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Set stability for switched Boolean networks with open-loop and closed-loop switching signals

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Abstract This paper mainly analyzes the set stability of switched Boolean networks (SBNs) by considering two different methods of switching signals in the form of open-loop and closed-loop switching. First, the definitions of set stability are proposed for SBNs with such switching signals. Then, we propose two different algorithms to find the largest open-loop switching invariant set and largest closed-loop switching invariant set. Furthermore, the corresponding set stability conditions for SBNs are given in each case. In addition, the constructive procedures are presented to design the switching signal sequence to achieve set stability under the two different methods of switching signals. Finally, the effectiveness of the proposed algorithms and research results are demonstrated using two numerical examples.

Keywords switched Boolean network, set stability, semi-tensor product, switching signal design

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1 Introduction

Since Boolean networks (BNs) were first proposed by Kauffman in 1969 as an important tool to model gene regulatory networks [1], it has attracted great attention in different theory and application domains [2,3]. In BNs, each Boolean variable expresses its state by 0 or 1 and updates its state in light of the Boolean function. Recently, the method of the semi-tensor product (STP) of matrices has been successfully applied to transform the BNs into a linear algebraic system that can be investigated by using the classical control methods [4]. Until now, this technique has been used to deal with many excellent studies on the control of BNs, including stability and stabilization [5–8], observation and controllability [9–14], optimal control [15, 16], and other control problems [17–26].

It is worth noting that the dynamics of BNs in practice often involves multiple instances of switching between different models [27]. For example, the growth and division of eukaryotic cells can be considered as a series of four event-triggered processes [28]. In addition, some established networks can be transformed into switched Boolean networks (SBNs). For instance, Boolean control networks (BCNs) can be deemed Boolean switched systems through the coding control input for the switching signal [10].

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Non-synchronous BNs can also be transformed into SBNs by composing all the Boolean functions [29]. Thus, the investigation of SBNs is very important. Over the years, numerous fascinating results have been presented regarding stability analysis and control design of SBNs [30–32] by resorting to the STP. Chen and Sun [33] researched the global stability and stabilization problems for SBNs by considering impulsive effects. Zhong et al. [34] investigated the synchronization of master-slave SBNs with delay. Li [35] considered arbitrary switching signals to solve the stability problem at a limit cycle of SBNs. Zhang et al. [36] discussed the finite automata approach for the observability of SBNs.

As is well known, stability is a commonly used property of control theory. Sometimes, a system cannot converge to a point, but to a few points of the state space, which can be considered as the main problem of set stability. Guo et al. [37] used invariant subsets to study the set stability and stabilization for BCNs, and pointed out that synchronization, stability and set stability are all special situations of the set stability and stabilization of BNs. Hence, these issues are significant. However, we also know that SBCNs [38] are more valuable than the previous models of BNs in certain theory and application areas. We consider whether we can design a switching signal rather than imposing control inputs to realize set stability in certain cases in the control of BNs. Unfortunately, there are no related results available on the set stability for SBNs considering open-loop and closed-loop switching signals, which is motivation for us to address this popular topic.

Motivated by the above discussions, this paper utilizes the method of STP to discuss the set stability problems for SBNs. The main contributions lie in the discussions on the switching invariant set and switching signals, which can be listed as follows.

(1) Two classes of algorithms are provided to calculate the largest open-loop switching invariant set and closed-loop switching invariant set, respectively.

(2) The set stability criteria for SBNs are given, and the constructive procedures are presented to design the open-loop and closed-loop switching signal sequence to achieve set stability, which is computationally tractable.

(3) We design open-loop switching signals to achieve set stability of SBNs, which is different from pointwise stability and global stability under arbitrary switching signals in [39]. The switching signal in our work depends only on time and is independent of the initial point, and after a certain time, the switching signal will remain unchanged. Our research results are of more practical significance.

2 Notation and preliminaries

Some notation and necessary preliminaries of STP are given as follows: $\mathbb{R}_{m\times n}$ denotes an $m \times n$-dimensional real matrix; $\mathbb{N}^*$ represents the set of positive integers; $\mathcal{D} := \{0,1\}$ and $\mathcal{D}^n = \underbrace{\mathcal{D}\times\mathcal{D}\times\cdots\times\mathcal{D}}_{n}$; δ_n^k denotes the i-th column of the $n\times n$ identity matrix I_n; $\Delta_n := \{\delta_n^k : k = 1,2,\ldots,n\}$, $\Delta := \Delta_2$; $\mathrm{Col}(L)$ denotes all the columns of matrix L and $\mathrm{Col}_i(L)$ denotes the i-th column of L; The logical matrix is $M{=}[\delta_n^{i_1}\ \delta_n^{i_2}\ \cdots\ \delta_n^{i_t}]$, which can be briefly expressed as $M = \delta_n[i_1, i_2, \ldots, i_t]$ and we define $\mathcal{L}_{m\times n}$ as the set of $m\times n$ logical matrix; $M\backslash N$ denotes the elements in set M get rid of the elements in set N.

Definition 1 ([4])**.** Let $A \in \mathbb{R}_{m\times n}$, $B \in \mathbb{R}_{p\times q}$, α is the least common multiple of n and p, then the STP of two matrices A and B is

$$A \ltimes B = (A \otimes I_{\alpha/n})(B \otimes I_{\alpha/p}), \tag{1}$$

where $\otimes$ is the Kronecker product.

If $n = p$, we can see that the STP of matrices A and B would become the traditional matrix product. Thus, we use the STP as the matrix product and the symbol $\ltimes$ is usually omitted if no confusion will arise. In addition, $W_{[m,n]} \in \mathcal{L}_{mn\times mn}$ denotes a swap matrix that can swap two vectors in their product, a detailed definition is given in [4].

By identifying $T = 1 \sim \delta_2^1$, $F = 0 \sim \delta_2^2$, the logical variable takes a value from $\mathcal{D} \sim \Delta = \{\delta_2^1, \delta_2^2\}$, where "$\sim$" denotes the same object in different forms.

Lemma 1 ([40]). Let $L(A_1, A_2, \ldots, A_n) : (\Delta)^n \mapsto \Delta$ be logical function with logical arguments A_1, $A_2, \ldots, A_n \in \Delta$, then there exists a unique structure matrix of L denoted by $M_L \in \mathcal{L}_{2\times 2^n}$, and the logical function can be expressed as

$$L(A_1, A_2, \ldots, A_n) = M_L A_1 \ltimes A_2 \ltimes \cdots \ltimes A_n. \tag{2}$$

Lemma 2 ([40]). Let $x = x_1 \ltimes x_2 \cdots x_n \in \Delta_{2^n}$ with $x_i \in \Delta, i = 1, 2, \ldots, n$, then

$$x^2 = \Phi_n x,$$

where $\Phi_n = \mathrm{diag}[\delta_{2^n}^1, \delta_{2^n}^2, \ldots, \delta_{2^n}^{2^n}] \in \mathcal{L}_{2^{2n}\times 2^n}$.

3 Main results

Consider the following SBN

$$\begin{cases} x_1(t+1) = f_1^{\sigma(t)}(x_1(t), x_2(t), \ldots, x_n(t)), \\ x_2(t+1) = f_2^{\sigma(t)}(x_1(t), x_2(t), \ldots, x_n(t)), \\ \quad\vdots \\ x_n(t+1) = f_n^{\sigma(t)}(x_1(t), x_2(t), \ldots, x_n(t)), \end{cases} \tag{3}$$

where $x_i \in \mathcal{D}$, $i = 1, 2, \ldots, n$ is Boolean variable, $f_i^{\sigma(t)}$: $\mathcal{D}^n \to \mathcal{D}$, $i = 1, 2, \ldots, n$ is logical function and the switching signal $\sigma : N \to W = \{1, 2, \ldots, \omega\}$ is a function of time.

Define $x(t) = \ltimes_{i=1}^n \in \Delta_{2^n}$, and $M_i^{\sigma(t)}$ is the structure matrix of $f_i^{\sigma(t)}$. By Lemma 1, Eq. (3) can be easily transformed into the following form:

$$\begin{cases} x_1(t+1) = M_1^{\sigma(t)} x(t), \\ x_2(t+1) = M_2^{\sigma(t)} x(t), \\ \quad\vdots \\ x_n(t+1) = M_n^{\sigma(t)} x(t). \end{cases} \tag{4}$$

Multiplying both sides of (4) leads to

$$x(t+1) = L_{\sigma(t)} x(t), \tag{5}$$

where $\mathrm{Col}_j(L_{\sigma(t)}) = \ltimes_{i=1}^n \mathrm{Col}_j(M_i^{\sigma(t)})$, $j = 1, 2, \ldots, 2^n$.

Indicating that $\sigma(t) = i \sim \delta_\omega^i$, $i \in W$, we obtain

$$\begin{aligned} x(t+1) =& [L_1\ L_2\ \cdots\ L_\omega]\sigma(t)x(t) \\ =& [L_1\ L_2\ \cdots\ L_\omega]W_{[2^n,\omega]}x(t)\sigma(t) \\ =& Fx(t)\sigma(t), \end{aligned} \tag{6}$$

where $L_j = \ltimes_{i=1}^n M_i^j \in \mathcal{L}_{2^n\times 2^n}$, $j = 1, 2, \ldots, \omega$ and $F = [L_1\ L_2\ \cdots\ L_\omega]W_{[2^n,\omega]} \in \mathcal{L}_{2^n\times\omega 2^n}$.

Split F into 2^n equal blocks as $F = [F_1\ F_2\ \cdots\ F_{2^n}]$. In Subsections 3.1 and 3.2, the set stability of (3) will be investigated based on the algebraic form (6).

3.1 Set stability in an open-loop switching signal

First, the definitions of set stability and the largest open-loop switching invariant set for SBN are given as follows.

Definition 2. Given a non-empty set $\mathcal{M} \subset \Delta_{2^n}$, define $x(t; x(0), \sigma)$ as the state vector of system (6) starting from the initial state $x(0)$ at switching signal σ. System (6) is $\mathcal{M}$-stable if $\exists\ T \in \mathbb{N}^*$ and $\sigma(T)$, such that $x(t; x(0), \sigma) \in \mathcal{M}$ holds for $\forall\ x(0) \in \Delta_{2^n}$ and $\forall\ t \geqslant T$.

Definition 3. The set $\widetilde{M} \subset \mathcal{M}$ is the open-loop switching invariant set of $\mathcal{M}$ for SBN (6), if $\exists\, 1 \leqslant \mu_0 \leqslant \omega$, such that $\mathrm{Col}_{\mu_0}(F_i) \in \widetilde{M}$ holds for $\forall\ \delta_{2^n}^i \in \widetilde{M}$. A set M^* is called the largest open-loop switching invariant set of $\mathcal{M}$ for SBN (6), if it contains the largest number of elements among all the open-loop switching invariant sets.

Let $\mathcal{M} = \{\delta_{2^n}^{i_1}, \delta_{2^n}^{i_2}, \ldots, \delta_{2^n}^{i_p}\}$, Algorithm 1 is constructed to find the largest open-loop switching invariant set M^*.

Algorithm 1

Step 1. If there exists a unique μ, for $\forall\ \delta_{2^n}^{i_j} \in \mathcal{M}$, $j = 1, 2, \ldots, p$, we have $\mathrm{Col}_\mu(F_{i_j}) \in \mathcal{M}$, let $M^* = \mathcal{M}$; otherwise, we set $\mu = \mu_0$ (μ_0 exists in Definition 3) and go to Step 2.

Step 2. Search $\delta_{2^n}^{i_j} \in \mathcal{M}$ such that for $1 \leqslant \mu \leqslant \omega$ in Step 1, $\mathrm{Col}_\mu(F_{i_j}) \notin \mathcal{M}$. We use the notation $M_1 = \mathcal{M}\backslash\delta_{2^n}^{i_j}$. If for $\forall\ \delta_{2^n}^{i_j} \in M_1$, we have $\mathrm{Col}_\mu(F_{i_j}) \in M_1$, let $M^* = M_1$; otherwise, go to Step 3.

Step 3. Search $\delta_{2^n}^{i_j} \in M_1$ such that for $1 \leqslant \mu \leqslant \omega$ in Step 1, $\mathrm{Col}_\mu(F_{i_j}) \notin M_1$. We use the notation $M_2 = M_1\backslash\delta_{2^n}^{i_j}$.

Step 4. Continue this similar process until we find the set M_k. There exists $1 \leqslant \mu \leqslant \omega$ such that for $\forall\ \delta_{2^n}^{i_j} \in M_k$, we have $\mathrm{Col}_\mu(F_{i_j}) \in M_k$. Let $M^* = M_k$ be the largest open-loop switching invariant set.

Proposition 1. By using Algorithm 1, it is easy to obtain the largest open-loop switching invariant set M^* of $\mathcal{M}$ for SBN (6).

Next, the stability of (6) is investigated by considering an open-loop switching signal sequence $\{(0, \sigma(0)), \ldots, (l, \sigma(l)), \ldots\}$. From (6), for $\forall\ \tau \in \mathbb{N}^*$, we have

$$\begin{aligned} x(\tau) =& Fx(\tau-1)\sigma(\tau-1) \\ =& F^2x(\tau-2)\sigma(\tau-2)\sigma(\tau-1) \\ =& F^\tau x(0)\sigma(0)\cdots\sigma(\tau-1). \end{aligned}$$

Split F^τ into 2^n equal blocks as $F^\tau = [F_1^\tau\ F_2^\tau\ \cdots\ F_{2^n}^\tau]$, we can obtain Theorem 1.

Theorem 1. System (6) is $\mathcal{M}$-stable in an open-loop switching signal iff

(1) $M^* \neq \emptyset$;

(2) $\exists\ 1 < \tau$ and $1 \leqslant \eta \leqslant \omega^\tau$, then $\mathrm{Col}_\eta(F_i^\tau) \in M^*$ holds for $\forall\ i = 1, 2, \ldots, 2^n$.

Moreover, if conditions (1) and (2) hold, we can design the following open-loop switching sequence:

$$\sigma(t) = \begin{cases} \mu^*(t), & 0 \leqslant t \leqslant \tau - 1, \\ \delta_\omega^\mu, & t \geqslant \tau, \end{cases} \tag{7}$$

where μ^* is determined by $\ltimes_{i=\tau-1}^0 \mu^*(i) = \delta_{\omega^\tau}^\eta$.

Proof. (Sufficiency) For $\forall\ x(0) = \delta_{2^n}^i \in \Delta_{2^n}$, $i = 1, 2, \ldots, 2^n$, postulate that (1) and (2) hold, and set $\ltimes_{i=\tau-1}^0 \sigma(i) = \delta_{\omega^\tau}^\eta$, then we have

$$x(\tau) = F^\tau x(0)\sigma(0)\cdots\sigma(\tau-1) = F_i^\tau \delta_{\omega^\tau}^\eta = \mathrm{Col}_\eta(F_i^\tau) \in M^*. \tag{8}$$

For $t = \tau + 1$ and $\sigma(\tau) = \delta_\omega^\mu$, by the above algorithm, we have

$$x(t) = x(\tau+1) = Fx(\tau)\sigma(\tau) = \mathrm{Col}_\mu(F_{i_j}) \in M^*,$$

where $x(\tau) = \delta_{2^n}^{i_j} \in M^*$, $1 \leqslant \mu \leqslant \omega$. Thus, system (6) is stable to $\mathcal{M}$ under the sequences (7) for $\forall\ t \geqslant \tau$, $\forall\ x(0) \in \Delta_{2^n}$.

(Necessity) Assume that system (6) is stable to $\mathcal{M}$ under open-loop switching sequence $\sigma(t)$, then for $\forall\ x(0) = \delta_{2^n}^i \in \Delta_{2^n}$, there exists $\tau > 0$ such that $x(\tau) = F^\tau x(0)\sigma(0)\cdots\sigma(\tau-1) \in M^*$ holds. Setting $\sigma(0) \ltimes \cdots \ltimes \sigma(\tau-1) = \delta_{\omega^\tau}^\eta$, then we have $x(t) = \mathrm{Col}_\eta(F_i^\tau) \in M^*$, which implies that condition (2) is true.

If condition (1) does not hold, that is, $M^* = \emptyset$, condition (2) is not satisfied. That is, for $\forall\ \tau > 1$ and $\forall\ 1 \leqslant \eta \leqslant \omega^\tau$, there exists $x(0) = \delta_{2^n}^i$, such that $\mathrm{Col}_\eta(F_i^\tau) \notin M^*$, which is contradictory. Hence, condition (1) holds.

Remark 1. Theorem 1 provides a method to design an open-loop switching sequence that stabilizes the system (6) to $\mathcal{M}$.

Remark 2. The open-loop switching here is only dependent on time and it changes to a constant when the system reaches set stability. In this paper, we assume that the system reaches set stability at time T. We give the conditions for set stability under open-loop switching, and consider how to design the switching signal $\sigma(0), \sigma(1), \ldots, \sigma(T-1), \sigma(T)$. Li et al. [39] also considered the stability problem of SBNs. It requires stability to be achieved under an arbitrary switching signal, thus it needs a stronger condition.

3.2 Set stability in the closed-loop switching signals

Now, we consider set stability for SBNs by choosing the proper closed-loop switching signals. At the same time, we provide a constructive program to design such switching signals.

Definition 4. The set $\widehat{M} \subset \mathcal{M}$ is the closed-loop switching invariant set of $\mathcal{M}$ for SBN (6), if for $\forall\, \delta_{2^n}^i \in \widehat{M}$, $\exists\ 1 \leqslant \mu_i \leqslant \omega$, such that $\mathrm{Col}_{\mu_i}(F_i) \in \widehat{M}$ holds. A set M' is called the largest closed-loop switching invariant set of $\mathcal{M}$ for SBN (6), if it contains the largest number of elements among all the closed-loop switching invariant sets.

Define $R(\delta_{2^n}^i)$ as the set whose elements can be steered to $\delta_{2^n}^i$ in one step. From (6), we have

$$R(\delta_{2^n}^i) = \{\delta_{2^n}^j : \delta_{2^n}^i \in \mathrm{Col}(F_j),\ j = 1, 2, \ldots, 2^n\}.$$

Then, we define $R(\mathcal{M}) = \bigcup_{j=1}^{p} R(\delta_{2^n}^{i_j})$. Algorithm 2 is given to find the largest closed-loop switching invariant set M'.

Algorithm 2

Step 1. If for $\forall\ \delta_{2^n}^{i_j} \in \mathcal{M}, j = 1, 2, \ldots, p$, there exists μ_{i_j} such that $\mathrm{Col}_{\mu_{i_j}}(F_{i_j}) \in \mathcal{M}$, let $M' = \mathcal{M}$; otherwise, go to Step 2.

Step 2. Search $\delta_{2^n}^{i_j} \in \mathcal{M}$ such that for $\forall\ 1 \leqslant \mu \leqslant \omega$, $\mathrm{Col}_{\mu}(F_{i_j}) \notin \mathcal{M}$. We use the notation $M_1 = \mathcal{M} \backslash \delta_{2^n}^{i_j}$.
If for $\forall\ \delta_{2^n}^{i_j} \in M_1$, there exists μ_{i_j} such that $\mathrm{Col}_{\mu_{i_j}}(F_{i_j}) \in M_1$, let $M' = M_1$; otherwise, go to Step 3.

Step 3. Search $\delta_{2^n}^{i_j} \in M_1$ such that for $\forall\ 1 \leqslant \mu \leqslant \omega$, $\mathrm{Col}_{\mu}(F_{i_j}) \notin M_1$. We use the notation $M_2 = M_1 \backslash \delta_{2^n}^{i_j}$.

Step 4. Continue this similar process until we find the set M_k such that for $\forall\ \delta_{2^n}^{i_j} \in M_k$, there exists μ_{i_j} such that $\mathrm{Col}_{\mu_{i_j}}(F_{i_j}) \in M_k$. Let $M' = M_k$ be the largest close-loop switching invariant set.

Proposition 2. By using Algorithm 2, we can easily find the largest closed-loop switching invariant set M' of $\mathcal{M}$ for SBN (6).

Now, we construct a sequence of vector sets by using the following procedure:

$$\begin{aligned} &S_0 = M', \\ &S_1 = R(S_0) \backslash S_0, \\ &S_2 = R(S_1) \backslash \bigcup_{i=0}^{1} S_i, \\ &\vdots \\ &S_m = R(S_{m-1}) \backslash \bigcup_{i=0}^{m-1} S_i. \end{aligned} \tag{9}$$

There is a positive integer $m \leqslant 2^n - |M'|$ such that $S_{m+1} = \emptyset$, $\sum_{i=0}^{m-1} |S_i| \leqslant 2^n$. From the above procedure, Theorem 2 can be obtained.

Theorem 2. System (6) is $\mathcal{M}$-stable under closed-loop switching signal $\sigma(t) = Kx(t)$ iff

(1) $M' \neq \emptyset$;

(2) $\exists\ m \leqslant 2^n - |M'|$ such that $\bigcup_{i=1}^{m} S_i = \Delta_{2^n}$ holds.

Proof. (Sufficiency) Supposing that (1) and (2) hold, we need to construct a state feedback transition matrix $K \in \mathcal{L}_{\omega \times 2^n}$ to realize $\mathcal{M}$-stable.

The procedure (9) and two conditions imply that for $\delta_{2^n}^i \in \Delta_{2^n}$, there exists a unique set $S_{l_i}, l_i \in \{0, 1, \ldots, m\}$ such that $\delta_{2^n}^i \in S_{l_i}$. For $\delta_{2^n}^i \notin M'$ we set

$$Q(i) = \{\delta_\omega^j : \mathrm{Col}_j(F_i) \in S_{l_i - 1},\ j = 1, 2, \ldots, \omega,\ l_i = 1, 2, \ldots, m\}.$$

In particular, when $\delta_{2^n}^i \in M'$, we set

$$Q(i) = \{\delta_\omega^j : \mathrm{Col}_j(F_i) \in S_0,\ j = 1, 2, \ldots, \omega\}.$$

Then the state feedback transition matrix $K = [Q_1, Q_2, \ldots, Q_{2^n}]$, where $Q_i \in Q(i)$, $i = 1, 2, \ldots, 2^n$, under which the SBN can become $\mathcal{M}$-stable.

(Neccessity) If system (6) is $\mathcal{M}$-stable under closed-loop switching $\sigma(t)$, there exists $T \geqslant 0$ for $\forall\, x(0) \in \Delta_{2^n}$, such that $x(t) \in \mathcal{M}$ for $\forall\ t \geqslant T$. Now, we prove the two conditions individually.

(1) If the condition (1) does not hold, that is, $M' = \emptyset$, condition (2) is not satisfied. That is, for $\forall\, m$, we have $\bigcup_{i=1}^m S_i \neq \Delta_{2^n}$, which is contrary to the fact that system (6) can be stabilized to $\mathcal{M}$ by a closed-loop switching sequence $\sigma(t) = Kx(t)$. Thus, condition (1) is met.

(2) Since system (6) is $\mathcal{M}$-stable for $\forall\ x(0) \in \Delta_{2^n}$, it is clear that we have $\bigcup_{i=1}^m S_i = \Delta_{2^n}$. Let m be the smallest positive integer such that $\bigcup_{i=1}^m S_i = \Delta_{2^n}$. Since there are $2^n - |M'|$ elements contained in $\Delta_{2^n} \backslash M'$, we have $m \leqslant 2^n - |M'|$.

Remark 3. Theorem 2 gives us a way to construct feedback transition matrix K that is computationally tractable.

Remark 4. System (6) is $\mathcal{M}$-stable under closed-loop switching $\sigma(t) = Kx(t)$, where $K = [Q_1, Q_2, \ldots, Q_{2^n}]$ is constructed in Theorem 2. Further, the total number of different closed-loop switchings is $\prod_{i=1}^{2^n} |Q(i)| = |Q(1)| \times |Q(2)| \times \cdots \times |Q(2^n)|$.

4 Numerical examples and computation

In this section, two numerical examples are provided to show how to check the set stability of SBN based on the proposed theoretical results.

Example 1. Consider the existing apoptosis network

$$\begin{cases} x_1(t+1) = \neg x_2(t) \wedge u(t), \\ x_2(t+1) = \neg x_1(t) \wedge x_3(t), \\ x_3(t+1) = x_2(t) \vee u(t), \end{cases} \tag{10}$$

where x_1, x_2, x_3 and u represent the level of concentration of inhibitor of apoptosis protein (IAP), caspase 3 (C3a), caspase 8 (C8a), and tumor necrosis factor (TNF), respectively.

By letting $u = 1$ and $u = 0$, Eq. (10) can be transformed into the following SBN [31]:

$$\begin{cases} x_1(t+1) = f_1^{\sigma(t)}(x_1(t), x_2(t), x_3(t)), \\ x_2(t+1) = f_2^{\sigma(t)}(x_1(t), x_2(t), x_3(t)), \\ x_3(t+1) = f_3^{\sigma(t)}(x_1(t), x_2(t), x_3(t)), \end{cases} \tag{11}$$

where $\sigma : N \mapsto W = \{1, 2\}$, $f_1^1 = \neg x_2$, $f_2^1 = \neg x_1 \wedge x_3$, and $f_3^1 = 1$ correspond to $u = 1$, $f_1^2 = 0$, $f_2^2 = \neg x_1 \wedge x_3$, and $f_3^1 = x_2$ correspond to $u = 0$.

From the above network model, $\{x_1 = 1, x_2 = 0\}$ indicates the cell survival, so it is interesting to observe the states of x_1 and x_2 at each time instance. Hence, we use the notation $\mathcal{M} = \{(1, 0, 1), (1, 0, 0)\}$. Next, we research the $\mathcal{M}$-stable of system (11), and there exist two key problems as follows that should be solved in this example.

(1) Can system (11) achieve $\mathcal{M}$-stable under an open-loop switching sequence?

(2) Can system (11) achieve $\mathcal{M}$-stable under closed-loop switching?

We convert the logical variables into vector forms. Setting $x(t) = \ltimes_{i=1}^{3} x_i(t)$, we can easily obtain the following algebraic form:

$$x(t+1) = Fx(t)\sigma(t),$$

where $F = \delta_8[7\ 7\ 7\ 7\ 3\ 8\ 3\ 8\ 5\ 5\ 7\ 7\ 1\ 6\ 3\ 8]$. Moreover, $\mathcal{M} = \{\delta_8^3, \delta_8^4\}$.

Splitting F into 2^3 blocks as $F = [F_1\ F_2\ \cdots\ F_{2^3}]$, we can obtain $F_1 = \delta_8[7\ 7]$, $F_2 = \delta_8[7\ 7]$, $F_3 = \delta_8[3\ 8]$, $F_4 = \delta_8[3\ 8]$, $F_5 = \delta_8[5\ 5]$, $F_6 = \delta_8[7\ 7]$, $F_7 = \delta_8[1\ 6]$, and $F_8 = \delta_8[3\ 8]$. Based on Algorithms 1 and 2, we can obtain $M^* = M' = \mathcal{M}$. For $x(0) = \delta_8^2$, we can see that for $\forall\ \sigma \in W = \{1, 2\}$ and for $\forall\ t > 0$, we have $x(t; \delta_8^2, \sigma) \notin \mathcal{M}$. Thus, the condition (2) of Theorem 1 is not satisfied, and system (11) cannot achieve $\mathcal{M}$-stable under an open-loop switching sequence.

Now, we consider problem (2). Based on procedure (9), a straightforward computation shows that

$$\begin{aligned} &S_0 = M' = \{\delta_8^3, \delta_8^4\},\\ &S_1 = \{\delta_8^8\},\\ &S_2 = \emptyset. \end{aligned} \tag{12}$$

Thus, condition (2) of Theorem 2 does not hold. Therefore, we cannot design a close-loop switching to achieve $\mathcal{M}$-stable, which is consistent with the bistability of the cell apoptosis networks [31].

Example 2. Consider the following SBN model in [39]:

$$\begin{cases} x_1(t+1) = f_1^{\sigma(t)}(x_1(t), x_2(t)),\\ x_2(t+1) = f_2^{\sigma(t)}(x_1(t), x_2(t)), \end{cases} \tag{13}$$

where $\sigma : N \mapsto W = \{1, 2, 3\}$ is the switching signal, and

$$\begin{aligned} &f_1^1(x_1, x_2) = \neg x_1 \wedge x_2, \quad f_1^2(x_1, x_2) = x_1 \bar{\vee} x_2, \quad f_1^3(x_1, x_2) = x_1 \bar{\vee} x_2,\\ &f_2^1(x_1, x_2) = 0, \quad f_2^2(x_1, x_2) = \neg(x_1 \to x_2), \quad f_2^3(x_1, x_2) = x_1 \bar{\vee} x_2. \end{aligned}$$

Our objective is to detect the stability of (13) with respect to the set $\mathcal{M} = \{(1, 1), (2, 2)\}$.

We also use the vector to express logical variables. Set $x(t) = \ltimes_{i=1}^{2} x_i(t)$, and we can easily obtain the following algebraic form of (13):

$$x(t+1) = Fx(t)\sigma(t),$$

where $F = \delta_4[4\ 4\ 4\ 4\ 1\ 1\ 2\ 2\ 1\ 4\ 4\ 4]$. Moreover, $\mathcal{M} = \{\delta_4^1, \delta_4^4\}$.

Splitting F into four blocks as $F = [F_1\ F_2\ F_3\ F_4]$, we obtain that $F_1 = \delta_4[4\ 4\ 4]$, $F_2 = \delta_4[4\ 1\ 1]$, $F_3 = \delta_4[2\ 2\ 1]$, and $F_4 = \delta_4[4\ 4\ 4]$, then we can easily obtain that $\mathrm{Col}(F_4) = \mathrm{Col}(F_1) \subset \mathcal{M}$. At the same time, we also know that $F^2 = \delta_4[4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 1\ 1\ 4\ 1\ 1\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4\ 4]$, $\mathrm{Col}(F^2) \subset \mathcal{M}$. Based on Algorithms 1 and 2, we can obtain $M^* = M' = \mathcal{M}$. Hence, system (13) is $\mathcal{M}$-stable under an open-loop switching signal, and the open-loop switching sequence can be designed as $\sigma(0) = \sigma(1) = \cdots = \sigma(t) \in \{\delta_3^1, \delta_3^2, \delta_3^3\}$, $t \geqslant 1$.

In the following, we consider whether or not system (13) is stable with respect to $\mathcal{M} = \{(1, 1), (2, 2)\}$. Based on procedure (9), a straightforward computation shows that

$$\begin{aligned} &S_0 = M' = \{\delta_4^1, \delta_4^4\},\\ &S_1 = \{\delta_4^2, \delta_4^3\}. \end{aligned} \tag{14}$$

This satisfies that $M' \neq \emptyset$ and $\bigcup_{i=1}^{2} S_i = \Delta_{2^2}$, and the closed-loop switching $\sigma(t) = Kx(t)$, where $K = [Q_1, Q_2, Q_3, Q_4]$, and $Q_1 = Q_2 = Q_4 \in \{\delta_3^1, \delta_3^2, \delta_3^3\}$, $Q_3 = \delta_3^3$. Moreover, the total number of different closed-loop switchings that can stabilize the SBN is 27.

Remark 5. To check $\mathcal{M}$-stable, we first calculate the largest open-loop switching invariant set and closed-loop switching invariant set based on Algorithms 1 and 2, respectively. Then, we verify the conditions of the theorems individually, and then design the corresponding switching signal. The largest open-loop switching invariant set is not unique, but the largest closed-loop switching invariant set is unique with our algorithms. In Example 1, we use the contradiction method to conclude that the system is not $\mathcal{M}$-stable. In Example 2, $\mathcal{M}$-stable under open-loop switching is equivalent to arbitrary switching, and the system is time-optimal $\mathcal{M}$-stable under a closed-loop switching signal.

Remark 6. Open-loop and closed-loop switchings are essentially different. For a given set, the largest open-loop switching invariant set is likely to be different from the closed-loop version. Even if the largest open-loop switching invariant set is the same as the closed-loop version, the design of switching signals is generally different. In general, closed-loop switching is more efficient than open-loop switching.

5 Conclusion

The set stability problems for SBNs with open-loop and closed-loop switching signals have been addressed by resorting to the method of STP. The corresponding set stability conditions for SBNs have been introduced based on the largest open-loop invariant set and closed-loop invariant set, respectively. The constructive procedures have been proposed to design an open-loop switching signal and closed-loop switching signal to realize set stability. Finally, the effectiveness of the proposed results have been demonstrated by using two numerical examples. Some meaningful and challenging studies could be extended to the study of using a pinning control technique to reduce the computation complexity in the future.

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