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Smooth fractal surfaces derived from bicubic rational fractal interpolation functions

Fangxun BAO[1], Xunxiang YAO[3], Qinghua SUN[1],
Yunfeng ZHANG[2*] & Caiming ZHANG[1]

[1]*School of Mathematics, Shandong University, Jinan* 250100*, China;*
[2]*School of Computer Science & Technology, Shandong University of Finance and Economics, Jinan* 250014*, China;*
[3]*Faculty of Engineering and Information Technology, University of Technology Sydney, Sydney* NSW2007*, Australia*

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Dear editor,

Fractal geometry is an important and active branch of nonlinear science. It has attracted more and more attention [1, 2]. As an important research field of fractal geometry, fractal interpolation was first introduced by Barnsley using a certain Iterated Function System (IFS). Then it was extended to enable interpolation of fractal surfaces and three dimensional objects [3, 4]. Fractal interpolation technique, which provides a good deterministic representation of complex phenomena, has become a powerful tool for dealing with the irregular data. It is widely used in natural scenery simulation, security analysis, image processing and so on. Meanwhile, fractal interpolation is fused with other methods, such as wavelet [5].

The use of IFS to represent surfaces has been explored by a number of researchers. Bouboulis et al. [6] provided a construction of smooth fractal surfaces using C^1 Hermite fractal interpolation functions (FIFs). Yun et al. [7] proposed a method to construct fractal surfaces by recurrent fractal curves. Ruan and Xu [8] presented a general framework to construct fractal interpolation surfaces (FISs) on rectangular grids. They introduced bilinear FISs without any restriction on interpolation points and vertical scaling factors. Most of current fractal function interpolation systems are polynomial-based. They may ignore the intrinsic form implied by the given data points, so that the fractal interpolation surfaces may have undesirable inflections or oscillations. The bivariate rational fractal interpolation, which honors the properties inherent in the data, has attracted the attention. For example, Ref. [9] developed a positive blending Hermite rational spline fractal interpolation scheme.

The aim of this study is to introduce a constructive method of shape modifiable fractal surfaces. We propose a type of bicubic rational spline FIFs with shape parameters. The basic idea of constructing rational FIFs is to treat them as the "fractal perturbation" of bicubic rational spline functions obtained via base functions. In addition, analytical properties of rational FIFs are investigated.

Bivariate IFS. Let $I = [a, b], J = [c, d]$ and $\Omega = I \times J$. Let $a = x_1 < x_2 < \cdots < x_N = b$ and $c = y_1 < y_2 < \cdots < y_M = d$, which are partitions of intervals I and J respectively. Set $I_i = [x_i, x_{i+1}], J_j = [y_j, y_{j+1}]$ and $\Omega_{i,j} = I_i \times J_j$. Denote $\mathcal{I} = \{1, 2, \ldots, N\}$, $\mathcal{J} = \{1, 2, \ldots, M\}$, $\mathscr{I} = \{1, 2, \ldots, N-1\}$, $\mathscr{J} = \{1, 2, \ldots, M-1\}$. For $i \in \mathcal{I}$ and $j \in \mathscr{J}$, define mappings $w_{i,j} : \Omega \times \mathbb{R} \to \Omega_{i,j} \times \mathbb{R}$,

$$w_{i,j}(x, y, z) = (\phi_i(x), \varphi_i(y), F_{i,j}(x, y, z)), \quad (1)$$

* Corresponding author (email: yfzhang@sdufe.edu.cn)

where $\phi_i(x) = a_i x + b_i, \varphi_i(y) = c_j y + d_j$, $a_i = \frac{x_{i+1}-x_i}{x_N-x_1}$, $b_i = \frac{x_N x_i - x_1 x_{i+1}}{x_N-x_1}$, $c_j = \frac{y_{j+1}-y_j}{y_M-y_1}$, $d_j = \frac{y_M y_j - y_1 y_{j+1}}{y_M-y_1}$, and $F_{i,j}$ are continuous mappings: $\Omega \times \mathbb{R} \to \mathbb{R}$, which satisfy

$$\begin{aligned} &F_{i,j}(x_1, y_1, z_{1,1}) = z_{i,j},\\ &F_{i,j}(x_1, y_M, z_{1,M}) = z_{i,j+1},\\ &F_{i,j}(x_N, y_1, z_{N,1}) = z_{i+1,j},\\ &F_{i,j}(x_N, y_M, z_{N,M}) = z_{i+1,j+1}. \end{aligned}$$

Then IFS (1) generates a unique attractor which is the graph of a continuous function $\Phi : \Omega \to \mathbb{R}$ satisfying $\Phi(x_i, y_j) = z_{i,j}$. Function Φ is called a FIF. And for $i \in \mathscr{I}$, $j \in \mathscr{J}$, the following relations hold.

$$\Phi(\phi_i(x), \varphi_j(y)) = F_{i,j}(x, y, \Phi(x, y)),$$
$$\Phi(x, y) = F_{i,j}(\phi_i^{-1}(x), \varphi_j^{-1}(y), \Phi(\phi_i^{-1}(x), \varphi_j^{-1}(y))).$$

Rational fractal surfaces. Let $\Omega = [a, b; c, d]$ be the plane region, and $\Delta = \{(x_i, y_j, f_{i,j}, d^*_{i,j}, d_{i,j}) : i \in \mathcal{I}; j \in \mathcal{J}\}$ be a given set of data points, where $a = x_1 < x_2 < \cdots < x_N = b$, $c = y_1 < y_2 < \cdots < y_M = d$ are knot spacings, $f_{i,j}$ represents the value of $f(x, y)$ at the point (x_i, y_j), and $d^*_{i,j}$ and $d_{i,j}$ are the chosen first-order partial derivatives $\frac{\partial f(x,y)}{\partial x}$ and $\frac{\partial f(x,y)}{\partial y}$ at the knots (x_i, y_j), respectively. Denote $h_i = x_{i+1} - x_i$, $l_j = y_{j+1} - y_j$, $H_N = x_N - x_1$, $L_M = y_M - y_1$, and for any point $(x, y) \in \Omega$, $\theta := \frac{x-x_1}{x_N-x_1}$, $\eta := \frac{y-y_1}{y_M-y_1}$.

Define

$$\begin{aligned} \omega_{0,0}(\theta, \alpha_{i,j}) &= \frac{(1-\theta)^2((1+\theta)\alpha_{i,j} + \theta)}{(1-\theta)\alpha_{i,j} + \theta},\\ \omega_{0,1}(\theta, \alpha_{i,j}) &= \frac{\theta^2((1-\theta)\alpha_{i,j} + 2 - \theta)}{(1-\theta)\alpha_{i,j} + \theta},\\ \omega_{1,0}(\theta, \alpha_{i,j}) &= \frac{\theta(1-\theta)^2\alpha_{i,j}}{(1-\theta)\alpha_{i,j} + \theta},\\ \omega_{1,1}(\theta, \alpha_{i,j}) &= \frac{-\theta^2(1-\theta)}{(1-\theta)\alpha_{i,j} + \theta}. \end{aligned}$$

We regard rational fractal surfaces as a "fractal perturbation" of rational surfaces obtained via rational base surfaces. Firstly, bicubic rational spline interpolation functions $P_{i,j}(\phi_i(x), \varphi_j(y))$ and perturbation base functions $B_{i,j}(x, y)$ are defined as follows.

$$\begin{aligned} P_{i,j}(\phi_i(x), \varphi_j(y)) = \sum_{s=0}^{1}\sum_{r=0}^{1}&[a_{r,s}(\theta, \eta) f_{i+r,j+s}\\ &+ b_{r,s}(\theta, \eta) h_i d^*_{i+r,j+s} + c_{r,s}(\theta, \eta) l_j d_{i+r,j+s}],\\ B_{i,j}(x, y) = \sum_{s=0}^{1}\sum_{r=0}^{1}&[a_{r,s}(\theta, \eta) f_{r(N-1)+1,s(M-1)+1}\\ &+ b_{r,s}(\theta, \eta) H_N d^*_{r(N-1)+1,s(M-1)+1}\\ &+ c_{r,s} L_M d_{r(N-1)+1,s(M-1)+1}], \end{aligned}$$

where

$$\begin{aligned} a_{r,s}(\theta, \eta) &= \omega_{0,r}(\theta, \alpha_{i,j+s})\omega_{0,s}(\eta, \beta_{i,j}),\\ b_{r,s}(\theta, \eta) &= \omega_{1,r}(\theta, \alpha_{i,j+s})\omega_{0,s}(\eta, \beta_{i,j}),\\ c_{r,s}(\theta, \eta) &= \omega_{0,r}(\theta, \alpha_{i,j+s})\omega_{1,s}(\eta, \beta_{i,j}). \end{aligned}$$

These terms $\{(a_{r,s}(\theta, \eta), b_{r,s}(\theta, \eta), c_{r,s}(\theta, \eta)) : r = 0, 1; s = 0, 1\}$ satisfy

$$\begin{aligned} &\sum_{s=0}^{1}\sum_{r=0}^{1} a_{r,s}(\theta, \eta) = 1,\\ &\sum_{s=0}^{1}\sum_{r=0}^{1} |b_{r,s}(\theta, \eta)| = \theta(1-\theta),\\ &\sum_{s=0}^{1}\sum_{r=0}^{1} |c_{r,s}(\theta, \eta)| = \eta(1-\eta). \end{aligned} \tag{2}$$

Next, we consider the following IFS:

$$\begin{cases} \phi_i(x) = a_i x + b_i,\\ \varphi_j(y) = c_j y + d_j,\\ F_{i,j}(x, y, z) = s_{i,j} z + P_{i,j}(\phi_i(x), \varphi_j(y))\\ \qquad\qquad -s_{i,j} B_{i,j}(x, y), \end{cases} \tag{3}$$

then this IFS admits a unique attractor G, and G is the graph of a C^1-continuous function $\Phi(x, y)$.

$$\begin{aligned} \Phi(\phi_i(x), \varphi_j(y)) &= F(x, y, \Phi(x, y))\\ &= s_{i,j}\Phi(x, y) + P_{i,j}(\phi_i(x), \varphi_j(y))\\ &\quad - s_{i,j} B_{i,j}(x, y), \end{aligned} \tag{4}$$

where $|s_{i,j}| < \min\{a_i, c_j\}$, and $s_{i,j}$ is scaling factor. The FIF defined by (4) is called a bicubic rational fractal interpolation function. Denote $h = \max\{h_i : i \in \mathscr{I}\}$, $l = \max\{l_j : j \in \mathscr{J}\}$, $M_0 = \max\{|f_{i,j}| : i \in \mathcal{I}; j \in \mathcal{J}\}$, $D^*_0 = \max\{|d^*_{i,j}| : i \in \mathcal{I}; j \in \mathcal{J}\}$, $D_0 = \max\{|d_{i,j}| : i \in \mathcal{I}; j \in \mathcal{J}\}$. Then the following inequalities can be derived.

$$\begin{aligned} &\|P_{i,j} - s_{i,j} B_{i,j}\|_\infty \leqslant (1 + |s|_\infty)M_0 + \frac{hD^*_0 + lD_0}{2},\\ &\|\Phi\|_\infty \leqslant \frac{1}{1 - |s|_\infty}\left[(1 + |s|_\infty)M_0 + \frac{1}{2}(hD^*_0 + lD_0)\right]. \end{aligned}$$

Denote

$$A = \Big(\omega_{0,0}(\theta, \alpha_{i,j})\ \omega_{0,1}(\theta, \alpha_{i,j})\ \omega_{1,0}(\theta, \alpha_{i,j})\ \omega_{1,1}(\theta, \alpha_{i,j})\Big),$$

$$B = \Big(\omega_{0,0}(\eta, \beta_{i,j})\ \omega_{0,1}(\eta, \beta_{i,j})\ \omega_{1,0}(\eta, \beta_{i,j})\ \omega_{1,1}(\eta, \beta_{i,j})\Big)^{\mathrm{T}},$$

$$F^* = \begin{bmatrix} E_{1,1} & E_{1,2} & D_{1,1} & D_{1,2}\\ E_{2,1} & E_{2,2} & D_{2,1} & D_{2,2}\\ D^*_{1,1} & D^*_{1,2} & 0 & 0\\ D^*_{2,1} & D^*_{2,2} & 0 & 0 \end{bmatrix},$$

where $E_{1,1} = f_{i,j} - s_{i,j}f_{1,1}$, $E_{1,2} = f_{i,j+1} - s_{i,j}f_{1,M}$, $E_{2,1} = f_{i+1,j} - s_{i,j}f_{N,1}$, $E_{2,2} = f_{i+1,j+1} - s_{i,j}f_{N,M}$, $D^*_{1,1} = h_i d^*_{i,j} - s_{i,j}H_N d^*_{1,1}$, $D^*_{1,2} = h_i d^*_{i,j+1} - s_{i,j}H_N d^*_{1,M}$, $D^*_{2,1} = h_i d^*_{i+1,j} - s_{i,j}H_N d^*_{N,1}$, $D^*_{2,2} = h_i d^*_{i+1,j+1} - s_{i,j}H_N d^*_{N,M}$, $D_{1,1} = l_j d_{i,j} - s_{i,j}L_M d_{1,1}$, $D_{1,2} = l_j d_{i,j+1} - s_{i,j}L_M d_{1,M}$, $D_{2,1} = l_j d_{i+1,j} - s_{i,j}L_M d_{N,1}$, $D_{2,2} = l_j d_{i+1,j+1} - s_{i,j}L_M d_{N,M}$. Moreover, if shape parameters satisfy $\alpha_{i,j} = \alpha_{i,j+1}$, the bicubic rational FIF $\Phi(x,y)$ can be represented as the following matrix form:

$$\Phi(\phi_i(x), \varphi_j(y)) = s_{i,j}\Phi(x,y) + AF^*B.$$

Remark 1. For the rational FIF $\Phi(x,y)$ defined by (4), if scaling factors $s_{i,j} = 0$ for all $i \in \mathscr{I}, j \in \mathscr{J}$, then $\Phi(x,y)$ coincides with the bivariate rational Hermite interpolant $P_{i,j}(x,y)$; If the shape parameters $\alpha_{i,j} = \beta_{i,j} = 1$, then $\Phi(x,y)$ reduces to Hermite FIF; If $s_{i,j} = 0$ and the shape parameters $\alpha_{i,j} = \beta_{i,j} = 1$, then $\Phi(x,y)$ reduces to the standard Hermite interpolant. It means that the rational FIF defined by (4) is more flexible and diversiform for the choice of a suitable interpolant.

Analysis properties. For the rational FIF defined by (4), the following theorems hold.

Theorem 1. For the IFS defined by (3), there exists a metric $\mathscr{D}$ on $\Omega \times \mathbb{R}$, equivalent to the Euclidean metric, such that the IFS is hyperbolic, and has a unique attractor G in $\mathbb{R}^3$.

Theorem 2. Let $f(x,y) \in C^r(\Omega), r = 1,2$, and $\Phi(x,y)$ be the rational FIF defined by (4), then

$$\begin{aligned}&\|f - \Phi\|_\infty \\ &\leqslant l^r \left\| \frac{\partial^r f}{\partial y^r} \right\|_\infty C + h\left(\left\| \frac{\partial f}{\partial x} \right\|_\infty + \left\| \frac{\partial P}{\partial x} \right\|_\infty \right) \\ &\quad + \frac{|s|_\infty}{1 - |s|_\infty}\left(2M_0 + \frac{h + H_N}{4} D^*_0 + \frac{l + L_M}{4} D_0 \right),\end{aligned}$$

where C is a positive constant.

Let $\hat{\Delta} = \{(x_i, y_j, \hat{f}_{i,j}, \hat{d}^*_{i,j}, \hat{d}_{i,j}) : i \in \mathcal{I}; j \in \mathcal{J}\}$ be another interpolation data set generated by perturbation of $f_{i,j}$, $d^*_{i,j}$ and $d_{i,j}$. We define an IFS $\{\hat{F}; (\phi_i(x), \varphi_j(y), \hat{F}_{i,j}(x,y,\hat{\Phi}(x,y))) : i \in \mathscr{I}; j \in \mathscr{J}\}$, where $\phi_i(x)$ and $\varphi_j(y)$ are defined as above, $\hat{F}_{i,j}(x,y,\hat{\Phi}(x,y)) = s_{i,j}\hat{\Phi}(x,y) + \hat{P}_{i,j}(\phi_i(x), \varphi_j(y)) - s_{i,j}\hat{B}_{i,j}(x,y)$, $\hat{P}_{i,j}(\phi_i(x), \varphi_j(y))$, and $\hat{B}_{i,j}(x,y)$ satisfy the corresponding join-up condition. The IFS determines an FIF, denoted by $\hat{\Phi}(x,y)$. Denote $\delta_f = \max\{|f_{i,j} - \hat{f}_{i,j}| : i \in \mathcal{I}; j \in \mathcal{J}\}$, $\delta_{d^*} = \max\{|d^*_{i,j} - \hat{d}^*_{i,j}| : i \in \mathcal{I}; j \in \mathcal{J}\}$, $\delta_d = \max\{|d_{i,j} - \hat{d}_{i,j}| : i \in \mathcal{I}; j \in \mathcal{J}\}$.

Theorem 3. Let Ω be a plane region, $\Delta = \{(x_i, y_j, f_{i,j}, d^*_{i,j}, d_{i,j}) : i \in \mathcal{I}; j \in \mathcal{J}\}$ and $\hat{\Delta} = \{(x_i, y_j, \hat{f}_{i,j}, \hat{d}^*_{i,j}, \hat{d}_{i,j}) : i \in \mathcal{I}; j \in \mathcal{J}\}$ be two sets of interpolation data points. $\Phi(x,y)$ and $\hat{\Phi}(x,y)$ are corresponding FIFs defined on Ω, then

$$\|\Phi - \hat{\Phi}\|_\infty \leqslant \frac{1 + |s|_\infty}{1 - |s|_\infty}\delta_f + \frac{h\delta_{d^*} + l\delta_d}{2(1 - |s|_\infty)}.$$

Conclusion. We proposed a constructive method of fractal surfaces. The key advantages of the presented rational FIF are as follows: (i) It gives good approximation to the original function, and is stable on the perturbation of interpolating data. (ii) It possesses good capacity of quasi-locality on IFS parameters. (iii) The developed rational spline IFS provides more flexibility and diversity for the choice of a suitable interpolants, which can recapture the traditional non-recursive shape modifiable interpolation, and provide shape properties of the interpolant and fractality of derivatives.

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Supporting information Appendixes A–E. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

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