(t|\theta-h)\mathrm{d}\theta$ with respect to t yields that

$$\begin{aligned}
&\mathrm{d}\int_t^{t+h}\Pi(t,\theta)\hat{\phi}(t|\theta-h)\mathrm{d}\theta\\
&= \Pi(t,t+h)\phi(t)\mathrm{d}t-\Pi(t,t)\hat{\phi}(t|t-h)\mathrm{d}t+\int_t^{t+h}\frac{\partial\Pi(t,\theta)}{\partial t}\hat{\phi}(t|\theta-h)\mathrm{d}\theta\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&+\int_t^{t+h}\Pi(t,\theta)\mathrm{E}[\mathrm{d}\phi(t)|\mathcal{F}_{\theta-h}]\mathrm{d}\theta\\
=&\bigg\{\Pi(t,t+h)\phi(t)-\Pi(t,t)\hat{\phi}(t|t-h)+\int_t^{t+h}\Pi(t,\theta)[M(t)+S_1(t)L(t)]\hat{\phi}(t|\theta-h)\mathrm{d}\theta\\
&+\int_t^{t+h}\Pi(t,\theta)S_1(t)\mathrm{E}\big[\Theta(t)|\mathcal{F}_{\theta-h}\big]\mathrm{d}\theta+\int_t^{t+h}\Pi(t,\theta)N_1(t)\mathrm{d}\theta w(t-h)\\
&+\int_t^{t+h}\Pi(t,\theta)S_2(t)\big[I-L(t)\bar{S}_2(t)\big]^{-1}\mathrm{E}[\bar{\Theta}(t)|\mathcal{F}_{\theta-h}]\mathrm{d}\theta+\int_t^{t+h}\Pi(t,\theta)S_2(t)\big[I\\
&-L(t)\bar{S}_2(t)\big]^{-1}L(t)\big[\bar{M}(t)+S_2'(t)L(t)\big]\hat{\phi}(t|\theta-h)\mathrm{d}\theta+\int_t^{t+h}\Pi(t,\theta)S_2(t)\big[I\\
&-L(t)\bar{S}_2(t)\big]^{-1}L(t)S_2'(t)\mathrm{E}[\Theta(t)|\mathcal{F}_{\theta-h}]\mathrm{d}\theta+\int_t^{t+h}\Pi(t,\theta)S_2(t)\big[I-L(t)\bar{S}_2(t)\big]^{-1}\\
&\times L(t)\bar{N}_1(t)\mathrm{d}\theta w(t-h)\bigg\}\mathrm{d}t-\int_t^{t+h}\bigg[\Pi(t,\theta)M(t)+\big\{L(t)S_1(t)+M'(t)\\
&+\big[L(t)S_2(t)+\bar{M}'(t)\big]\big[I-L(t)\bar{S}_2(t)\big]^{-1}L(t)S_2'(t)\big\}\Pi(t,\theta)+\Pi(t,\theta)S_2(t)\big[I\\
&-L(t)\bar{S}_2(t)\big]^{-1}L(t)\bar{M}(t)+\Pi(t,\theta)\big\{S_1(t)+S_2(t)\big[I-L(t)\bar{S}_2(t)\big]^{-1}L(t)S_2'(t)\big\}L(t)\\
&-\int_\theta^{t+h}\Pi(t,\theta)\big\{S_1(t)+S_2(t)\big[I-L(t)\bar{S}_2(t)\big]^{-1}L(t)S_2'(t)\big\}\Pi(t,\tau)\mathrm{d}\tau\\
&-\int_\theta^{t+h}\Pi(t,\tau)\big\{S_1(t)+S_2(t)\big[I-L(t)\bar{S}_2(t)\big]^{-1}L(t)S_2'(t)\big\}\Pi(t,\theta)\mathrm{d}\tau\bigg]\hat{\phi}(t|\theta-h)\mathrm{d}\theta\mathrm{d}t,
\end{aligned}
$$

where (30), (32), (26), (22) have been used in the derivation of the second equality. In addition, it can be obtained that

$$
\begin{aligned}
&\int_t^{t+h}\Pi(t,\theta)\big\{S_1(t)+S_2(t)[I-L(t)\bar{S}_2(t)]^{-1}L(t)S_2'(t)\big\}\int_t^{t+h}\Pi(t,\tau)\mathrm{E}\big[\hat{\phi}(t|\tau-h)|\mathcal{F}_{\theta-h}\big]\mathrm{d}\tau\mathrm{d}\theta\\
=&\int_t^{t+h}\Pi(t,\theta)\big\{S_1(t)+S_2[I-L(t)\bar{S}_2(t)]^{-1}L(t)S_2'(t)\big\}\int_\theta^{t+h}\Pi(t,\tau)\hat{\phi}(t|\theta-h)\mathrm{d}\tau\mathrm{d}\theta\\
&+\int_t^{t+h}\Pi(t,\theta)\big\{S_1(t)+S_2(t)[I-L(t)\bar{S}_2(t)]^{-1}L(t)S_2'(t)\big\}\int_t^{\theta}\Pi(t,\tau)\mathrm{E}\big[\hat{\phi}(t|\tau-h)|\mathcal{F}_{\theta-h}\big]\mathrm{d}\tau\mathrm{d}\theta\\
=&\int_t^{t+h}\int_\theta^{t+h}\Pi(t,\theta)\big\{S_1(t)+S_2(t)[I-L(t)\bar{S}_2(t)]^{-1}L(t)S_2'(t)\big\}\Pi(t,\tau)\hat{\phi}(t|\theta-h)\mathrm{d}\tau\mathrm{d}\theta\\
&+\int_t^{t+h}\int_\tau^{t+h}\Pi(t,\theta)\big\{S_1(t)+S_2(t)[I-L(t)\bar{S}_2(t)]^{-1}L(t)S_2'(t)\big\}\Pi(t,\tau)\hat{\phi}(t|\tau-h)\mathrm{d}\theta\mathrm{d}\tau\\
=&\int_t^{t+h}\int_\theta^{t+h}\Pi(t,\theta)\big\{S_1(t)+S_2(t)[I-L(t)\bar{S}_2(t)]^{-1}L(t)S_2'(t)\big\}\Pi(t,\tau)\hat{\phi}(t|\theta-h)\mathrm{d}\tau\mathrm{d}\theta\\
&+\int_t^{t+h}\int_\theta^{t+h}\Pi(t,\tau)\big\{S_1(t)+S_2(t)[I-L(t)\bar{S}_2(t)]^{-1}L(t)S_2'(t)\big\}\Pi(t,\theta)\hat{\phi}(t|\theta-h)\mathrm{d}\tau\mathrm{d}\theta.
\end{aligned}
$$

Combining with (25)–(28), it is easy to verify that $(-\int_t^{\min(T,t+h)}\Pi(t,\theta)\hat{\phi}(t|\theta-h)\mathrm{d}\theta,0)$ satisfies (34) and (35). Thus, $\Theta(t)=-\int_t^{\min(T,t+h)}\Pi(t,\theta)\hat{\phi}(t|\theta-h)\mathrm{d}\theta$. The proof is now completed.

Based on (33) in the above discussion, we now give the strictly positive definiteness of the matrix $\Omega(t)$ in the following result.

Lemma 5. Assume that there exists a unique open-loop strategy for the stochastic differential game and (25) admits a solution, then the matrix $\Omega(t)$ is positive definite.

Proof. From (14) and (15), applying Itô's formula to $p_2'(t)x(t)$ and taking expectation yield that

$$\begin{aligned}\mathrm{E}\int_0^T \mathrm{d}\big[p_2'(t)x(t)\big] =& -\mathrm{E}\int_0^T \big\{\big[D'(t)p_2(t)+\bar{D}'(t)q_2(t)+Q_2x(t)\big]'x(t)+p_2'(t)\big[D(t)x(t)\\&+E(t)w(t-h)+F_1(t)\zeta_1(t)+F_2(t)\bar{\zeta}_1(t)\big]\mathrm{d}t+q_2'(t)\big[\bar{D}(t)x(t)+\bar{E}(t)w(t-h)\\&+F_2'(t)\zeta_1(t)+\bar{F}_2(t)\bar{\zeta}_1(t)\big]\big\}\mathrm{d}t\\=&\ \mathrm{E}\int_0^T \big\{-x'(t)Q_2x(t)+w'(t-h)\big[E'(t)p_2(t)+\bar{E}'(t)q_2(t)\big]\mathrm{d}t+\zeta_1'(t)\big[F_1(t)p_2(t)\\&+F_2(t)q_2(t)\big]+\bar{\zeta}_1'(t)\big[F_2'(t)p_2(t)+\bar{F}_2(t)q_2(t)\big]\big\}\mathrm{d}t. \end{aligned} \tag{36}$$

Similarly, it yields that

$$\begin{aligned}\mathrm{E}\int_0^T \mathrm{d}\big[\zeta_1'(t)\xi(t)\big] =&\ \mathrm{E}\int_0^T \big\{-w'(t-h)\bar{\bar{D}}(t)\xi(t)+\zeta_1'(t)\big[F_1(t)p_2(t)\\&+F_2(t)q_2(t)\big]+\bar{\zeta}_1'(t)\big[F_2'(t)p_2(t)+\bar{F}_2(t)q_2(t)\big]\big\}\mathrm{d}t.\end{aligned}$$

Thus,

$$\begin{aligned}\mathrm{E}\int_0^T \mathrm{d}\big[p_2'(t)x(t)\big] =&\ \mathrm{E}\int_0^T \big\{-x'(t)Q_2x(t)+w'(t-h)\big[E'(t)p_2(t)\\&+\bar{E}'(t)q_2(t)+\bar{\bar{D}}(t)\xi(t)\big]\mathrm{d}t+\mathrm{d}\big[\zeta_1'(t)\xi(t)\big]\big\}. \end{aligned}\tag{37}$$

The optimal cost of the leader can be reformulated as (noting (32))

$$\begin{aligned}J_2 =&\ \mathrm{E}\big[p_2(0)'x_0\big]+\mathrm{E}\int_0^T w'(t-h)\big[R_2w(t-h)+E'(t)p_2(t)+\bar{E}'(t)q_2(t)\\&+\bar{\bar{D}}(t)\xi(t)\big]\mathrm{d}t+\mathrm{E}\int_0^T \mathrm{d}[\zeta_1'(t)\xi(t)]\\=&\ \mathrm{E}\big[p_2(0)'x_0\big]+\mathrm{E}\int_0^T w'(t-h)\big\{\Omega(t)w(t-h)+N_2'(t)\phi(t)+N_1'(t)\psi(t)\\&+\bar{N}_1'(t)\big[I-L(t)\bar{S}_2(t)\big]^{-1}L(t)\big[\bar{M}(t)\phi(t)+S_2'(t)\psi(t)\big]\big\}\mathrm{d}t.\end{aligned}$$

Similarly, we have

$$\begin{aligned}J_2^t \triangleq&\ \mathrm{E}\left[\int_t^T x'(s)Q_2x(s)\mathrm{d}s+\int_t^{T-h} w'(s)R_2w(s)\mathrm{d}s+x(T)'H_2x(T)\right]\\=&\ \mathrm{E}\big[p_2(t)'x(t)\big]+\mathrm{E}\int_t^T w'(s-h)\big\{\Omega(s)w(s-h)+N_2'(s)\phi(s)+N_1'(s)\psi(s)\\&+\bar{N}_1'(s)\big[I-L(s)\bar{S}_2(s)\big]^{-1}L(s)\big[\bar{M}(s)\phi(s)+S_2'(s)\psi(s)\big]\big\}\mathrm{d}s.\end{aligned}\tag{38}$$

Let the initial time be t and the initial values be chosen as $x(t)=0$, $w(t+s)=0$, $s\in[-h,0]$. Then, the unique optimal solution with respect to these initial values is $w(s-h)=0$, $s\in[t,T]$ and the corresponding optimal cost is zero. This implies that any nonzero controller w will lead to a strictly positive cost. Select $w(s-h)=0$, $s\in(t+\varepsilon,T)$ and let $w(s-h)$, $s\in[t,t+\varepsilon]$ be arbitrarily nonzero where ε is a sufficiently small positive constant. In this case, $\phi(t)=0$ from (22) and $\psi(s)=0$, $s\in[t+\varepsilon,T]$ from (23). Then, (38) is reduced to

$$\begin{aligned}J_2^t =&\ \mathrm{E}\int_t^{t+\varepsilon} w'(s-h)\big\{\Omega(s)w(s-h)+N_2'(s)\phi(s)+N_1'(s)\psi(s)\\&+\bar{N}_1'(s)\big[I-L(s)\bar{S}_2(s)\big]^{-1}L(s)\big[\bar{M}(s)\phi(s)+S_2'(s)\psi(s)\big]\big\}\mathrm{d}s.\end{aligned}$$

Noting that $\phi(t) = 0$, $\psi(t+\varepsilon) = 0$ and ε is sufficiently small, the above equation is reformulated as $J_2^t = \varepsilon \mathrm{E} w'(t-h)\Omega(t)w(t-h)$ which is positive from the uniqueness of the open-loop solution for the game. This implies that $\Omega(t) > 0$. The proof is completed.

From Lemma 4 and (33), it yields that

$$\begin{aligned} w(t-h) &= -\Omega(t)^{-1}\mathrm{E}\Bigg[\big\{N_2'(t) + \bar{N}_1'(t)\big[I - L(t)\bar{S}_2(t)\big]^{-1}L(t)\bar{M}(t)\big\}\phi(t) + \big\{N_1'(t) + \bar{N}_1'(t)\big[I \\ &\quad -L(t)\bar{S}_2(t)\big]^{-1}L(t)S_2'(t)\big\}\left[L(t)\phi(t) - \int_t^{t+h}\Pi(t,\theta)\hat{\phi}(t|\theta-h)\mathrm{d}\theta\right]\Bigg|\mathcal{F}_{t-h}\Bigg] \\ &= -\Omega(t)^{-1}\mathrm{E}\Bigg[\big\{N_2'(t) + \bar{N}_1'(t)\big[I - L(t)\bar{S}_2(t)\big]^{-1}L(t)\bar{M}(t)\big\} + \big\{N_1'(t) + \bar{N}_1'(t)\big[I - L(t)\bar{S}_2(t)\big]^{-1} \\ &\quad \times L(t)S_2'(t)\big\}\left[L(t) - \int_t^{t+h}\Pi(t,\theta)\mathrm{d}\theta\right]\Bigg]\hat{\phi}(t|t-h), \end{aligned}$$

that is,

$$w(t-h) = -\Omega(t)^{-1}\Lambda(t)\hat{\phi}(t|t-h). \tag{39}$$

3.4 Solution to stochastic game

We are now in the position to state the main results for the open-loop strategy.

Theorem 1. Assume that there exists a unique open-loop strategy for the leader-follower stochastic game (1)–(3). Provided that the coupled equation (25)–(28) admits a solution, the open-loop strategy is given by

$$u(t) = K_1^u(t)\phi(t) + \int_t^{\min(T,t+h)} K_2^u(t,\theta)\hat{\phi}(t|\theta-h)\mathrm{d}\theta + K_3^u(t)\hat{\phi}(t|t-h), \tag{40}$$

$$w(t-h) = K^w(t)\hat{\phi}(t|t-h), \tag{41}$$

where

$$\begin{aligned} &K^w(t) = -\Omega(t)^{-1}\Lambda(t), \\ &K_1^u(t) = -\gamma_1(t)^{-1}\{[0\ B'P_1(t) + \bar{B}'P_1(t)\bar{A}] + [B'\ 0]L(t) \\ &\qquad +[\bar{B}'\ 0][I - L(t)\bar{S}_2(t)]^{-1}L(t)[\bar{M}(t) + S_2'(t)L(t)]\}, \\ &K_2^u(t,\theta) = \gamma_1(t)^{-1}\{[B'\ 0]\Pi(t,\theta) + [\bar{B}'\ 0][I - L(t)\bar{S}_2(t)]^{-1}L(t)S_2'(t)\Pi(t,\theta)\}, \\ &K_3^u(t) = \gamma_1(t)^{-1}\{[\bar{B}'\ 0][I - L(t)\bar{S}_2(t)]^{-1}L(t)\bar{N}_1(t) + \bar{B}'P_1(t)\bar{C}\}\Omega(t)^{-1}\Lambda(t). \end{aligned}$$

Proof. From (39), the optimal controller of $w(t-h)$ is as (41). Furthermore, in view of (7) and Lemma 4, we have

$$\begin{aligned} u(t) &= -\gamma_1(t)^{-1}\left\{[B'P_1(t) + \bar{B}'P_1(t)\bar{A}]x(t) + B'\zeta_1(t) + \bar{B}'P_1(t)\bar{C}w(t-h) + \bar{B}'\bar{\zeta}_1(t)\right\} \\ &= -\gamma_1(t)^{-1}\left\{[0\ B'P_1(t) + \bar{B}'P_1(t)\bar{A}]\phi(t) + [B'\ 0]\psi(t) + [\bar{B}'\ 0]\bar{\psi}(t) + \bar{B}'P_1(t)\bar{C}w(t-h)\right\} \\ &= -\gamma_1(t)^{-1}\Bigg\{[0\ B'P_1(t) + \bar{B}'P_1(t)\bar{A}]\phi(t) + [B'\ 0]L(t)\phi(t) - [B'\ 0]\int_t^{t+h}\Pi(t,\theta)\hat{\phi}(t|\theta-h)\mathrm{d}\theta \\ &\quad +[\bar{B}'\ 0][I - L(t)\bar{S}_2(t)]^{-1}\Bigg\{L(t)[\bar{M}(t) + S_2'(t)L(t)]\phi(t) - L(t)S_2'(t)\int_t^{t+h}\Pi(t,\theta)\hat{\phi}(t|\theta-h)\mathrm{d}\theta \\ &\quad +L(t)\bar{N}_1(t)w(t-h)\Bigg\} + \bar{B}'P_1(t)\bar{C}w(t-h)\Bigg\} \end{aligned}$$

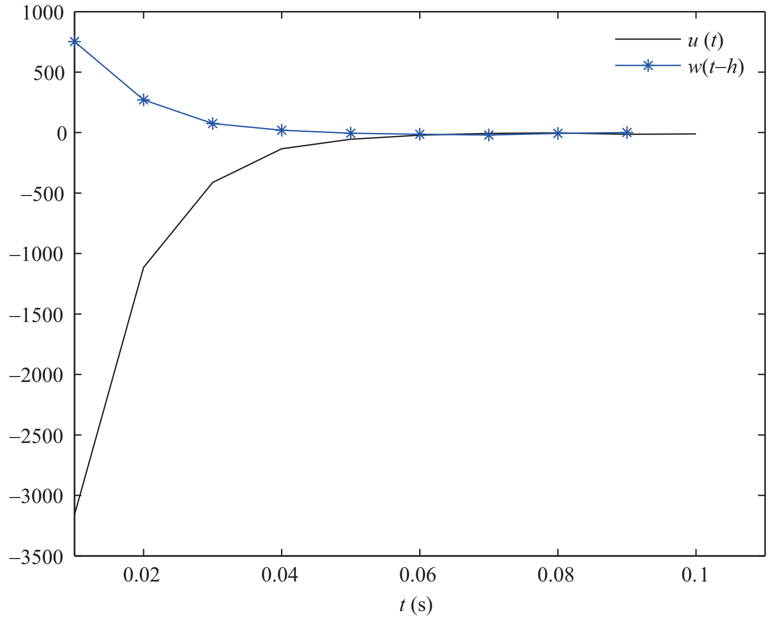

Figure 1 (Color online) The open-loop Stackelberg solution.

$$= -\gamma_1(t)^{-1}\Big\{\big([0\ B'P_1(t)+\bar{B}'P_1(t)\bar{A}]+[B'\ 0]L(t)+[\bar{B}'\ 0][I-L(t)\bar{S}_2(t)]^{-1}L(t)$$

$$\times[\bar{M}(t)+S_2'(t)L(t)]\big)\phi(t)-\int_t^{t+h}\big([B'\ 0]\Pi(t,\theta)+[\bar{B}'\ 0][I-L(t)\bar{S}_2(t)]^{-1}L(t)S_2'(t)\Pi(t,\theta)\big)$$

$$\times\hat{\phi}(t|\theta-h)\mathrm{d}\theta-\big([\bar{B}'\ 0][I-L(t)\bar{S}_2(t)]^{-1}L(t)\bar{N}_1(t)+\bar{B}'P_1(t)\bar{C}\big)\Omega(t)^{-1}\Lambda(t)\hat{\phi}(t|t-h)\Big\},$$

which is exactly (40). The proof is now completed.

Remark 2. Compared to the results in [17], the main contribution of this paper is to introduce the new variable $\zeta_2(t)$ captured the information from the leader's controller $w(s-h)$ $(s \geqslant t)$. Accordingly, the nonhomogeneous relationship (29) is established while it is homogeneous for the delay-free case.

3.5 Simulation

Let $A = 0.12$, $\bar{A} = 1$, $B = 0.17$, $\bar{B} = 0.21$, $C = 0.1$, $\bar{C} = 0.3$, $T = 0.1$, $h = 0.01$, $H_1 = H_2 = 1$, $Q_1 = Q_2 = R_1 = R_2 = 0.1$. From Theorem 1 and using the discretization technique, the open-loop solution is given by Figure 1.

4 Conclusion

The open-loop strategy for a leader-follower stochastic differential game with time delay appearing in the leader's control has been studied in this paper. The main contribution of the paper is to illustrate the explicitly optimal controller in terms of the decoupled and symmetric Riccati equations based on the stochastic maximum principle. The key technique is to establish the nonhomogeneous relationship between the forward variables and the backward ones. It is highly desirable to study the closed-loop strategy for our problem, as well as the solvability of the Riccati equations. These topics will be considered in our future study.

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References

1 Basar T, Olsder G J. Dynamic Noncooperative Game Theory. New York: Academic Press, 1995
2 Isaacs R. Differential Games: A Mathematical Theory With Applicaitons to Warfare and Pursuit, Control and Optimization. Hoboken: John Wiley and Sons, 1999
3 Starr A W, Ho Y C. Nonzero-sum differential games. J Optim Theory Appl, 1969, 3: 184–206
4 Shi J T, Wang G C, Xiong J. Leader-follower stochastic differential game with asymmetric information and applications. Automatica, 2016, 63: 60–73
5 Mu Y F, Guo L. How cooperation arises from rational players? Sci China Inf Sci, 2013, 56: 112201
6 Mukaidani H. Dynamic games for stochastic systems with delay. Asian J Control, 2013, 3: 1251–1260
7 Zhang H S, Xu J J. Control for Itô stochastic systems with input delay. IEEE Trans Autom Control, 2017, 62: 350–365
8 Wang T X, Shi Y F. Linear quadratic stochastic integral games and related topics. Sci China Math, 2015, 58: 2405–2420
9 Freiling G, Jank G, Lee S R. Existence and uniqueness of open-loop Stackelberg equilibria in linear-quadratic differential games. J Optim Theory Appl, 2001, 110: 515–544
10 Papavassilopoulos G P, Cruz J B. Nonclassical control problems and Stackelberg games. IEEE Trans Autom Control, 1979, 24: 155–166
11 Simaan M, Cruz J B. On the Stackelberg strategy in nonzero-sum games. J Optim Theory Appl, 1973, 11: 533–555
12 Basar T. Stochastic stagewise Stackelberg strategies for linear quadratic systems. In: Stochastic Control Theory and Stochastic Differential Systems. Berlin: Springer, 1979
13 Bensoussan A, Chen S, Sethi S P. The maximum principle for global solutions of stochastic Stackelberg differential games. SIAM J Control Optim, 2015, 53: 1956–1981
14 Hamadéne S. Nonzero sum linear-quadratic stochastic differential games and backward-forward equations. Stoch Anal Appl, 1999, 17: 117–130
15 Hu Y. N-person differential games governed by semilinear stochastic evolution systems. Appl Math Optim, 1991, 24: 257–271
16 Wu Z. Forward-backward stochastic differential equations, linear quadratic stochastic optimal control and nonzero sum differential games. J Syst Sci Complex, 2005, 18: 179–192
17 Yong J M. A leader-follower stochastic linear quadratic differential game. SIAM J Control Optim, 2002, 41: 1015–1041
18 Shi J T, Wang G C, Xiong J. Linear-quadratic stochastic Stackelberg differential game with asymmetric information. Sci China Inf Sci, 2017, 60: 092202
19 Dong X W, Li Q D, Ren Z, et al. Formation-containment control for high-order linear time-invariant multi-agent systems with time delays. J Franklin Inst, 2015, 352: 3564–3584
20 Dong X W, Han L, Li Q D, et al. Containment analysis and design for general linear multi-agent systems with time-varying delays. Neurocomputing, 2016, 173: 2062–2068
21 Mukaidani H, Unno M, Yamamoto T, et al. Nash strategy for Markov jump stochastic delay systems. In: Proceedings of the 52nd IEEE Conference on Decision and Control, Florence, 2013. 1198–1203
22 Chen L, Wu Z. Maximum principle for the stochastic optimal control problem with delay and application. Automatica, 2010, 46: 1074–1080
23 Huang J H, Li X, Shi J T. Forward-backward linear quadratic stochastic optimal control problem with delay. Syst Control Lett, 2012, 61: 623–630
24 Øksendal B, Sulem A. A maximum principle for optimal control of stochastic systems with delay, with applications to finance. In: Optimal Control and Partial Differential Equations — Innovations and Applications. Amsterdam: IOS Press, 2000
25 Wang H X, Zhang H S. LQ control for Itô-type stochastic systems with input delays. Automatica, 2013, 49: 3538–3549
26 Yu Z Y. The stochastic maximum principle for optimal control problems of delay systems involving continuous and impulse controls. Automatica, 2012, 48: 2420–2432
27 Xu J J, Zhang H S. Sufficient and necessary open-loop Stackelberg strategy for two-player game with time delay. IEEE Trans Cybern, 2016, 46: 438–449
28 Yong J M, Zhou X Y. Stochatic Controls: Hamiltonian Systems and HJB Equations. Berlin: Springer, 1999
29 Rami M A, Chen X, Moore J B, et al. Solvability and asymptotic behavior of generalized Riccati equations arising in indefinite stochastic LQ controls. IEEE Trans Autom Control, 2001, 46: 428–440
30 Tadmor G, Mirkin L. H^∞ control and estimation with preview-part I: matrix ARE solutions in continuous time. IEEE Trans Autom Control, 2005, 50: 19–28